Conformally Invariant Random Geometry on Manifolds of Even Dimensions

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based on joint work
Conformally invariant random fields, Liouville Quantum Gravity measures, and random Paneitz operators on Riemannian manifolds of even dimension
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Random Riemannian Geometry and Conformal Invariance

Original Goal

"Average" over the set of all (compact) Riemannian manifolds (M, g) of dimension n, i.e. find a probability measure $\mathbf{P}$ on this set.

Simplification: fix a reference space (M, g) and ask for average over the set

$$\left\{ \left(M, e^{2h} g \right) : \; h \in \mathcal{C}(M) \right\}.$$

Modified Goal

Associate to each (M, g) a probability measure $\mathbf{P}_{M,g}$ on "fields" (continuous functions, distributions) on M such that

- $\mathbf{P}_{M,g'} = \mathbf{P}_{M,g}$ if $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}(M)$
- $h \overset{(d)}{=} h' \circ \Phi$ if $\Phi : M \to M'$ is an isometry and h and h' are distributed according to $\mathbf{P}_{M,g}$ and $\mathbf{P}_{M',g'}$, resp.

For the sequel fix M.

Typically, $\mathbf{P}_g$ is a Gaussian field, informally given as

$$d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\,dh$$

with (non-existing) uniform distribution dh on $\mathcal{C}(M)$, normalizing constant Z_g, and bilinear form $\mathfrak{e}_g(u,v) = (u, Av)_{L^2}$.
Rigorous definition (on spaces of distributions rather than continuous functions) via Bochner–Minlos Theorem

$$\int e^{i\langle u,h\rangle}\,d\mathbf{P}_g(h) = \exp\Big(-\frac{1}{2}\mathfrak{k}_g(u,u)\Big)$$

where $\mathfrak{k}_g(u,v) := (u, A^{-1}v)_{L^2} = \iint u(x)v(y)\,k(x,y)\,dx\,dy$ Green energy dual to $\mathfrak{e}_g$. Then $\mathbb{E}[h(x)] = 0$, $\mathbb{E}[h(x)h(y)] = k(x,y)$.

Random Riemannian Geometry and Conformal Invariance

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Conformal Invariance Requirement

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall \varphi,\ \forall u.$$

In case $n = 2$, celebrated property of the *Dirichlet energy*

$$\mathfrak{e}_g(u,u) := \int_M \big|\nabla_g u\big|^2\, d\,\mathrm{vol}_g\,.$$

Random Riemannian Geometry and Conformal Invariance

In case $n = 2$:

Gaussian Free Field [Sheffield, Miller, ...],

$$d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh \tag{1}$$

with conformally invariant Dirichlet energy

$$\mathfrak{e}_g(u,u) = \int_M \big|\nabla_g u\big|^2 \, d\,\mathrm{vol}_g = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall\varphi,\ \forall u.$$

Liouville Quantum Gravity: random measure

$$e^{\gamma h(x) - \frac{\gamma^2}{2}\mathbf{E}h(x)^2}\, d\,\mathrm{vol}(x)$$

rigorously defined as weak limit of RHS with h replaced by regular approximations $(h_\ell)_{\ell\in\mathbb{N}}$

Links to Schramm–Loewner evolution [Lawler/Schramm/Werner, ...], convergence to Brownian map: universal scaling limit of planar random graphs [LeGall, Miermond]

How to give a rigorous meaning to (1), how to find a instructive interpretation?
Consider particular case **n=1**, say $M = [0, 1]$, $h(0) = 0$:

$$d\mathbf{P}(h) = \frac{1}{Z} \exp\Big(- \frac{1}{2} \int_0^1 h'(t)^2 dt \Big)\, dh$$

