Sensitivity w.r.t. Hurst parameter of functionals of diffusions driven by fractional Brownian motions

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SDE, 2022

Outline

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Objective: Sensitivity analysis w.r.t. the long-range/memory noise parameter for probability distributions of functionals of solutions to SDEs.

Given $H \in (\frac{1}{4}, 1)$:

$$X_t^H = x_0 + \int_0^t b(X_s^H)\,\mathrm{d}s + \int_0^t \sigma(X_s^H) \circ \mathrm{d}B_s^H,$$

where the stochastic integral is to be precised.
For $H = \frac{1}{2}$, classical MARKOVIAN SDE in the Stratonovich sense:

$$X_t = x_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s) \circ \mathrm{d}B_s$$

Two sensitivity problems when the Hurst parameter H of the noise tends to the critical Brownian parameter $H = \frac{1}{2}$ from above OR from below :

- Smooth functionals: Time marginal probability distributions of X^H.
- Irregular functionals: Laplace transforms of the first passage times of X^H at given thresholds.

Motivations: neuroscience, risk analyses, stochastic computational models, etc.

Outline

Smooth functionals: Sensitivity of marginal distributions

Theorem 1

Suppose that b and σ are smooth enough and σ is strongly elliptic. Suppose that φ is bounded and Hölder continuous of order $2+\beta$ for some $\beta > 0$. Then, for any $T > 0$, there exists $C_T > 0$ such that for any $H \in [\frac{1}{4}, 1)$:

$$\sup_{t \in [0,T]} \left| \mathbb{E}\varphi(X_t^H) - \mathbb{E}\varphi(X_t) \right| \leq C_T \left| H - \tfrac{1}{2} \right|$$

Irregular functionals: Sensitivity of first exit time Laplace transforms

Set $\mu := |\frac{b}{\sigma}|_\infty$, $\mathcal{R}(\lambda) := \sqrt{2\lambda + \mu^2} - \mu$, $\widetilde{\lambda} := \lambda - |\widetilde{b}'|_\infty$.
Let F be the (increasing) Lamperti transform for (X_t) and

$$\boxed{\mathcal{M}_p(x_0, \lambda) := \sup_{s \in \mathbb{R}_+} \left(e^{-\frac{1}{2}\widetilde{\lambda} ps} \ \mathbb{E} \ e^{-|F(1)-F(X_s^H)| \ p \ \mathcal{R}(\lambda)} \right).}$$

Theorem 2 (INFORMAL)

Let $x_0 < 1$. One has:

$$\left| \mathbb{E}\left(e^{-\lambda \tau_X^H}\right) - \mathbb{E}\left(e^{-\lambda \tau_X}\right) \right|$$
$$\leq C_H \, |H - \tfrac{1}{2}| \, \frac{(1+\lambda)^2}{1 \wedge \widetilde{\lambda}^4} \left(\mathcal{M}_1(\mathbb{Y} - y_0, \lambda) + (\mathcal{M}_2(\dots))^{\frac{H \wedge \frac{1}{2}}{6}} + (\mathcal{M}_4(\dots))^{\frac{H \wedge \frac{1}{2}}{12}} \right)$$

Technical difficulties: accurate controls w.r.t. $|H - \frac{1}{2}|$, $1 - x_0$, λ.

Irregular functionals: An estimate on $\mathcal{M}_p(\mathbb{Y} - y_0, \lambda)$

Reminder: for $H = \frac{1}{2}$, X^H=BM. The corresponding Laplace transform is $\boxed{e^{-(1-x_0)\sqrt{2\lambda}}}$. Thus one needs a similar estimate on $\mathcal{M}_p(x_0, \lambda)$.

Recall:

$$\mathcal{M}_p(x_0, \lambda) := \sup_{s \in \mathbb{R}_+} \left(e^{-\frac{1}{2}\widetilde{\lambda} ps} \, \mathbb{E} \, e^{-|F(1) - F(X_s^H)| p \mathcal{R}(\lambda)} \right)$$

