Optimal control problems with generalized mean-field dynamics and viscosity solution to Master Bellman equation

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Outline

1. Objective of the talk
2. Formulation of the mean-field stochastic control problems
3. Dynamic programming principle
4. Master Bellman equation and viscosity solution
5. Main results

Brief state of the art

Mean-field problems:

1) Mean-Field SDEs have been intensively studied for a longer time as limit equ. for systems with a large number of particles (propagation of chaos)(Bossy, Méléard, Sznitman, Talay,...);

2) Mean-Field Games and related topics, since 2006-2007 by J.M.Lasry and P.L.Lions, Huang-Caines-Malhamé (2006) (Nash certainty equivalence principle); Mean field game systems:

$$\begin{cases} \textbf{i)} & -\partial_t u - v\triangle u + H(x, Du, m) = 0 \ \text{ in } (0,T)\times\mathbb{R}^d \text{ HJB equ.} \\ \textbf{ii)} & \partial_t m - v\triangle m - \text{div}(H_p(x, Du, m)m) = 0 \ \text{ in } (0,T)\times\mathbb{R}^d \text{ continuity equ.} \\ \textbf{iii)} & m(0) = m_0,\ u(x,T) = G(x, m(T)) \quad \text{ in } \mathbb{R}^d \end{cases}$$

Master equation evaluated for $U = U(t, x, m)$:

$$\begin{cases} -\partial_t U - v\triangle_x U + H(x, D_x U, m) - v\int_{\mathbb{R}^d} \text{div}_y D_m U(t,x,m,y) m(dy) \\ \qquad + \int_{\mathbb{R}^d} D_m U(t,x,m,y)\cdot D_p H(y, D_x U, m) m(dy) = 0 \\ U(T,x,m) = G(x,m) \quad \text{ in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

Mean-field problems:

3) +) Mean-Field BSDEs/FBSDEs and associated nonlocal PDEs:

- Prel. works: B., Djehiche, Li, Peng (AOP2009); B., Li, Peng (SPA2009);
- Classical solution of non-local PDE related with the mean-field SDE:
  B., Li, Peng, Rainer (AOP2017 (Arxiv2014)):

$$
\begin{aligned}
&0 = \partial_t V(t,x,\mu) + \partial_x V(t,x,\mu) b(x,\mu) + \tfrac{1}{2}\partial^2_{xx} V(t,x,\mu)\sigma^2(x,\mu) \\
&\quad + \int_{\mathbb{R}} \Big[(\partial_\mu V)(t,x,\mu,y) b(y,\mu) + \frac{1}{2}\partial_y(\partial_\mu V)(t,x,\mu,y)\sigma^2(y,\mu)\Big]\mu(dy), \\
&V(T,x,\mu) = \Phi(x,\mu), \qquad (t,x,\mu) \in [0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}).
\end{aligned}
$$

- For the case with mean-field SDE with jumps: Hao, Li (NODEA2016);
- For the case with the mean-field FBSDE with jumps: Li (SPA2017);
- For the case with the mean-field BDSDE and related nonlocal semi-linear backward SPDEs: B., Li, Xing (Arxiv2021);

Brief state of the art

Dynamic programming approach:

+) For classical case:

• For SDGs: B., Li (2008, SICON);

• For SDGs with jumps: B., Hu, Li (2011, SPA);

+) Classical case, but fully coupled FBSDEs with jumps: Li, Wei (2014, AMO);

• For stochastic control problems with jumps: Li, Peng (2009, NA);

• Stoch. control for fully coupled FBSDEs: Li, Wei (2014, SICON);

+) For mean-field stochastic optimal control problems:

The objective is to characterize the value function of the mean-field control problem as a viscosity solution of a second order PDE on Wasserstein space, known as Master Bellman equation. The viscosity theory of this kind of PDEs is still at a rather early stage.

• **Consider definition by "lifting" :**

The following works adopt the notion of viscosity solution from Crandall-Lions and adapt it to the Wasserstein space by lifting to L^2. The uniqueness is established for this lifted Bellman equation.

$\rightsquigarrow$ Pham, Wei (2018, ESAIM: COCV):

Controlled mean-field stochastic system:

$$dX_t^u = b(t, X_t^u, u_t, \mathbb{P}_{(X_t^u, u_t)})dt + \sigma(t, X_t^u, u_t, \mathbb{P}_{(X_t^u, u_t)})dW_t,\ t \in [0, T],$$

where $u_t := \tilde{u}(t, X_t^u, \mathbb{P}_{X_t^u})$.

It is assumed a priori that the controls were of Markovian feedback type.

Brief state of the art

$\rightsquigarrow$ Bayraktar, Cosso, and Pham (2018, TAMS): Controlled mean-field SDEs:

$$X_s^{t,\xi,u} = \xi + \int_t^s b(r, X_r^{t,\xi,u}, \mathbb{P}_{X_r^{t,\xi,u}}, u_r)dr + \int_t^s \sigma(r, X_r^{t,\xi,u}, \mathbb{P}_{X_r^{t,\xi,u}}, u_r)dW_r,$$

$$X_s^{t,x,\xi,u} = x + \int_t^s b(r, X_r^{t,x,\xi,u}, \mathbb{P}_{X_r^{t,\xi,u}}, u_r)dr + \int_t^s \sigma(r, X_r^{t,x,\xi,u}, \mathbb{P}_{X_r^{t,\xi,u}}, u_r)dW_r.$$

They study control problems for **open-loop controls**, but without the dependence of the law on the control, and they proved a so-called Randomized DPP, based on a characterisation of the value function through an auxiliary intensity control problem for a Poisson random measure.

$\rightsquigarrow$ Pham and Wei (2017, SICON),
$\rightsquigarrow$ Cosso and Pham (2019, JMPA),
...

Brief state of the art

- **Consider "intrinsic" definition:**

$\rightsquigarrow$ Burzoni, Ignazio, Reppen and Soner (2020, SICON):
Controlled mean -field stochastic system with jumps:

$$dX_t^u = b(t, \mathbb{P}_{X_t^u}, u_t)dt + \sigma(t, \mathbb{P}_{X_t^u}, u_t)dW_t + dJ_t,\ t \in [0, T],$$

where J is a purely discontinuous process.

The authors considered deterministic control processes only depending on the time. They studied viscosity solutions for a particular class of integro-differential Master equations. The uniqueness of viscosity solutions has been proved on Wasserstein spaces of probability measures which have finite exponential moments.

$\rightsquigarrow$ Cosso et al. (2024, TAMS): Controlled mean-field stochastic system:
$dX_t^u = b(t, X_t^u, u_t, \mathbb{P}_{X_t^u})dt + \sigma(t, X_t^u, u_t)dW_t,\ t \in [0, T].$
By using refinements of early ideas from the Crandall-Lions theory of viscosity solutions, they proved the uniqueness of the viscosity solutions on $\mathcal{P}_2(\mathbb{R}^d)$, but only for coefficients which do not depend on the law of the control.

1. Objective of the talk

We develop a dynamic programming approach to study an optimal control problem with generalized mean-field dynamics with considering:

- Open-loop controls;
- Coefficients which depend on the joint law of state processes and controls;
- Dynamics of both a "mean-field player" and a representative "individual player".