Random Riemannian Geometry and Conformal Invariance

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- = Wiener measure
- = limit of distribution for the values $(h_j)_{j=1,\dots,N}$ at points $(t_j)_{j=1,\dots,N}$ of partition of $[0,1]$, chosen according to the Dirichlet energy of the linear interpolation
- = distribution of

$$h := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \psi_j\, \xi_j = \sum_{j\in\mathbb{N}} \frac{\pi}{j} \sin\Big(\frac{j}{\pi}\,\cdot\Big)\, \xi_j$$

  with eigenfunctions $(\psi_j)_{j\in\mathbb{N}}$, eigenvalues $(\nu_j)_{j\in\mathbb{N}}$ for the negative (Dirichlet) Laplacian on $[0,1]$ and iid $\mathcal{N}(0,1)$ random variables $(\xi_j)_{j\in\mathbb{N}}$.

This yields probability measure on functions $h : [0,1] \to \mathbb{R}$ of class $\mathcal{C}^{1/2-\epsilon}$.

Similar construction in **n=2** yields probability measure on distributions h on M of class $H^{-\epsilon}$.

Random Riemannian Geometry and Conformal Invariance

Gaussian fields $d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(-\frac{1}{2}\mathfrak{e}_g(h,h)\Big)\, dh$ with conformally invariant energy

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \qquad \forall\varphi,\ \forall u.$$

In $n \neq 2$, Dirichlet energy no longer conformally invariant:

$$\mathfrak{e}_{e^{2\varphi}g}(u,u) = \int_M \left|\nabla_g u\right|^2 e^{(n-2)\varphi}\, d\,\mathrm{vol}_g\,.$$

In $n = 4$, more promising: bi-Laplacian energy

$$\tilde{\mathfrak{e}}_g(u,u) := \int_M \left(\Delta_g u\right)^2 d\,\mathrm{vol}_g\,.$$

Still not conformally invariant but close to:

$$\tilde{\mathfrak{e}}_{e^{2\varphi}g}(u,u) := \int_M \left(\Delta_g u + 2\nabla_g\varphi\,\nabla_g u\right)^2 d\,\mathrm{vol}_g = \tilde{\mathfrak{e}}_g(u,u) + \text{ low order terms.}$$

Paneitz:

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 - 2\,\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\,\mathrm{scal}_g \cdot |\nabla_g u|^2\right] d\,\mathrm{vol}_g$$

is conformally invariant.

Co-Polyharmonic Energy on n-Manifolds

Assume from now on that (M, g) is n-dimensional smooth, compact, connected Riemannian manifold without boundary, n even.

Graham/Jenne/Mason/Sparling.

The co-polyharmonic energy

$$\mathfrak{e}_g(u,v) = c\int_M \big(-\Delta_g\big)^{n/4} u \cdot \big(-\Delta_g\big)^{n/4} v \; d\,\mathrm{vol}_g + \text{low order terms}$$

is conformally invariant.

$\mathfrak{e}_g(u,v) = \int_M p_g u \cdot v \; d\,\mathrm{vol}_g$ with co-polyharmonic operator

$$p_g u := c\,(-\Delta)^{n/2} u + \text{ low order terms}$$

Choose $c = \frac{2}{\Gamma(n/2)\,(4\pi)^{n/2}} =: a_n$.

Co-Polyharmonic Energy on n-Manifolds

Integrable functions (or distributions) u on M will be called grounded if $\int_M u \, d\,\mathrm{vol}_g = 0$ (or $\langle u, \mathbf{1}\rangle = \mathbf{0}$, resp.).

Grounded Sobolev spaces $\mathring{H}^s(M,g) = (-\Delta_g)^{-s/2}\mathring{L}^2(M,\mathrm{vol}_g)$ for $s \in \mathbb{R}$,
usual Sobolev spaces $H^s(M,g) = (1-\Delta)^{-s/2}L^2(M,\mathrm{vol}_g) = \mathring{H}^s(M,g) \oplus \mathbb{R}\cdot\mathbf{1}$

Laplacian $-\Delta : H^s \to \mathring{H}^{s-2}$; grounded Green operator $\mathring{\mathsf{G}}_g : \mathring{H}^s \to \mathring{H}^{s+2}$.