Theorem 3

Let $q := p\mathcal{R}(\lambda)$ and $m := F(1) - F(x_0)$. For any $p > 0$ and $\lambda > |\widetilde{b}'|_\infty$ one has

$$\mathcal{M}_p(x_0, \lambda) \leq C \left(e^{-\frac{q}{2} m} + e^{-\frac{\widetilde{\lambda}}{2} \Psi_q^H(m)} + \exp\left(-c \; m^{\frac{2}{1+2H}} \; \widetilde{\lambda}^{\frac{2H}{1+2H}} \right) + \exp\left(-\widetilde{\lambda} \frac{m}{2\mu} \right) \right),$$

where

$$\Psi_q^H(m) := \frac{m}{\mu + q} \, \mathbb{I}_{\left[\left(\frac{m}{\mu+q}\right)^{2H-1} < 1\right]} + \left(\frac{m}{\mu + q} \right)^{\frac{1}{H}} \, \mathbb{I}_{\left[\left(\frac{m}{\mu+q}\right)^{2H-1} \geq 1\right]}$$

Outline

Covariance of Fractional BMs

fBM with Hurst parameter $H \in (0,1)$: self-similar Gaussian process with stationary increments and covariance function

$$R_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})$$

One has

$$R_H(s,t) = \int_0^{s\wedge t} K_H(s,u)\ K_H(t,u)\ du$$

where, for $0 < r < s$ and some well chosen χ_H,

$$K_H(s,r) := \chi_H \left\{ \left(\frac{s(s-r)}{r}\right)^{H-\frac{1}{2}} - (H-\tfrac{1}{2})\ r^{\frac{1}{2}-H} \int_r^s \theta^{H-\frac{3}{2}} (\theta - r)^{H-\frac{1}{2}}\ d\theta \right\}$$

Volterra representation:

For some standard Brownian motion $\mathbf{B} \equiv \mathbf{B}^{\frac{1}{2}}$,

$$\forall t \geq 0, \quad \boxed{B_t^H = \int_0^t K_H(t,u)\ d\mathbf{B}_u}$$

Malliavin calculus

Set

$$K_H^*\varphi(s) = K_H(T,s)\,\varphi(s) + (H-\tfrac{1}{2})\,\chi_H \int_s^T \left(\frac{\theta}{s}\right)^{H-\frac{1}{2}} (\theta-s)^{H-\frac{3}{2}} \,(\varphi(\theta)-\varphi(s))\,d\theta$$

The Cameron-Martin space $\mathcal{H}_H$ is defined as the completion of the space of simple functions for the scalar product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_H} = \langle K_H^*\varphi, K_H^*\psi \rangle_{L^2[0,T]}$$

Let **D** be the Malliavin derivative w.r.t. **B**. On $\mathcal{H}_H$ define

$$D^H := (K_H^*)^{-1}\mathbf{D}$$

and the Skorokhod integral $\delta_H^{(T)}$ by duality:

$$\mathbb{E}\left(\langle u, D^H F \rangle_{\mathcal{H}_H}\right) = \mathbb{E}\left(F\ \delta_H^{(T)}(u)\right)$$

For any u such that $K_H^* u \in \text{dom}\ \delta$ one has

$$\delta_H^{(T)}(u) = \boldsymbol{\delta}(K_H^* u)$$

Stratonovich integrals

Except when $H = \frac{1}{2}$, the fBm is not a semimartingale.
The Stratonovich integral $\int u \circ \mathrm{d}B^H$ is the limit in probability (if it exists) of

$$\frac{1}{2\epsilon}\int_0^T u_s(B^H_{(s+\epsilon)\wedge T} - B^H_{(s-\epsilon)\vee 0})\mathrm{d}s$$

when $\epsilon \to 0$.
If $u \in \mathbb{D}^{1,2}(|\mathcal{H}_H|)$ and if the following limit in probability exists:

$$\mathrm{Tr}\, D^H u := \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^T \langle D^H_{\cdot} u_s, \mathbf{1}_{[s-\epsilon,s+\epsilon]\cap[0,T]}\rangle_{\mathcal{H}_H}\; ds$$

then the Stratonovich integral exists and

$$\int_0^T u_s \circ\; dB^H_s = \delta_H(u) + \mathrm{Tr}\, D^H u$$

(see Nualart).