We characterize the value function as **the unique viscosity solution of a second order PDE on Wasserstein space**, by adapting the intrinsic notion of viscosity solutions in Burzoni et al. [2020].

We consider:

• $(\Omega, \mathcal{F}, \mathbb{P})$ – the classical Wiener space; the driving Brownian Motion B is the coordinate process on $\Omega := C_0([0,T];\mathbb{R}^d))$: $T > 0$ - a fixed horizon; $\mathcal{F} = \mathcal{B}(\Omega) \vee \mathcal{N}_{\mathbb{P}}$; $\mathbb{P}$ - Wiener measure.

• $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ – the filtration generated by $B = (B_s)_{s\in[0,T]}$ and augmented by all $\mathbb{P}$-null sets.

• For $k \geq 1$, $\mathcal{P}_2(\mathbb{R}^k)$ – the space of the probability measures over $\mathbb{R}^k$ with finite second moment and endowed with the 2-Wasserstein metric:

$$\mathcal{W}_2(\mu,\nu) := \inf\Big\{\Big(\int_{\mathbb{R}^k\times\mathbb{R}^k} |x-y|^2\rho(dxdy)\Big)^{\frac{1}{2}},\ \rho\in\Pi_{\pi,\nu}\Big\},\ \mu,\nu\in\mathcal{P}_2(\mathbb{R}^k),$$

where $\Pi_{\mu,\nu}=\big\{\rho\in\mathcal{P}_2(\mathbb{R}^{2k})$ with $\rho(\cdot\times\mathbb{R}^k)=\mu,\ \rho(\mathbb{R}^k\times\cdot)=\nu\big\}$.

Note: $(\mathcal{P}_2(\mathbb{R}^k), \mathcal{W}_2)$ is a complete separable space.

Spaces we work with:

- $L^2(\mathcal{F}_t;\mathbb{R}^n)$ is the set of $\mathbb{R}^n$-valued, $\mathcal{F}_t$-measurable random variables $\zeta:\Omega\to\mathbb{R}^n$ such that $E[|\zeta|^2]<\infty$.
- $L^2_{\mathbb{F}}([0,T];\mathbb{R}^n)$ is the set of $\mathbb{R}^n$-valued, $\mathbb{F}$-progressively measurable processes $\phi:\Omega\times[0,T]\to\mathbb{R}^n$, with $E\Big[\int_0^T|\phi_t|^2dt\Big]<+\infty$.
- $\mathcal{S}^2(0,T;\mathbb{R}^n)$ is the set of $\mathbb{F}$-adapted continuous processes $\phi:\Omega\times[0,T]$ $\to\mathbb{R}^n$ satisfying $E\Big[\sup_{0\le s\le T}|\phi_s|^2\Big]<\infty$.

For simplicity, we write $L^2(\mathcal{F}_t):=L^2(\mathcal{F}_t;\mathbb{R})$, $L^2_{\mathbb{F}}([0,T]):=L^2_{\mathbb{F}}([0,T];\mathbb{R})$.

The **dynamics** of our stochastic control problem are the following controlled mean-field SDEs:

$$\begin{aligned} X_s^{t,\xi,u^2} =& \ \xi + \int_t^s b_1(r, (X_r^{t,\xi,u^2}, u_r^2), \mathbb{P}_{(X_r^{t,\xi,u^2}, u_r^2)})dr \\ &+ \int_t^s \sigma_1(r, (X_r^{t,\xi,u^2}, u_r^2), \mathbb{P}_{(X_r^{t,\xi,u^2}, u_r^2)})dB_r, \quad s \in [t,T], \end{aligned} \tag{2.1}$$

$$\begin{aligned} X_s^{t,x,\xi,u} =& \ x + \int_t^s b_2(r, (X_r^{t,x,\xi,u}, u_r^1), \mathbb{P}_{(X_r^{t,\xi,u^2}, u_r^2)})dr \\ &+ \int_t^s \sigma_2(r, (X_r^{t,x,\xi,u}, u_r^1), \mathbb{P}_{(X_r^{t,\xi,u^2}, u_r^2)})dB_r, \quad s \in [t,T], \end{aligned} \tag{2.2}$$

where $t \in [0,T]$, $x \in \mathbb{R}$, $\xi \in L^2(\mathcal{F}_t; \mathbb{R})$, $u = (u^1, u^2) \in \mathcal{U}_{t,T} := \mathcal{U}_{t,T}^0 \times \mathcal{U}_{t,T}^0$. Admissible controls $u^1, u^2 \in \mathcal{U}_{t,T}^0$ are $\mathbb{F}$-adapted stochastic processes on $[t,T]$ taking values in a compact set $U \subset \mathbb{R}^n$.

Remark: Interpretation of the dynamics

• Equation (2.1) can be interpreted as the dynamics of an agent (referred to as the "mean-field player"), who plays collectively with using a "collective control" u^2. It describes the average over the states of all agents.

• Equation (2.2) describes the dynamics of an individual agent who faces the "mean-field player", and u^1 is the "individual control" played by this individual agent.

• In other words, (2.1) characterizes the evolution of the law $\mathbb{P}_{(X^{t,\zeta,u^2},u^2)}$, while (2.2) describes the associated trajectories of an individual agent with initial condition $X_t^{t,x,\zeta,u} = x$.

We shall make the following standard assumptions.

Assumption 2.1.

(i) $b_1, b_2, \sigma_1, \sigma_2$ are continuous and, for simplicity, bounded.

(ii) Lipschitz continuity: There exists a constant $C > 0$, such that

$$\left|\phi(t,(x,u),\mathbb{P}_{(\zeta,\eta)})-\phi(t,(x',u),\mathbb{P}_{(\zeta',\eta)})\right| \leq C(|x-x'|+\mathcal{W}_2(\mathbb{P}_{(\zeta,\eta)},\mathbb{P}_{(\zeta',\eta)})),$$

for all $t \in [0,T],\ x,x' \in \mathbb{R}^n,\ u \in U,\ \zeta,\zeta' \in L^2(\mathcal{F}_t;\mathbb{R}^n),\ \eta \in L^2(\mathcal{F}_t;U)$, $\phi = b_1, b_2, \sigma_1, \sigma_2$.

Under Assumption 2.1, there exists a unique pair of solutions $\left(X_s^{t,\zeta,u^2}, X_s^{t,x,\zeta,u}\right)_{s\in[t,T]} \in \mathcal{S}^2(0,T;\mathbb{R}^n)\times\mathcal{S}^2(0,T;\mathbb{R}^n)$ to the equations (2.1) and (2.2) (see, e.g., Buckdahn, Li, Peng and Rainer [2017]).