Definition

The n-manifold (M,g) is called admissible if $\mathfrak{e}_g > 0$ on $\mathring{H}^{n/2}(M)$.

Large classes of n-manifolds are admissible. For instance in $n = 4$:

- all compact Einstein 4-manifolds with $\mathrm{Ric} \geq 0$ are admissible.
- all compact hyperbolic 4-manifolds with spectral gap $\lambda_1 > 2$ are admissible.

For the sequel, we always assume that (M,g) is admissible.

Two Key Properties of the Co-Polyharmonic Green Kernel

Define co-polyharmonic Green operator

$$\mathsf{k}_g := \mathsf{p}_g^{-1} : H^{-n}(M) \to \mathring{L}^2(M)$$

and associated bilinear form with domain $H^{-n/2}(M)$ by

$$\mathfrak{k}_g(u, v) := \langle u, \mathsf{k}_g v \rangle_{L^2}.$$

Theorem

k_g is an integral operator with an integral kernel k_g which is grounded, symmetric, and satisfies

$$\Big| k_g(x, y) + \log d_g(x, y) \Big| \leq C_0.$$

Theorem

Assume that $g' := e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then the co-polyharmonic Green kernel $k_{g'}$ for the metric g' is given by

$$k_{g'}(x, y) = \ k_g(x, y) - \frac{1}{2}\bar{\phi}(x) - \frac{1}{2}\bar{\phi}(y)$$

with $\bar{\phi} \in \mathcal{C}^\infty(M)$ obtained by re-grounding.

Co-Polyharmonic Gaussian Field – Definition, Construction

Definition

A co-polyharmonic Gaussian field on (M, g) is a linear family $\{\langle h, u\rangle\}_{u\in H^{-n/2}}$ of centered Gaussian random variables (defined on some probability space) with

$$\mathbf{E}\big[\langle h, u\rangle\,\langle h, v\rangle\big] = \mathfrak{k}_g(u, v) \qquad \forall u, v \in H^{-n/2}(M).$$

Interpretation: $\mathbf{E}\big[h(x)\big] = 0, \qquad \mathbf{E}\big[h(x)\,h(y)\big] = k_g(x, y) \qquad (\forall x, y)$

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Let a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ be given and an i.i.d. sequence $(\xi_j)_{j\in\mathbb{N}}$ of $\mathcal{N}(0, 1)$ random variables. Furthermore, let $(\psi_j)_{j\in\mathbb{N}_0}$ and $(\nu_j)_{j\in\mathbb{N}_0}$ denote the sequences of eigenfunctions and eigenvalues for p_g (counted with multiplicities).

Theorem

A co-polyharmonic field is given by

$$h := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

A co-polyharmonic Gaussian field on (M, g) can be regarded as a random variable with values in $\mathring{H}^{-\epsilon}(M)$ for any $\epsilon > 0$.

Co-Polyharmonic Gaussian Field – Smooth Approximation

Theorem

A co-polyharmonic field is given by

$$h := \sum_{j\in\mathbb{N}} \xi_j \cdot \sqrt{\mathsf{k}_g}\, \psi_j = \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

More precisely,

1. *For each $\ell \in \mathbb{N}$, a centered Gaussian random variable h_ℓ with values in $\mathcal{C}^\infty(M)$ is given by*

$$h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2}\, \xi_j\, \psi_j.$$

2. *The convergence $h_\ell \to h$ holds in $L^2(\mathbf{P}) \times H^{-\epsilon}(M)$ for every $\epsilon > 0$. In particular, for a.e. ω and every $\epsilon > 0$,*

$$h^\omega \in H^{-\epsilon}(M),$$

3. *For every $u \in H^{-n/2}(M)$, the family $(\langle u, h_\ell\rangle)_{\ell\in\mathbb{N}}$ is a centered $L^2(\mathbf{P})$-bounded martingale and*

$$\langle u, h_\ell\rangle \to \langle u, h\rangle \quad \textit{in } L^2(\mathbf{P}) \textit{ as } \ell \to \infty.$$

Co-Polyharmonic Gaussian Field – Discrete Approximation

Let M be the continuous torus $\mathbb{T}^n \cong [0,1)^n$ and consider its discrete approximations $\mathbb{T}^n_L \cong \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^n$ for $L \in \mathbb{N}$.