Outline

Strong solutions to SDEs

$$X_t^H = x_0 + \int_0^t b(X_s^H)\ ds + \int_0^t \sigma(X_s^H) \circ\ dB_s^H$$

- For $H > \frac{1}{2}$: solutions in the sense of Young (Nualart and Rascanu)
- For $H \in (\frac{1}{4}, \frac{1}{2})$: solutions in the sense of Alòs, León, Nualart

Proposition (Lamperti transform)

Let $H \in (\frac{1}{4}, 1)$. Suppose b, σ smooth enough and $\sigma(x) \geq \sigma_0 > 0$. Let $F(x) := \int_0^x \frac{1}{\sigma(z)}\, \mathrm{d}z$ and $\tilde{b} = \frac{b \circ F^{-1}}{\sigma \circ F^{-1}}$. Then $Y^H := F(X^H)$ solves

$$Y_t^H = y_0 + B_t^H + \int_0^t \tilde{b}(Y_s^H)\ ds \quad \text{with} \quad y_0 := F(x_0)$$

Remark: The proof is based on the fact that the diffusion coefficient of the target process Y^H is 1.

Proposition

$$D_r^H Y_t^H = \mathbf{1}_{[0,t]}(r) \exp\left(\int_r^t b'(Y_\theta^H) \mathrm{d}\theta \right)$$

Proposition

Let $H \in (0,1) \setminus \{\frac{1}{2}\}$. From $\mathbf{D} = K_H^* \, D^H$:

$$\begin{cases} \forall r > t, \; |\mathbf{D}_r Y_t^H| = 0, \\ \forall r \leq t, \; |\mathbf{D}_r Y_t^H| \leq C \, e^{|\widetilde{b}'|_\infty (t-r)} \left\{ |K_H(t,r)| + (t-r)^{H+\frac{1}{2}} \, \mathbb{I}_{\{H<\frac{1}{2}\}} \right\}. \end{cases}$$

In addition, for any $r \leq s < t$,

$$|\mathbf{D}_r Y_t^H - \mathbf{D}_r Y_s^H| \leq C \, e^{|\widetilde{b}'|_\infty (t-r)} \{ |K_H(t,r) - K_H(s,r)| \\ + (t-s) \left(|K_H(s,r)| + (s-r)^{H+\frac{1}{2}} \, \mathbb{I}_{\{H<\frac{1}{2}\}} \right) \Big\}.$$

Sobolev regularity – Itô - Skorokhod's formula

For any smooth enough function g, $\text{Tr}\, D^H g(Y^H)$ exists and

$$G(Y_t^H) = G(Y_0^H) + \int_0^t G'(Y_s^H)\ \tilde{b}(Y_s^H)\ ds + \int_0^t G'(Y_s^H) \circ\ dB_s^H$$
$$= G(Y_0^H) + \int_0^t G'(Y_s^H)\ \tilde{b}(Y_s^H)\ ds + \delta_H\left(\mathbf{1}_{[0,t]}(\cdot) G'(Y_\cdot^H)\right)$$
$$+ \text{Tr}\left[D^H \partial_y G(\cdot, Y_\cdot^H)\right]_t$$

where

$$\text{Tr}\left[D^H \partial_y G(\cdot, Y_\cdot^H)\right]_t$$
$$:= \int_0^t \partial_y^2 G(s, Y_s^H) \left(Hs^{2H-1} + \int_0^s \partial_s K_H(s,r) \int_0^s \mathbf{D}_r \widetilde{b}(Y_v^H)\, dv\ dr\right) ds$$

Remark: The proof is based on the fact that the diffusion coefficient of the source process Y^H is 1.

Smooth functionals: Sensitivity of marginal distributions

Theorem 1

Suppose that b and σ are smooth enough, σ is strongly elliptic and ψ is bounded and Hölder continuous of order $2+\beta$ for some $\beta > 0$. Then, for any $T > 0$, there exists $C_T > 0$ such that for any $H \in [\frac{1}{4}, 1)$:

$$\sup_{t\in[0,T]} \left|\mathbb{E}\psi(X_t^H) - \mathbb{E}\psi(X_t)\right| \leq C_T \left|H - \tfrac{1}{2}\right|$$

Equivalently,

$$\sup_{t\in[0,T]} \left|\mathbb{E}\varphi(Y_t^H) - \mathbb{E}\varphi(Y_t)\right| \leq C_T \left|H - \tfrac{1}{2}\right|$$

with $\varphi := \psi \circ F^{-1}$.