Moreover, for every $p \geq 2$, we have the following L^p-**estimates**:
There exists $C_p \in \mathbb{R}^+$ such that, for all $t \in [0,T]$, $u = (u^1, u^2) \in \mathcal{U}_{t,T}$, and $x, x' \in \mathbb{R}^n$, $\zeta, \zeta' \in L^2(\mathcal{F}_t; \mathbb{R}^n)$,

$$E\Big[\sup_{s\in[t,T]}\Big|X_s^{t,\zeta,u^2} - X_s^{t,\zeta',u^2}\Big|^p \,|\mathcal{F}_t\Big] \leq C_p|\zeta-\zeta'|^p,$$

$$E\Big[\sup_{s\in[t,T]}\Big|X_s^{t,x,\zeta,u} - X_s^{t,x',\zeta',u}\Big|^p \,|\mathcal{F}_t\Big] \leq C_p\Big(|x-x'|^p + |\zeta-\zeta'|^p\Big).$$

From the uniqueness of the solutions of the both equations, we also have the following **flow property**: For all $0 \leq t < t+\delta \leq T$, $x \in \mathbb{R}^n$, $\zeta \in L^2(\mathcal{F}_t; \mathbb{R}^n)$, $u = (u^1, u^2) \in \mathcal{U}_{t,T}$:

$$\Big(X_s^{t+\delta, X_{t+\delta}^{t,x,\zeta,u}, X_{t+\delta}^{t,\zeta,u^2}, u}, X_s^{t+\delta, X_{t+\delta}^{t,\zeta,u^2}, u^2}\Big) = \Big(X_s^{t,x,\zeta,u}, X_s^{t,\zeta,u^2}\Big), s \in [t+\delta, T], \mathbb{P}\text{-a.s.}$$

Assumption 2.2.

Let $\Phi : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$ be Lipschitz, i.e., for some constant $C > 0$ we have, for all $x,\ x' \in \mathbb{R}^n$, $\mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^n)$,

$$\left|\Phi(x,\mu) - \Phi\left(x',\mu'\right)\right| \le C\Big(\left|x - x'\right| + \mathcal{W}_2(\mu,\mu')\Big).$$

Given the control processes $u = \left(u^1, u^2\right) \in \mathcal{U}_{t,T}$, we introduce the **cost functional** of our mean-field stochastic control problem:

$$J(t,x,\zeta,u) := E\Big[\Phi\Big(X_T^{t,x,\zeta,u}, \mathbb{P}_{X_T^{t,\zeta,u^2}}\Big)\Big|\mathcal{F}_t\Big],$$

where $(t,x) \in [0,T] \times \mathbb{R}^n,\ \zeta \in L^2(\mathcal{F}_t;\mathbb{R}^n),\ \left(X^{t,\zeta,u^2}, X^{t,x,\zeta,u}\right)$ are the solutions of (2.1) and (2.2).

Remark. $X^{t,x,\zeta,u}$ is in general not independent of $\mathcal{F}_t$, and therefore $J(t,x,\zeta,u)$ is an $\mathcal{F}_t$-measurable random variable.

The definition of the value function:

Suppose that both the "mean-field player" and the "individual player" try to minimize the cost, for all $(t,x)\in[0,T]\times\mathbb{R}^n$, $\zeta\in L^2(\mathcal{F}_t;\mathbb{R}^n)$,

$$\begin{aligned}V(t,x,\zeta) &:= \operatorname*{essinf}_{u\in\mathcal{U}_{t,T}} J(t,x,\zeta,u)\\ &= \operatorname*{essinf}_{u^2\in\mathcal{U}^0_{t,T}}\left(\operatorname*{essinf}_{u^1\in\mathcal{U}^0_{t,T}} J\left(t,x,\zeta,(u^1,u^2)\right)\right).\end{aligned}$$

Set $W\left(t,x,\zeta,u^2\right) := \operatorname*{essinf}_{u^1\in\mathcal{U}^0_{t,T}} J\left(t,x,\zeta,(u^1,u^2)\right)$, $(t,x)\in[0,T]\times\mathbb{R}^n$, $u^2\in\mathcal{U}^0_{t,T}$. Then

$$V(t,x,\zeta) = \operatorname*{essinf}_{u^2\in\mathcal{U}^0_{t,T}} W\left(t,x,\zeta,u^2\right).$$

3. Dynamic programming principle

Recall that $W(t, x, \zeta, u^2)$ is an $\mathcal{F}_t$-measurable random variable. The following lemma shows that it is even deterministic.

Lemma 3.1.

For any $(t, x) \in [0, T] \times \mathbb{R}^n$, $\zeta \in L^2(\mathcal{F}_t; \mathbb{R}^n)$, $u^2 \in \mathcal{U}^0_{t,T}$, $W\left(t, x, \zeta, u^2\right)$ is deterministic. Moreover, by standard estimates, $\exists\ C > 0$ s.t.

$$|W(t, x, \zeta, u^2) - W(t, x', \zeta', u^2)| \leq C\big(|x - x'| + \big(E[|\zeta - \zeta'|^2]\big)^{\frac{1}{2}}\big),$$

for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $\zeta, \zeta' \in L^2(\mathcal{F}_t; \mathbb{R}^n)$, $u^2 \in \mathcal{U}^0_{t,T}$.

The proof of this lemma uses a standard argument from Buckdahn and Li [2008] with a novel, subtle approach.

The above lemma, combined with the definition of the value function V yields that also

$$V(t, x, \zeta) = \inf_{u^2 \in \mathcal{U}^0_{t,T}} W\left(t, x, \zeta, u^2\right)$$

is deterministic, for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $\zeta \in L^2(\mathcal{F}_t; \mathbb{R}^n)$.

3. Dynamic programming principle

Next, we prove that **the value function** $V(t, x, \zeta)$ **does not depend on** ζ **itself but only on its law** $\mathbb{P}_\zeta$**.** For this, we have first to study some auxiliary results.

Consider the following space of elementary control processes:

$$\mathcal{U}^e_{t,T} := \Big\{ u^2 = \sum_{i,j=0}^{N-1} \mathbf{1}_{A_{i,j}} \zeta_{i,j} \mathbf{1}_{(t_i, t_{i+1}]} \Big| \; N \geq 1,\ t = t_0 \leq \cdots \leq t_N = T,\ A_{i,j} \in \mathcal{F}_t,$$

$$\zeta_{i,j} \in L^2(\mathcal{F}^t_{t_i}; U),\ i = 0, \cdots, N-1,\ (A_{i,j})_{j=0}^{N-1} \text{ is a decomposition of } \Omega \Big\},$$

where $\mathcal{F}^t_s := \sigma\{B_r - B_t,\ r \in [t, s]\},\ s \in [t, T]$.

Lemma 3.2.

$\mathcal{U}^e_{t,T}$ is dense in $L^2_{\mathbb{F}}([t, T]; U)\big(= \mathcal{U}^0_{t,T}\big)$ with respect to the L^2-norm over $[t, T] \times \Omega$.

3. Dynamic programming principle

Then standard arguments allow to show:

$$V(t,x,\zeta) = \inf_{u^2 \in \mathcal{U}^e_{t,T}} W\left(t,x,\zeta,u^2\right).$$

Now we have the following result:

Lemma 3.4.