Co-polyharmonic Gaussian Field on the discrete torus $\mathbb{T}^n_L$

Centered Gaussian field $(h_L(v))_{v\in\mathbb{T}^n_L}$ with covariance function

$$k_L(u,v) = \frac{1}{a_n}\,\mathring{G}^{n/2}_L(u,v) = \frac{1}{a_n}\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/2}_{L,z}}\cdot\cos\Big(2\pi\, z\cdot(v-u)\Big)$$

where $\lambda_{L,z} = 4L^2\sum_{k=1}^n \sin^2\big(\pi z_k/L\big)$ and $\mathbb{Z}^n_L = \{z\in\mathbb{Z}^n : 0 < \|z\|_\infty < L/2\}$.

Given iid standard normals $(\xi_z)_{z\in\mathbb{Z}^n_L}$ and Fourier basis functions $\varphi_z(x) = \frac{1}{\sqrt{2}}\cos(2\pi xz)$ and $\varphi_{-z}(x) = \frac{1}{\sqrt{2}}\sin(2\pi xz)$, a Co-polyharmonic Gaussian Field is given as

$$h_L = \frac{1}{\sqrt{a_n}}\sum_{z\in\mathbb{Z}^n_L\setminus\{0\}} \frac{1}{\lambda^{n/4}_{L,z}}\cdot\xi_z\,\varphi_z.$$

Given white noise on $\mathbb{T}^n_L$, i.e. iid centered Gaussian variables $(\Xi(v))_{v\in\mathbb{T}^n_L}$ with variance $L^{n/2}$, then

$$h_L = \frac{1}{\sqrt{a_n}}\,\mathring{G}^{n/4}_L\,\Xi.$$

Co-Polyharmonic Gaussian Field – Discrete Approximation

The law of the "ungrounded" Polyharmonic Gaussian Field is given explicitly as

$$c_n \exp\left(-\frac{a_n}{2N}\left\|(-\Delta_L)^{n/4}h\right\|^2\right) d\mathcal{L}^N(h)$$

on $\mathbb{R}^N \cong \mathbb{R}^{\mathbb{T}_L^n}$ where $N = L^n$.

Theorem

- *Convergence of fields $h_L \to h$ as $L \to \infty$: tested against $f \in \bigcup_{s>n/2} H^s(\mathbb{T}^n)$*
- *Convergence of Fourier extension of h_L to h: in each $H^{-\epsilon}(\mathbb{T}^n)$ and also tested against $f \in H^{-n/2}(\mathbb{T}^n)$*

Co-Polyharmonic Gaussian Field – Conformal Invariance

The co-polyharmonic Gaussian field for general (M, g) is conformally invariant modulo re-grounding.

Theorem

Let $h : \Omega \to H^{-\epsilon}(M)$ denote a co-polyharmonic Gaussian field for (M, g) and let $g' = e^{2\varphi} g$ with $\varphi \in \mathcal{C}^\infty(M)$. Then

$$h' := h - \frac{1}{\mathrm{vol}_{g'}(M)} \langle h, \mathbf{1} \rangle_{\mathrm{vol}_{g'}}$$

is a co-polyharmonic Gaussian field for (M, g').

Liouville Geometry

Liouville Geometry

Fix an admissible manifold (M, g) and a co-polyharmonic Gaussian field $h : \Omega \to \mathfrak{D}'$. Our naive goal is to study the 'random geometry' (M, g_h) obtained by the random conformal transformation,

$$g_h = e^{2h} g \,,$$

and in particular to study the associated 'random volume measure' given as

$$\mu^h(x) = e^{nh(x)} \operatorname{vol}_g(x)$$

and the 'random metric' (or 'random distance') as

$$d^h(x, y) = \inf_{\varphi} \int_0^1 e^{h(\varphi(t))} |\varphi'(t)| dt$$

Due to the singular nature of the noise h, however, both of these objects will be degenerate – as long as no additional renormalization is built in.