Sketch of the proof of Theorem 1 ($H > \frac{1}{2}$)

Consider the parabolic PDE with initial condition φ at time $t \in (0, T]$:

$$\begin{cases} \partial_s u(s,y) + \tilde{b}(y)\ \partial_y u(s,y) + \frac{1}{2}\ \partial_{yy} u(s,y) = 0, \quad (s,y) \in [0,t) \times \mathbb{R}, \\ u(t,y) = \varphi(y), \quad y \in \mathbb{R}. \end{cases}$$

Reminder:

In the pure Brownian case $H = \frac{1}{2}$, by Itô's formula,

$$u(t, Y_t) = u(0, y_0) + \int_0^t \partial_y u(s, Y_s)\ d\mathbf{B}_s + \int_0^t \Big(\partial_s u(s, Y_s) + \tilde{b}(Y_s)\ \partial_y u(s, Y_s) + \frac{1}{2}\ \partial_{yy} u(s, Y_s) \Big)\ ds,$$

from which, by martingale property, $\mathbb{E}\varphi(Y_t) = \mathbb{E}u(t, Y_t) = u(0, y_0)$.

Use the Itô - Skorokhod's formula formula and an explicit representation of Tr when $H > \frac{1}{2}$:

$$u(t, Y_t^H) = u(0, y_0) + \int_0^t \left(\partial_s u(s, Y_s^H) + \partial_y u(s, Y_s^H)\ \tilde{b}(Y_s^H)\right)\ ds$$
$$+ \delta_H \left(\mathbf{1}_{[0,t]}\ \partial_y u(\cdot, Y_\cdot^H)\right)$$
$$+ H\ (2H-1)\ \int_0^t \int_0^s (s-r)^{2H-2} D_r^H Y_s^H\ \partial_{yy}^2 u(s, Y_s^H)\ dr\ ds$$

Use the PDE solved by u and the fact that the Skorokhod integral has zero mean . Use also that $D_r^H Y_s^H = D_r^H Y_s^H - 1 + 1$.

$$\mathbb{E}\varphi(Y_t^H) - \mathbb{E}\varphi(Y_t) = \mathbb{E} \int_0^t \partial_{yy}^2 u(s, Y_s^H) \left(Hs^{2H-1} - \tfrac{1}{2}\right)\ ds$$
$$+ H\ (2H-1)\ \mathbb{E} \int_0^t \int_0^s (s-r)^{2H-2} (D_r^H Y_s^H - 1)\ dr$$
$$\partial_{yy}^2 u(s, Y_s^H)\ ds$$
$$=: \Delta_H^1 + \Delta_H^2$$

For $H > \frac{1}{2}$:

We bound $|\Delta_H^1|$ as follows:

$$|\Delta_H^1| = \left| \int_0^t \partial_{yy}^2 u(s, Y_s^H) \left(Hs^{2H-1} - \tfrac{1}{2}\right) \; ds \right| \leq C \; \left(H - \tfrac{1}{2}\right)$$

To bound $|\Delta_H^2|$, use the above Proposition for $D_r^H Y_t^H$:

$$\begin{aligned} |\Delta_H^2| &\leq C \; (2H-1) \int_0^t \int_0^s (s-r)^{2H-2} \; (s-r) \; |\partial_{yy}^2 u(s, Y_s^H)| \; dr \; ds \\ &\leq C \; \left(H - \tfrac{1}{2}\right) \|\partial_{yy}^2 u\|_\infty \int_0^t \int_0^s (s-r)^{2H-1} \; dr \; ds \\ &\leq C\left(H - \tfrac{1}{2}\right) \end{aligned}$$

For $H < \frac{1}{2}$: Similar methodology, much more complex calculations (because Tr has no simple explicit form, see below).

Irregular functionals: Sensitivity of first exit time Laplace transforms

Theorem 2 (INFORMAL)

Let $x_0 < 1$. One has:

$$\left|\mathbb{E}\left(e^{-\lambda\tau_X^H}\right) - \mathbb{E}\left(e^{-\lambda\tau_{\mathbf{X}}}\right)\right|$$
$$\leq C_H\,|H-\tfrac{1}{2}|\,\frac{(1+\lambda)^2}{1\wedge\widetilde{\lambda}^4}\left(\mathcal{M}_1(\mathbb{Y}-y_0,\lambda)+(\mathcal{M}_2(\dots))^{\frac{H\wedge\frac{1}{2}}{6}}+(\mathcal{M}_4(\dots))^{\frac{H\wedge\frac{1}{2}}{12}}\right)$$

Sketch of the proof of Theorem 2

Starting point: After having used the Lamperti transform F,

$$\mathbb{E}\left(e^{-\lambda\tau_H^X(x_0)}\right) - \mathbb{E}\left(e^{-\lambda\tau_{1/2}^X(x_0)}\right) = \mathbb{E}\left(e^{-\lambda\tau_H^Y(F(x_0))}\right) - \mathbb{E}\left(e^{-\lambda\tau_{1/2}^Y(F(x_0))}\right).$$

Set $y_0 := F(x_0)$ and $\mathbb{Y} := F(1)$.