For every $\zeta, \zeta' \in L^2\left(\mathcal{F}_t; \mathbb{R}^n\right)$ with $\mathbb{P}_\zeta = \mathbb{P}_{\zeta'}$, and every $u^2 \in \mathcal{U}^e_{t,T}$, there exists $u^{2\prime} \in \mathcal{U}^e_{t,T}$ such that

$$J\left(t,x,\zeta',\left(u^1,u^{2\prime}\right)\right) = J\left(t,x,\zeta,\left(u^1,u^2\right)\right), \quad \mathbb{P}\text{-a.s.},$$

for all $u^1 \in \mathcal{U}^0_{t,T},\ x \in \mathbb{R}^n$, and, in particular,

$$W\left(t,x,\zeta',u^{2\prime}\right) := \operatorname*{essinf}_{u^1 \in \mathcal{U}^0_{t,T}} J\left(t,x,\zeta',\left(u^1,u^{2\prime}\right)\right)$$
$$= \operatorname*{essinf}_{u^1 \in \mathcal{U}^0_{t,T}} J\left(t,x,\zeta,\left(u^1,u^2\right)\right) = W\left(t,x,\zeta,u^2\right).$$

3. Dynamic programming principle

From the above lemma we deduce now easily:

Proposition 3.2.

Let $(t, x) \in [0, T] \times \mathbb{R}^n$, $\zeta \in L^2(\mathcal{F}_t; \mathbb{R}^n)$. Then, for all $\zeta' \in L^2(\mathcal{F}_t; \mathbb{R}^n)$ with $\mathbb{P}_\zeta = \mathbb{P}_{\zeta'}$, we have

$$V(t, x, \zeta) = V(t, x, \zeta'),$$

i.e., V depends on ζ only through $\mathbb{P}_\zeta$. We write:

$$V(t, x, \mathbb{P}_\zeta) := V(t, x, \zeta),$$

where $V : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$. Moreover, by standard estimates, there exists $C \in \mathbb{R}_+$, such that

$$\left| V(t, x, \mathbb{P}_\zeta) - V(t, \widetilde{x}, \mathbb{P}_{\widetilde{\zeta}}) \right| \leq C\left(|x - \widetilde{x}| + \mathcal{W}_2(\mathbb{P}_\zeta, \mathbb{P}_{\widetilde{\zeta}}) \right),$$

for all $t \in [0, T],\ x, \widetilde{x} \in \mathbb{R}^n,\ \zeta, \widetilde{\zeta} \in L^2(\mathcal{F}_t; \mathbb{R}^n)$.

Study of $W(t,x,\zeta,u^2)$:

Recall that:

• $W(t,\cdot,\cdot,\cdot):\ \mathbb{R}\times L^2(\mathcal{F}_t;\mathbb{R}^n)\times\mathcal{U}^0_{t,T}\to\mathbb{R}$ is defined by

$$W\left(t,x,\zeta,u^2\right)=\operatorname*{essinf}_{u^1\in\mathcal{U}^0_{t,T}} J\left(t,x,\zeta,\left(u^1,u^2\right)\right)$$
$$=\operatorname*{essinf}_{u^1\in\mathcal{U}^0_{t,T}} E\left[\Phi\left(X_T^{t,x,\zeta,(u^1,u^2)},\mathbb{P}_{X_T^{t,\zeta,u^2}}\right)\Big|\mathcal{F}_t\right];$$

• For any $u^2\in\mathcal{U}^0_{t,T}$, $W(t,x,\zeta,u^2)$ is deterministic;

• From standard estimates:

$$|W(t,x,\zeta,u^2)-W(t,x',\zeta',u^2)|\le C\left(|x-x'|+\left(E[|\zeta-\zeta'|^2]\right)^{\frac{1}{2}}\right),$$

for all $t\in[0,T]$, $x,x'\in\mathbb{R}^n$, $\zeta,\zeta'\in L^2(\mathcal{F}_t;\mathbb{R}^n)$, $u^2\in\mathcal{U}^0_{t,T}$.

3. Dynamic programming principle

Using the above properties of W, we obtain the following DPP for W.

Theorem 3.1. (DPP for W)

For all $0 \leq t < t+\delta \leq T,\ u^2 \in \mathcal{U}^0_{t,T},\ x \in \mathbb{R}^n,\ \zeta \in L^2(\mathcal{F}_t;\mathbb{R}^n)$,

$$W(t,x,\zeta,u^2) = \operatorname*{essinf}_{u^1 \in \mathcal{U}^0_{t,t+\delta}} E\Big[W\big(t+\delta, X^{t,x,\zeta,(u^1,u^2)}_{t+\delta}, X^{t,\zeta,u^2}_{t+\delta}, u^2\big)\Big|\mathcal{F}_t\Big].$$

The proof of this proposition uses a standard argument from Theorem 3.1 of Buckdahn and Li [2008].

Remark.

From the proof of Theorem 3.1, as W is deterministic, we obtain, for all $t < t + \delta \le T$,

$$W(t,x,\zeta,u^2) = \inf_{u^1 \in \mathcal{U}^0_{t,t+\delta}} E\left[W\left(t+\delta, X^{t,x,\zeta,(u^1,u^2)}_{t+\delta}, X^{t,\zeta,u^2}_{t+\delta}, u^2\right)\right]. \qquad (3.1)$$

In particular, for $t + \delta = T$,

$$W(t,x,\zeta,u^2) = \inf_{u^1 \in \mathcal{U}^0_{t,T}} E\left[\Phi\left(X^{t,x,\zeta,(u^1,u^2)}_T, \mathbb{P}_{X^{t,\zeta,u^2}_T}\right)\right],$$

and therefore, for the value function $V(t,x,\mathbb{P}_\zeta) = V(t,x,\zeta)$:

$$V(t,x,\mathbb{P}_\zeta) = V(t,x,\zeta) = \operatorname*{essinf}_{(u^1,u^2)\in\mathcal{U}_{t,T}} E\left[\Phi\left(X^{t,x,\zeta,(u^1,u^2)}_T, \mathbb{P}_{X^{t,\zeta,u^2}_T}\right)\Big|\mathcal{F}_t\right]$$

$$= \inf_{u^2 \in \mathcal{U}^0_{t,T}} W(t,x,\zeta,u^2) = \inf_{(u^1,u^2)\in\mathcal{U}_{t,T}} E\left[\Phi\left(X^{t,x,\zeta,(u^1,u^2)}_T, \mathbb{P}_{X^{t,\zeta,u^2}_T}\right)\right].$$

Study of $V(t,x,\mathbb{P}_\zeta)$:

As the following inequality shows, we get the one-sided DPP for V:
For $0\le t<t+\delta\le T$, $x\in\mathbb{R}^n$, $\zeta\in L^2(\mathcal{F}_t;\mathbb{R}^n)$,

$$
\begin{aligned}
V(t,x,\mathbb{P}_\zeta) &\overset{V\ \text{Def}}{=} \inf_{u^2\in\mathcal{U}^0_{t,T}} W(t,x,\zeta,u^2)\\
&\overset{(3.1)}{=} \inf_{u^2\in\mathcal{U}^0_{t,T}}\Big(\inf_{u^1\in\mathcal{U}^0_{t,t+\delta}} E\big[W\big(t+\delta,X^{t,x,\zeta,(u^1,u^2)}_{t+\delta},X^{t,\zeta,u^2}_{t+\delta},u^2\big)\big]\Big)\\
&\overset{W\ \text{Def}}{\ge} \inf_{u^2\in\mathcal{U}^0_{t,T}}\Big(\inf_{u^1\in\mathcal{U}^0_{t,t+\delta}} E\big[V\big(t+\delta,X^{t,x,\zeta,(u^1,u^2)}_{t+\delta},X^{t,\zeta,u^2}_{t+\delta}\big)\big]\Big)\\
&\overset{\text{Indep. of } u^2|_{(t+\delta,T]}}{=} \inf_{(u^1,u^2)\in\mathcal{U}_{t,t+\delta}} E\big[V\big(t+\delta,X^{t,x,\zeta,(u^1,u^2)}_{t+\delta},\mathbb{P}_{X^{t,\zeta,u^2}_{t+\delta}}\big)\big].
\end{aligned}
$$

But we can not get the above inequality in the opposite direction.