Liouville Geometry

In n=2:
Replacing h by finite-dimensional noise approximations h_ℓ as before and proper renormalization leads (for sufficiently small $\gamma \in \mathbb{R}$) to sequences of random measures (μ^{h_ℓ}) and random distances (d^{h_ℓ}) on M which converge as $\ell \to \infty$ to non-trivial limit objects

Duplantier/Sheffield 2011, Rhodes/Vargas 2014

$$\mu^h(x) = \lim_{\ell\to\infty} e^{\gamma h_\ell(x) - \frac{\gamma^2}{2}\mathbf{E}[h_\ell(x)^2]} \, \mathrm{vol}_g(x).$$

Ding/Dubedat/Dunlap/Falconet 2020, Gwynne/Miller 2021

$$d^h(x,y) = \lim_{\ell\to\infty} \frac{1}{\lambda_{\gamma,\ell}} \inf_{\varphi} \int_0^1 e^{\gamma\, h_\ell(\varphi(t))} |\varphi'(t)| dt.$$

Miller/Sheffield 2020/21

For the particular value $\gamma = \sqrt{8/3}$, the random metric measure space (M, d^h, μ^h) is isometric in distribution to the Brownian map
$=$ scaling limit of random triangulations (Le Gall 2013) or quadrangulations (Miermont 2013) of the sphere.

Liouville Quantum Gravity Measure

Let M as before be a compact manifold of even dimension and h the co-polyharmonic Gaussian field.
For $\ell \in \mathbb{N}$ define a random measure $\mu_\ell = \rho_\ell \, \mathrm{vol}_g$ on M with density

$$\rho_\ell(x) := \exp\Big(\gamma h_\ell(x) - \frac{\gamma^2}{2} k_\ell(x,x)\Big)$$

where as before $h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2} \xi_j \psi_j$ and $k_\ell(x,x) := \mathbf{E}\big[h_\ell^2(x)\big] = \sum_{j=1}^{\ell} \nu_j^{-1} \psi_j^2(x)$.
Based on Kahane 1986, Shamov 2016:

Theorem

If $|\gamma| < \sqrt{2n}$, then there exists a random measure μ on M with $\mu_\ell \to \mu$. More precisely, for every $u \in \mathcal{C}(M)$,

$$\int_M u \, d\mu_\ell \longrightarrow \int_M u \, d\mu \quad \textit{in } L^1(\mathbf{P}) \textit{ and } \mathbf{P}\textit{-a.s. as } \ell \to \infty.$$

The random measure $\mu := \lim_{\ell \to \infty} \mu_\ell$ is called *Liouville Quantum Gravity measure*.

Liouville Quantum Gravity Measure

Theorem

If $|\gamma| < \sqrt{n}$, then for every $u \in \mathcal{C}_b(M)$,

$$\left(Y_\ell\right)_{\ell\in\mathbb{N}} := \left(\int_M u\, d\mu_\ell\right)_{\ell\in\mathbb{N}} \quad \textit{is } L^2\textit{-bounded martingale}$$

Proof: Assume $0 \le u \le 1$. Then

$$\begin{aligned}
\sup_\ell \mathbf{E}\left[{Y_\ell}^2\right] &= \sup_\ell \mathbf{E}\left[\iint e^{\gamma h_\ell(x)+\gamma h_\ell(y)-\frac{\gamma^2}{2}k_\ell(x,x)-\frac{\gamma^2}{2}k_\ell(y,y)}\cdot u(x)u(y)\,dx\,dy\right] \\
&= \sup_\ell \iint e^{\gamma^2 k_\ell(x,y)}\cdot u(x)u(y)\,dx\,dy\Big] \\
&\le \iint e^{\gamma^2 k(x,y)}\,dx\,dy \\
&= \iint \frac{1}{d(x,y)^{\gamma^2}}\,dx\,dy + \mathcal{O}(1)
\end{aligned}$$

due to the log divergence of k. The latter integral is finite if and only if $\gamma^2 < n$.