The second-order ODE satisfied by $\mathbf{W}_\lambda(y_0) := \mathbb{E}\left(e^{-\lambda \tau^Y_{1/2}(y_0)}\right)$ is

$$\begin{cases} \tilde{b}(y)\, \mathbf{W}_\lambda'(y) + \frac{1}{2}\, \mathbf{W}_\lambda''(y) & = \lambda\, \mathbf{W}_\lambda(y),\ \ y < \mathbb{Y}\ , \\ \mathbf{W}_\lambda(\mathbb{Y}) & = 1\ , \\ \lim_{y \to -\infty} \mathbf{W}_\lambda(y) & = 0\ . \end{cases}$$

Arbitrarily fix $N > 0$ and apply Itô's formula: for any $0 < t \leq \tau^Y_H \wedge N$,

$$\begin{aligned} e^{-\lambda t}\mathbf{W}_\lambda(Y^H_t) - \mathbf{W}_\lambda(y_0) = \int_0^t e^{-\lambda s} \left(\tilde{b}(Y^H_s)\, \mathbf{W}_\lambda'(Y^H_s) - \lambda\, \mathbf{W}_\lambda(Y^H_s)\right)\, \mathrm{d}s \\ + \delta_H(\mathbf{1}_{(0,t)}\mathbf{W}_\lambda'(Y^H)) \\ + \operatorname{Tr} D^H(\mathbf{1}_{[0,t]}(\cdot)\, \mathbf{W}_\lambda'(Y^H_\cdot)) \end{aligned}$$

Set

$$\Delta(s,H) := Hs^{2H-1} - \frac{1}{2} + \int_0^s \partial_s K_H(s,r) \int_0^s \mathbf{D}_r \widetilde{b}(Y_v^H)\, dv\, dr$$

We have obtained

$$\mathbb{E}\Big(\mathbf{W}_\lambda(Y^H_{N\wedge\tau_H})\, e^{-\lambda(N\wedge\tau_H)}\Big) - \mathbf{W}_\lambda(y_0) = \mathbb{E}\Big[\int_0^{N\wedge\tau_H} \mathbf{W}''_\lambda(Y_s^H)\, \Delta(s,H)\, e^{-\lambda s}\, ds\Big] + \mathbb{E}\Big[\, \delta_H^{(N)}\big(\mathbf{1}_{[0,t]}\, \mathbf{W}'_\lambda(Y_\cdot^H)\, e^{-\lambda \cdot}\big)\Big|_{t=\tau_H\wedge N}\Big]$$

Finally,

$$\mathbb{E}\left(e^{-\lambda\tau_H}\right) - \mathbb{E}\left(e^{-\lambda\tau_{\frac{1}{2}}}\right) = \mathbb{E}\left[\int_0^{\tau_H} \Delta(s,H)\ \mathbf{W}''_\lambda(Y_s^H)\ e^{-\lambda s}\ ds\right]$$

$$+ \lim_{N\to+\infty} \mathbb{E}\left[\delta_H^{(N)}\left(\mathbf{1}_{[0,t]}\ \mathbf{W}'_\lambda(Y_\cdot^H)\ e^{-\lambda\cdot}\right)\Big|_{t=\tau_H\wedge N}\right]$$

$$=: I_1(\lambda) + I_2(\lambda)$$

To estimate $I_1(\lambda)$: A crucial lemma

Lemma

$$\forall H \in (\tfrac{1}{4}, 1), \quad \left| \int_0^s \partial_s K_H(s,r) \int_0^s \mathbf{D}_r \widetilde{b}(Y_v^H)\, dv\, dr \right| \\ \leq C_H \left|H - \tfrac{1}{2}\right| e^{s|\widetilde{b}'|_\infty} (s^{2H} + s^{2H+1}) \text{ a.s.}$$

Therefore,

$$|\Delta(s,H)| \leq \left|Hs^{2H-1} - \tfrac{1}{2}\right| + C_H \left|H - \tfrac{1}{2}\right| e^{s|\widetilde{b}'|_\infty} (s^{2H} + s^{2H+1}) \text{ a.s.}$$

To estimate $I_2(\lambda)$

Recall: $\delta_H^{(N)}(u) = \boldsymbol{\delta}(K_H^* u)$.