3. Dynamic programming principle

The value function ϑ:

For $\theta,\ \zeta \in L^2(\mathcal{F}_t;\mathbb{R}^n)$, we introduce a new definition of the value function:

$$\vartheta(t,\theta,\mathbb{P}_\zeta) := \inf_{u^2\in\mathcal{U}^0_{t,T}} E\Big[W(t,\theta,\zeta,u^2)\Big] = \inf_{u^2\in\mathcal{U}^0_{t,T}} E\Big[\operatorname*{essinf}_{u^1\in\mathcal{U}^0_{t,T}} E\Big[\Phi\big(X_T^{t,\theta,\zeta,(u^1,u^2)},\mathbb{P}_{X_T^{t,\zeta,u^2}}\big)\Big|\mathcal{F}_t\Big]\Big]. \tag{3.2}$$

We can see:

- The function ϑ is obviously deterministic;
- $\vartheta(t,\theta,\mathbb{P}_\zeta)$ depends on θ only through the law $\mathbb{P}_\theta$. $\Big($Indeed,

$$\vartheta(t,\theta,\mathbb{P}_\zeta) = \inf_{u^2\in\mathcal{U}^0_{t,T}} \int_{\mathbb{R}^n} W(t,x,\zeta,u^2)\mathbb{P}_\theta(dx).\Big)$$

This allows to write

$$\vartheta(t,\mathbb{P}_\theta,\mathbb{P}_\zeta) := \vartheta(t,\theta,\mathbb{P}_\zeta),$$

and to consider ϑ as a function over $[0,T]\times\mathcal{P}_2(\mathbb{R}^n)\times\mathcal{P}_2(\mathbb{R}^n)$.

3. Dynamic programming principle

From Lemma 3.1 and a standard argument we have the following estimate for ϑ: There exists a constant $C > 0$ such that

$$|\vartheta(t,\mathbb{P}_\theta,\mathbb{P}_\zeta) - \vartheta(t,\mathbb{P}_{\theta'},\mathbb{P}_{\zeta'})| \leq C\big(\mathcal{W}_2(\mathbb{P}_\theta,\mathbb{P}_{\theta'}) + \mathcal{W}_2(\mathbb{P}_\zeta,\mathbb{P}_{\zeta'})\big), \qquad (3.3)$$

for all $t \in [0,T]$, $\theta,\ \theta',\ \zeta,\ \zeta' \in L^2(\mathcal{F}_t;\mathbb{R}^n)$.

Remark.

Notice that, for $(t,x) \in [0,T] \times \mathbb{R}^n,\ \zeta \in L^2(\mathcal{F}_t;\mathbb{R}^n)$,

$$\vartheta(t,x,\mathbb{P}_\zeta)(:= \vartheta(t,\delta_x,\mathbb{P}_\zeta)) = V(t,x,\mathbb{P}_\zeta),$$

where δ_x denotes the Dirac measure at x, which means that a description of $\vartheta(t,\mathbb{P}_\theta,\mathbb{P}_\zeta),\ (\theta,\zeta) \in L^2(\mathcal{F}_t;\mathbb{R}^n) \times L^2(\mathcal{F}_t;\mathbb{R}^n)$, as a solution of a PDE also characterizes $V(t,x,\mathbb{P}_\zeta)$. This allows to characterize V through studying ϑ.

Let us now study the new value function ϑ. To begin with, we prove that ϑ obeys the following DPP:

Theorem 3.2. (DPP for ϑ)

For any $0 \leq t < t+\delta \leq T,\ \theta, \zeta \in L^2(\mathcal{F}_t; \mathbb{R}^n)$,

$$\vartheta(t, \mathbb{P}_\theta, \mathbb{P}_\zeta) = \inf_{u \in \mathcal{U}_{t,t+\delta}} \vartheta\Big(t+\delta, \mathbb{P}_{X_{t+\delta}^{t,\theta,\zeta,u}}, \mathbb{P}_{X_{t+\delta}^{t,\zeta,u^2}}\Big). \tag{3.4}$$

By using the properties of W, especially the DPP for W, and standard arguments from Buckdahn and Li [2008], we prove Theorem 3.2.

Using the continuity properties of $\vartheta(t,\cdot,\cdot)$ on $\mathcal{P}^2(\mathbb{R}^d)\times\mathcal{P}^2(\mathbb{R}^d)$ and the DPP for ϑ, we also prove the continuity of ϑ with respect to t.

Proposition 3.3.

The value function ϑ is $\frac{1}{2}$-Hölder continuous in t: There exists a constant C such that, for every $t,t'\in[0,T]$, $\theta,\zeta\in L^2(\mathcal{F}_t;\mathbb{R}^n)$,

$$\left|\vartheta(t,\mathbb{P}_\theta,\mathbb{P}_\zeta)-\vartheta\left(t',\mathbb{P}_\theta,\mathbb{P}_\zeta\right)\right|\le C\big(1+(E[|\zeta|^2])^{\frac{1}{2}}+(E[|\theta|^2])^{\frac{1}{2}}\big)\left|t-t'\right|^{\frac{1}{2}}.$$

4. Master Bellman equation and viscosity solution

Let us begin with the notion of derivative w.r.t. the measure over the Wasserstein space.

Definition 4.1. (Carmona and Delarue [2018])

A function $\varphi : \mathcal{P}_2\left(\mathbb{R}^k\right) \to \mathbb{R}$ is said to have a linear functional derivative if there exists a function

$$\frac{\delta \varphi}{\delta \mu} : \mathcal{P}_2\left(\mathbb{R}^k\right) \times \mathbb{R}^k \ni (\mu, x) \mapsto \frac{\delta \varphi}{\delta \mu}(\mu)(x) \in \mathbb{R},$$

which is continuous with respect to the product topology ($\mathcal{P}_2\left(\mathbb{R}^k\right)$ is equipped with the 2-Wasserstein distance) such that, for any bounded subset $\mathcal{K} \subset \mathcal{P}_2\left(\mathbb{R}^k\right)$, the function $\mathbb{R}^d \ni x \mapsto [\delta\varphi/\delta\mu](\mu)(x)$ is at most of quadratic growth in x, uniformly in μ, for $\mu \in \mathcal{K}$, and for all μ and μ' in $\mathcal{P}_2\left(\mathbb{R}^k\right)$, it holds:

$$\varphi\left(\mu'\right) - \varphi(\mu) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta \varphi}{\delta \mu}\left(r\mu' + (1-r)\mu\right)(x) d(\mu' - \mu)(x) dr.$$

This leads to the definition of the L-derivative of φ (the derivative introduced by P.L. Lions [2013]).