Liouville Quantum Gravity Measure

Theorem

If $\gamma < \sqrt{2}$ then a.s. the LQG measure μ does not charge sets of vanishing H^1-capacity

$\longrightarrow$ Dirichlet form $\int_M |\nabla u|^2 d\,\mathrm{vol}_g$ on $L^2(M, \mu)$
$\longrightarrow$ Liouville Brownian motion (random time change of BM)

A key property of the Liouville Quantum Gravity measure is its quasi-invariance under conformal transformations.

Theorem

Let μ be the Liouville Quantum Gravity measure for (M, g), and μ' be the Liouville Quantum Gravity measure for (M, g') where $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then

$$\mu' \overset{(\mathrm{d})}{=} e^{-\gamma\xi + \frac{\gamma^2}{2}\bar{\varphi} + n\varphi}\,\mu$$

where $\xi := \frac{1}{v'}\langle h, e^{n\varphi}\rangle$ and $\bar{\varphi} := \frac{2}{v'}\,\mathrm{k}_g(e^{n\varphi}) - \frac{1}{v'^2}\,\mathfrak{k}_g(e^{n\varphi}, e^{n\varphi})$ with $v' :=$ $\mathrm{vol}_{g'}(M)$.

Polyakov-Liouville Measure

Polyakov-Liouville Measure and Conformal Field Theory

For n=2: In 20016-2019 Rhodes–Vargas with David, Garban, and Kupiainen provided a rigorous definition to the Polyakov–Liouville measure $\boldsymbol{\nu}_g^*$, informally given as

$$\frac{1}{Z_g} \exp\Big(-S_g(h)\Big)\, dh$$

with (non-existing) uniform distribution dh on the set of fields and Polyakov–Liouville action

$$S_g(h) := \int_M \Big(\frac{1}{4\pi}\,\big|\nabla h\big|^2 + \Theta\, R_g h + m e^{\gamma h}\Big)\, d\,\mathrm{vol}_g \tag{2}$$

— and thus established conformal field theory on 2-dimensional spaces.

This ansatz for the measure $\boldsymbol{\nu}_g^*$ reflects the coupling of the gravitational field with a matter field. It can be regarded as quantization of the the classical Einstein–Hilbert action or, more precisely, of its coupling with a matter field.

- Minimizers h of the action functional (with appropriate choice of constants) satisfy Liouville equation,
  the weighted metric $g' = e^{2h} g$ thus has constant curvature $R_{g'}$.
- Semiclassical limit ($\gamma \to 0$): Polyakov–Liouville measure concentrates on surfaces of constant curvature (Lacoin/Rhodes/Vargas 2019+).

Polyakov-Liouville Measure and Conformal Field Theory

Ansatz for Polyakov–Liouville measure action in arbitrary even dimensions

$$S_g(h) := \int_M \left(\frac{1}{2} \left| \sqrt{\mathsf{p}_g}\, h \right|^2 + \Theta\, Q_g h + \frac{\Theta^*}{\mathrm{vol}_g(M)} h + m e^{\gamma h} \right) d\,\mathrm{vol}_g\,. \tag{3}$$

Here p_g is the co-polyharmonic operator, Q_g denotes Branson's curvature, and $m, \Theta, \Theta^*, \gamma$ are parameters.
In the case $n = 4$, $Q_g = -\frac{1}{6}\Delta_g \mathrm{scal}_g - \frac{1}{2}|\mathrm{Ric}_g|^2 + \frac{1}{6}\mathrm{scal}_g^2$. In general, total Q-curvature is conformally invariant, and if $g' = e^{2\varphi} g$ then

$$e^{n\varphi} Q_{g'} = Q_g + \frac{1}{a_n}\, \mathsf{p}_g \varphi.$$

Polyakov-Liouville Measure and Conformal Field Theory

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$$e^{n\varphi} Q_{g'} = Q_g + \frac{1}{a_n}\, \mathsf{p}_g \varphi.$$