Notice that $\mathbf{W}'_\lambda(Y^H_\cdot)$ is adapted . Therefore, by the Optional Sampling Theorem for standard Brownian integrals,

$$\mathbb{E}\left[\boldsymbol{\delta}\left(\mathbf{1}_{[0,t]} e^{-\lambda \cdot} \mathbf{W}'_\lambda(Y^H_\cdot)\right)\Big|_{t=\tau_H \wedge N}\right] = 0$$

However, $K_H^*(\mathbf{1}_{[0,t]}(\bullet)\ e^{-\lambda \bullet}\ \mathbf{W}'_\lambda(Y^H_\bullet))$ is NOT adapted . One thus has to estimate

$$\boxed{\mathbb{E}\left[\ \boldsymbol{\delta}\left(\ (K_H^* - \mathrm{Id})\left(\mathbf{1}_{[0,t]}(\bullet)\ e^{-\lambda \bullet}\ \mathbf{W}'_\lambda(Y^H_\bullet)\right)(\cdot)\ \right)\Big|_{t=\tau_H \wedge N}\ \right]}$$

and let N tend to infinity.
Then, 40 pages of calculations...

To estimate $I_2(\lambda)$

Key idea:

- Localize in time, use the explicit expression for $(K_H^* - \mathrm{Id})$
- Use Meyer's inequalities for Skorokhod integrals and (essentially) reduce the calculation to the estimation of

$$\int_0^N \int_0^N \mathbb{E}((\mathbf{D}_r(U_t^{(N)}(v) - U_s^{(N)}(v)))^2)\ dr\ dv$$

where

$$\begin{aligned} U_t^{(N)}(v) - U_s^{(N)}(v) &= \mathbb{I}_{(s,t]}(v)\ (K_H(t,v) - 1)\ \mathbf{W}'_\lambda(Y_v^H)\ e^{-\lambda v} \\ &\quad + \int_v^t \partial_\theta K_H(\theta, v)\ (\mathbb{I}_{(s,t)}(\theta)\ \mathbf{W}'_\lambda(Y_\theta^H)\ e^{-\lambda\theta} \\ &\quad - \mathbb{I}_{(s,t)}(v)\ \mathbf{W}'_\lambda(Y_v^H)\ e^{-\lambda v})\ d\theta \end{aligned}$$

- Taylor expansions

A crucial observation

Recall that $\widetilde{\lambda} := \lambda - |\widetilde{b}'|_\infty$.
One has:

$$
\begin{aligned}
|\mathbf{D}_r \mathbf{W}'_\lambda(Y^H_\theta)| \; e^{-\lambda\theta} &\leq \mathbb{I}_{\{r\leq\theta\}} \; |\mathbf{W}''_\lambda(Y^H_\theta)| \; e^{-\lambda\theta+|\widetilde{b}'|_\infty\theta} \\
&\qquad \Big(K_H(\theta,r) + C\,(\theta-r)^{H+\frac{1}{2}}\;\mathbb{I}_{\{H<\frac{1}{2}\}}\Big) \\
&\leq C\;\mathbb{I}_{\{r\leq\theta\}}\;(1+\lambda)\;\; e^{-\widetilde{\lambda}\theta-|\mathbb{Y}-Y^H_\theta|\mathcal{R}(\lambda)} \\
&\qquad (K_H(\theta,r) + (\theta-r)^{H+\frac{1}{2}}\;\mathbb{I}_{\{H<\frac{1}{2}\}}),
\end{aligned}
$$

from which

$$
\begin{aligned}
&\mathbb{E}((\mathbf{D}_r \mathbf{W}'_\lambda(Y^H_\theta))^2)\; e^{-2\lambda\theta} \\
&\leq C\;\mathbb{I}_{\{r\leq\theta\}}\;(1+\lambda)^2\;\mathcal{M}_2(\mathbb{Y}-y_0,\lambda)\;\Big(|K_H(\theta,r)|^2 + (\theta-r)^{2H+1}\;\mathbb{I}_{\{H<\frac{1}{2}\}}\Big)\; e^{-\widetilde{\lambda}\theta}
\end{aligned}
$$

Outline

Open questions

Notice that the pure Brownian model is robust.

Open questions:

- Get rid of the ellipticity condition: Change the PDE. But . . .
- Examine multi-dimensional models. But . . .
- Invert Laplace transforms and get information on the robustness of densities