Definition 4.2. (Carmona and Delarue [2018])

If $\dfrac{\delta\varphi}{\delta\mu}$ is of class $\mathcal{C}^1$ with respect to the second variable, the L-derivative $\partial_\mu\varphi : \mathcal{P}\left(\mathbb{R}^k\right) \times \mathbb{R}^k \to \mathbb{R}^k$ is defined by

$$\partial_\mu\varphi(\mu,x) := \partial_x\Big(\frac{\delta\varphi}{\delta\mu}\Big)(\mu,x), \quad (\mu,x) \in \mathcal{P}_2(\mathbb{R}^k) \times \mathbb{R}^k.$$

For any $\mu \in \mathcal{P}_2(\mathbb{R}^k)$ and any μ-integrable function $f : \mathbb{R}^k \to \mathbb{R}$, we use the notation

$$\langle \mu, f \rangle := \int_{\mathbb{R}^k} f(x)\mu(dx).$$

For simplicity of notation, we restrict ourselves to dimension $n = d = k = 1$ in what follows.

In this section, we study our optimal control problem on the space $\mathcal{O} := [0,T) \times \mathcal{M} \times \mathcal{M}$, where $\mathcal{M} \subset \mathcal{P}_2(\mathbb{R})$ is the set of probability measures with δ-exponential moments, i.e.,

$$\mathcal{M} := \{\mu \in \mathcal{P}_2(\mathbb{R}) : \langle \mu, \exp(\delta|\cdot|)\rangle = \int_{\mathbb{R}} \exp(\delta|x|)\mu(\mathrm{d}x) < \infty\}, \quad (4.1)$$

where $\delta > 0$ is an arbitrary given constant. We endow $\mathcal{M}$ with the topology of weak convergence $\sigma(\mathcal{M}, C_b(\mathbb{R}))$, where $C_b(\mathbb{R})$ is the space of continuous and bounded functions on $\mathbb{R}$.

Remark. The space $\mathcal{O}$ has a suitable σ-**compact structure**, which allows to establish uniqueness in the next section. This form of $\mathcal{O}$ is crucial to obtain uniform integrability of the viscosity test functions as well as some continuity properties of the Hamiltonian.

4. Master Bellman equation and viscosity solution

Master Bellman equation:

For any $(t, \mu_1, \mu_2) \in [0, T) \times \mathcal{M} \times \mathcal{M}$,

$$\begin{cases} -\partial_t \vartheta(t, \mu_1, \mu_2) - \mathcal{H}\big(t, \mu_1, \mu_2, \partial_{\mu_1} \vartheta(t, \mu_1, \mu_2; \cdot), \partial_{\mu_2} \vartheta(t, \mu_1, \mu_2; \cdot)\big) = 0, \\ \vartheta(T, \mu_1, \mu_2) = \langle \mu_1, \Phi(\cdot, \mu_2) \rangle, \end{cases} \tag{4.2}$$

where, with the notation

$$\Pi_\mu := \big\{\gamma \in \mathcal{P}_2(\mathbb{R} \times U) : \gamma(\cdot \times U) = \mu\big\},\ \ \mu \in \mathcal{P}_2(\mathbb{R}),$$

the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}\big(t, \mu_1, \mu_2, p_1, p_2\big)$$
$$:= \inf\Big\{\langle \gamma_1, \mathcal{L}_t^{\mu_1, \mu_2, \gamma_2}[p_1] \rangle + \langle \gamma_2, \overline{\mathcal{L}}_t^{\mu_1, \mu_2, \gamma_2}[p_2] \rangle :\ \gamma_i \in \Pi_{\mu_i},\ i = 1, 2\Big\},$$

$p_1, p_2 \in C^1(\mathbb{R})$, with, for $(y, v) \in \mathbb{R} \times U$,

$$\mathcal{L}_t^{\mu_1, \mu_2, \gamma_2}[p_1](y, v) := p_1(y) b_2(t, (y, v), \gamma_2) + \frac{1}{2} \partial_y p_1(y) (\sigma_2(t, (y, v), \gamma_2))^2,$$

$$\overline{\mathcal{L}}_t^{\mu_1, \mu_2, \gamma_2}[p_2](y, v) := p_2(y) b_1(t, (y, v), \gamma_2) + \frac{1}{2} \partial_y p_2(y) (\sigma_1(t, (y, v), \gamma_2))^2.$$

Remark.

When the coefficients b_1 and σ_1 do not depend on the control u^2, then $\vartheta(t,\delta_x,\mu)=V(t,x,\mu)=W(t,x,\mu)$, and from PDE (4.2) we get the following PDE related with $V(t,x,\mu)$: For $(t,x,\mu)\in[0,T)\times\mathbb{R}\times\mathcal{M}$,

$$\begin{cases} -\partial_t V(t,x,\mu)-\inf\limits_{u\in U}\left\{\partial_x V(t,x,\mu)b_2(t,(x,u),\mu)+\frac{1}{2}\partial_x^2 V(t,x,\mu)(\sigma_2(t,(x,u),\mu))^2\right\} \\ -\int_{\mathbb{R}}\partial_\mu V(t,x,\mu;y)b_1(t,y,\mu)\mu(dy)-\int_{\mathbb{R}}\frac{1}{2}\partial_y\partial_\mu V(t,x,\mu;y)(\sigma_1(t,y,\mu))^2\mu(dy)=0, \\ V(T,x,\mu)=\Phi(x,\mu). \end{cases} \tag{4.3}$$

• If $b_1=\sigma_1=0$, and b_2,σ_2,Φ do not depend on the law, PDE (4.3) is just the classical HJB equation.

• If $b_1=b_2$, and $\sigma_1=\sigma_2$, i.e., all the coefficients are free of controls, then PDE (4.2) is just the mean-field PDE obtained in Buckdahn, Li, Peng and Rainer [2017].

For δ as in (4.1), we consider the function

$$e_\delta(x) := \exp\left(\delta(\sqrt{x^2+1}-1)\right), \quad x \in \mathbb{R}.$$

For $N \in \mathbb{N}$ and δ as in (4.1), let

$$\mathcal{O}_N := \left\{(t,\mu_1,\mu_2) \in [0,T) \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \mid \langle \mu_i, e_\delta \rangle \le N e^{K^* t},\ i = 1,2\right\},$$

$$\overline{\mathcal{O}}_N := \left\{(t,\mu_1,\mu_2) \in [0,T] \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \mid \langle \mu_i, e_\delta \rangle \le N e^{K^* t},\ i = 1,2\right\},$$

where K^* is a given positive constant which is derived from the proof of Lemma 4.1 further down, in order to ensure that $\mathcal{O}_N$ is invariant for the dynamics (2.1)-(2.2). Note that $\mathcal{O} = \cup_{N=1}^{\infty} \mathcal{O}_N$ and $\overline{\mathcal{O}} = \cup_{N=1}^{\infty} \overline{\mathcal{O}}_N$.

For a constant b and δ as in (4.1), we put

$$\mathcal{M}_b := \{\mu \in \mathcal{P}_2(\mathbb{R}) \mid \langle \mu, e_\delta \rangle \le b\}.$$

Lemma 4.1.