Remark: Minimizers of S_g satisfy

$$\mathsf{p}_g h + \Theta Q_g + \frac{\Theta^*}{\mathrm{vol}_g(M)} + m\gamma e^{\gamma h} = 0.$$

Choose $\Theta^* = 0$, $\Theta = \frac{n a_n}{\gamma}$, $m = -\frac{n a_n}{\gamma^2}\bar{Q}$ for some $\bar{Q} \in \mathbb{R}$ and put $\varphi = \frac{\gamma}{n} h$.
Then this reads as

$$\frac{1}{a_n}\, \mathsf{p}_g \varphi + Q_g = e^{n\varphi} \bar{Q}.$$

In other words, $g' = e^{2\varphi} g$ is a metric of constant Branson curvature $Q_{g'} = \bar{Q}$.

Polyakov-Liouville Measure and Conformal Field Theory

Informal ansatz

$$\boldsymbol{\nu}^*_g(dh) = \frac{1}{Z^*_g} \exp\left(- \int_M \left(\frac{1}{2} \left| \sqrt{\mathsf{p}_g}\, h \right|^2 + \Theta\, Q_g h + \frac{\Theta^*}{\mathrm{vol}_g(M)} h + m e^{\gamma h} \right) d\,\mathrm{vol}_g \right) dh$$

Rigorous

$$d\boldsymbol{\nu}^*_g(h) := \exp\Big(- \Theta \langle h, Q_g \rangle - \Theta^* \langle h \rangle_g - m\, \mu^h(M) \Big)\, d\widehat{\boldsymbol{\nu}}_g(h)$$

with $d\widehat{\boldsymbol{\nu}}_g =$ law of ungrounded co-polyharmonic Gaussian field $=$ image of $d\boldsymbol{\nu}_g(h) \otimes d\mathcal{L}^1(t)$ under map $(h, t) \mapsto h + t$, informally characterized as

$$d\widehat{\boldsymbol{\nu}}_g(h) = \frac{1}{Z_g} \exp\Big(- \frac{1}{2} \mathfrak{e}_g(h, h) \Big) dh,$$

and μ^h denotes the Liouville Quantum Gravity measure on the n-manifold M.

Polyakov-Liouville Measure and Conformal Field Theory

$$d\boldsymbol{\nu}^*_g(h) := \exp\Big(-\Theta\langle h, Q_g\rangle - \Theta^*\langle h\rangle_g - m\,\mu^h(M)\Big)\, d\widehat{\boldsymbol{\nu}}_g(h)$$

with $d\widehat{\boldsymbol{\nu}}_g =$ law of ungrounded co-polyharmonic Gaussian field and $\mu^h =$ Liouville Quantum Gravity measure.

Theorem

Assume that $0 < \gamma < \sqrt{2n}$ and $\Theta\, Q(M) + \Theta^* < 0$. Then $\boldsymbol{\nu}^*_g$ is a finite measure.

Theorem

If $\Theta = a_n \frac{n}{\gamma}$, and $\Theta^* = \gamma$, then $\boldsymbol{\nu}^*_g$ is conformally quasi-invariant modulo shift:

$$\boldsymbol{\nu}^*_{e^{2\varphi} g} = Z(g,\varphi) \cdot T_* \boldsymbol{\nu}^*_g \qquad \forall \varphi \tag{4}$$

with explicitly given conformal anomaly $Z(g,\varphi)$.

For $n = 2$: David, Kupiainen, Rhodes, Vargas '16 for surfaces of genus 0, David, Rhodes, Vargas '16 for surfaces of genus 1, and Garban, Rhodes, Vargas '19 for surfaces of higher genus.