Under Assumption 2.1 , for all $N \in \mathbb{N}$, the set $\mathcal{O}_N$ is invariant for the SDEs (2.1)-(2.2), namely,

$$(t, \mathbb{P}_\theta, \mathbb{P}_\zeta) \in \mathcal{O}_N \Longrightarrow \left(s, \mathbb{P}_{X_s^{t,\theta,\zeta,u}}, \mathbb{P}_{X_s^{t,\zeta,u^2}}\right) \in \mathcal{O}_N,$$

for all $t \in [0,T]$, $s \in [t,T]$, $\theta, \zeta \in L^2(\mathcal{F}_t)$, and $u = (u^1, u^2) \in \mathcal{U}_{t,T}$, where $\left(X_s^{t,\zeta,u^2}, X_s^{t,\theta,\zeta,u}\right)_{s\in[t,T]}$ is the solution to (2.1)-(2.2) with initial condition $\left(X_t^{t,\zeta,u^2}, X_t^{t,\theta,\zeta,u}\right) = (\zeta, \theta)$.

This implies that for any given initial law $(t, \mu_1, \mu_2) \in \mathcal{O}_N$, we may restrict the Master Bellman equation (4.2) to $\mathcal{O}_N$.

4. Master Bellman equation and viscosity solution

As $\mathcal{P}_2(\mathbb{R})$ itself is not σ-compact, the importance of $\overline{\mathcal{O}}_N$ stems also from the following fact:

Lemma 4.2.

For $N \in \mathbb{N}$, $\overline{\mathcal{O}}_N$ is a compact subset of $[0, T] \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$.

Now we give **the definition of a test function** and **that of a viscosity solution to the Master Bellman equation (4.2)**, which were first introduced in Burzoni et al. [2020]. We adapt them here to our framework.

Definition 4.3.

A cylindrical function is a mapping of the form $(t, \mu_1, \mu_2) \mapsto F(t, \langle \mu_1, f_1 \rangle, \langle \mu_2, f_2 \rangle)$ for some functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ and $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This function is called cylindrical polynomial, if f_1, f_2 are polynomials and F is continuously differentiable.

We extend the above class to its linear span. For any polynomial f, we denote the degree of f by $\deg(f)$.

Definition 4.4. (Test Functions)

For $E \subset \overline{\mathcal{O}}$, a viscosity test function on E is a function of the form

$$\varphi(t, \mu_1, \mu_2) = \sum_{j=1}^{\infty} \varphi_j(t, \mu_1, \mu_2), \quad (t, \mu_1, \mu_2) \in E,$$

where $\{\varphi_j\}_{j\in\mathbb{N}}$ is a sequence of cylindrical polynomials which is absolutely convergent at every (t, μ_1, μ_2) and, for $i = 1, 2$,

$$\lim_{M\to\infty} \sum_{j=M}^{\infty} \sup_{(t,\mu_1,\mu_2)\in\mathcal{O}_N} \sum_{i=1,2} \Big(\langle \mu_i, |\partial_{\mu_i}\varphi_j(t, \mu_1, \mu_2; \cdot)| \rangle + \langle \mu_i, |\partial_y \partial_{\mu_i}\varphi_j(t, \mu_1, \mu_2; \cdot)| \rangle \Big) = 0. \tag{4.4}$$

Let Γ_E be the set of all viscosity test functions on E.

Remark. Condition (4.4) is used for proving the continuity of $(t, \mu_1, \mu_2) \mapsto \mathcal{H}(t, \mu_1, \mu_2, \partial_{\mu_1}\varphi(t, \mu_1, \mu_2; \cdot), \partial_{\mu_2}\varphi(t, \mu_1, \mu_2; \cdot))$ in Proposition 4.1, when $\varphi \in \Gamma_{\mathcal{O}_N}$.

Definition 4.5.

For $E \subseteq \overline{\mathcal{O}}$ and $(t,\mu_1,\mu_2) \in E$ with $t < T$, the superjet of a function $u : E \to \mathbb{R}$ at (t,μ_1,μ_2) is given by

$$J_E^+ u(t,\mu_1,\mu_2) := \Big\{ \big(\partial_t \varphi(t,\mu_1,\mu_2), \partial_{\mu_1}\varphi(t,\mu_1,\mu_2;\cdot), \partial_{\mu_2}\varphi(t,\mu_1,\mu_2;\cdot)\big) \ \Big| \ \varphi \in \Gamma_E, (u-\varphi)(t,\mu_1,\mu_2) = \max_E (u-\varphi) \Big\}.$$

The subjet of u at (t,μ_1,μ_2) is defined as

$$J_E^- u(t,\mu_1,\mu_2) := -J_E^+(-u(t,\mu_1,\mu_2)).$$

Definition 4.6. (Viscosity Solution)

A continuous function $u : \mathcal{O}_N \to \mathbb{R}$ with $u(T, \mu_1, \mu_2) = \langle \mu_1, \Phi(\cdot, \mu_2) \rangle$ is called a viscosity subsolution of (4.2) on $\mathcal{O}_N$ if, for every $(t, \mu_1, \mu_2) \in \mathcal{O}_N$,

$$-\pi_t - \mathcal{H}\left(t, \mu_1, \mu_2, \pi_{\mu_1}, \pi_{\mu_2}\right) \leq 0, \quad (\pi_t, \pi_{\mu_1}, \pi_{\mu_2}) \in J^+_{\mathcal{O}_N} u(t, \mu_1, \mu_2).$$

A continuous function $u : \mathcal{O}_N \to \mathbb{R}$ with $u(T, \mu_1, \mu_2) = \langle \mu_1, \Phi(\cdot, \mu_2) \rangle$ is called a viscosity supersolution of (4.2) on $\mathcal{O}_N$ if for every $(t, \mu_1, \mu_2) \in \mathcal{O}_N$,

$$-\pi_t - \mathcal{H}\left(t, \mu_1, \mu_2, \pi_{\mu_1}, \pi_{\mu_2}\right) \geq 0, \quad (\pi_t, \pi_{\mu_1}, \pi_{\mu_2}) \in J^-_{\mathcal{O}_N} u(t, \mu_1, \mu_2).$$

A viscosity solution of (4.2) is a function on $\mathcal{O}$ that is both a subsolution and a supersolution of (4.2) on $\mathcal{O}_N$, for every $N \in \mathbb{N}$.

In what follows we show that $(t,\mu_1,\mu_2) \mapsto \mathcal{H}\big(t,\mu_1,\mu_2,\partial_{\mu_1}\varphi,\partial_{\mu_2}\varphi\big)$ is continuous on $\mathcal{O}_N$, for any $\varphi \in \Gamma_{\mathcal{O}_N}$.

Proposition 4.1.

Under Assumption 2.1, for every $\varphi \in \Gamma_{\mathcal{O}_N}$, the mapping

$$(t,\mu_1,\mu_2) \mapsto \mathcal{H}\big(t,\mu_1,\mu_2,\partial_{\mu_1}\varphi(t,\mu_1,\mu_2;\cdot),\partial_{\mu_2}\varphi(t,\mu_1,\mu_2;\cdot)\big)$$

is continuous on $\mathcal{O}_N$.

Lemma 4.5.

Under the Assumptions 2.1 and 2.2, for all N, the value function ϑ is bounded on $\mathcal{O}_N$.

Let us come to our main results: One states that **the value function ϑ is a viscosity solution of (4.2) on $\mathcal{O}$**, and the other shows that **the comparison theorem for (4.2) holds**.

Theorem 5.1.

Let the Assumptions 2.1 and 2.2 hold true. Then, for all $N \in \mathbb{N}$, the value function ϑ is both a viscosity sub- and a supersolution to (4.2) on $\mathcal{O}_N$, and so $\mathcal{O}$.

The proof of this theorem is rather subtle and technical, and so it is omitted here.

The remaining part of this section is devoted to a comparison theorem for the value function ϑ. For this purpose, we first need to introduce some definitions and notations.

Definition 5.1.

We say that a set of polynomials $\mathcal{X}$ has the $(*)$-property, if it satisfies

$$\text{for all } g \in \mathcal{X},\ g^{(i)} \in \mathcal{X},\ 0 \leq i \leq \deg(g),$$

where $g^{(i)}$ is the i-th derivative of g, and $\deg(g)$ denotes the degree of polynomial g. Let $\sum$ be the collection of all sets of polynomials that have the $(*)$-property.

For f a real polynomial we set

$$\mathcal{X}(f) := \bigcap_{\mathcal{X} \in \Sigma, f \in \mathcal{X}} \mathcal{X}.$$

5. Main results

We can easily check the following properties of $\mathcal{X}(f)$.

Lemma 5.1.

For every polynomial f, we have:
(a) $\mathcal{X}(f)$ is the smallest set of polynomials with the $(*)$-property that includes f;
(b) For every $g \in \mathcal{X}(f)$, $\mathcal{X}(g) \subset \mathcal{X}(f)$;
(c) $\mathcal{X}(f)$ is finite.

Let $\Theta := \bigcup_{j=1}^{\infty} \mathcal{X}(\psi_j)$, where $\psi_j(x) = x^j,\ x \in \mathbb{R}$. Then:

a) Θ is countable; b) $\{\psi_j\}_{j=1}^{\infty} \subset \Theta$; c) For any $f \in \Theta$, $\mathcal{X}(f) \subset \Theta$.

Let $\{f_j\}_{j=1}^{\infty}$ be an enumeration of Θ. We define the finite index set I_j:

$$I_j = \{ i \mid f_i \in \mathcal{X}(f_j) \},\ j \geq 1.$$

Then, for all $i \in I_j$, we have $\mathcal{X}(f_i) \subset \mathcal{X}(f_j)$ and, therefore, $I_i \subseteq I_j$.

Moreover, we define for $b > 0$ and $j \in \mathbb{N}$

$$c_j(b) := \Big(\sum_{k \in I_j} 2^k\Big)^{-1} \Big(\sum_{k \in I_j} s_k(b)\Big)^{-2},$$

where $s_j(b) := 1 + \sup_{\mu \in \mathcal{M}_b} \langle \mu, f_j \rangle$. We observe that it follows from Lemma 4.3 that $1 \le s_j(b) < \infty$, for all $j \in \mathbb{N}$. And we can see:

- Since $f_j \in \mathcal{X}(f_j)$, we have $j \in I_j$, and, thus, $c_j(b) \le 2^{-j}$. Hence, $\sum_{j=1}^{\infty} c_j(b) \le 1$.
- For $i \in I_j$, from $I_i \subset I_j$ we get $c_j(b) \le c_i(b)$.
- By the definition of $s_j(b)$ and $c_j(b)$,

$$\sum_{j=1}^{\infty} c_j(b) \langle \mu, f_j \rangle^2 \le 1, \quad \mu \in \mathcal{M}_b. \tag{5.1}$$

In the proof of comparison theorem, c_j and its properties are used to introduce a distance-like function d, in order to construct test functions.

5. Main results

To prove the comparison theorem, we need to impose an additional assumption. Let $N \geq 1$, and define

$$\mathcal{K}_N := \{\gamma \in \mathcal{P}_2(\mathbb{R} \times U) \mid \mu := \gamma(\cdot \times U) \in \mathcal{M}_{Ne^{K^*T}}\}.$$

Note that $\mathcal{K}_N$ is compact and $\overline{\mathcal{O}}_N \subset [0,T] \times \{\mu = \gamma(\cdot \times U) \mid \gamma \in \mathcal{K}_N\}^2$.

Assumption 5.1(N).

There exists a constant $\kappa_0 > 0$ and a finite set $\mathcal{I} \subset \mathbb{N}$ (possibly depending on N) such that for all $s,\ s' \in [t,T],\ x \in \mathbb{R},\ u \in U$, and $\gamma,\ \gamma' \in \mathcal{K}_N$ satisfying $\gamma(\mathbb{R} \times \cdot) = \gamma'(\mathbb{R} \times \cdot)$,

$$\big|\phi(s,(x,u),\gamma) - \phi(s',(x,u),\gamma')\big| \leq \kappa_0(|s-s'| + \sum_{i \in \mathcal{I}} |\langle \mu - \mu', x^i \rangle|), \quad (5.2)$$

where $\mu = \gamma(\cdot \times U), \mu' = \gamma'(\cdot \times U)$, for $\phi = b_1, b_2, \sigma_1, \sigma_2$, resp.

Remark.

Assumption 5.1(N) is a form of **Lipschitz continuity on $\overline{\mathcal{O}}_N$ w.r.t. cylindrical functions of the measure arguments**. Moreover, (5.2) also implies, that for some constant $C_{N,\mathcal{I}} > 0$,

$$|\varphi(s,(x,u),\gamma) - \varphi(s',(x,u),\gamma')| \leq K_0(|s-s'| + C_{N,\mathcal{I}}\mathcal{W}_2(\gamma,\gamma')),$$

for all $s,\ s' \in [t,T],\ u \in U,\ \gamma,\ \gamma' \in \mathcal{K}_N$ with $\gamma(\mathbb{R}\times\cdot) = \gamma'(\mathbb{R}\times\cdot)$. This latter relation shows that, if φ is independent of (x,u), Assumption 5.1(N) implies Assumption 2.1 on $[0,T]\times\mathcal{K}_N$.

Theorem 5.2. (Comparison Theorem)

We suppose that

$$b_j(t,(y,v),\gamma) = b_j(t,\gamma), \quad \sigma_j(t,(y,v),\gamma) = \sigma(t,\gamma), \tag{5.3}$$

$(t,(y,v),\gamma) \in [0,T] \times (\mathbb{R} \times U) \times \mathcal{P}_2(\mathbb{R} \times U)$, $j = 1,2$, i.e., the coefficients b_j, σ_j are independent of (y,v). Let Assumptions 2.2 and 5.1(N) hold on $[0,T] \times \mathcal{K}_N$. Let $u \in C(\mathcal{O}_N)$ be a viscosity subsolution to HJB equation (4.2) on $\mathcal{O}_N$ and $v \in C(\mathcal{O}_N)$ be a viscosity supersolution to HJB equation (4.2) on $\mathcal{O}_N$, satisfying $u(T,\mu_1,\mu_2) \le v(T,\mu_1,\mu_2)$, for any $(T,\mu_1,\mu_2) \in \overline{\mathcal{O}}_N$. Then $u \le v$ on $\overline{\mathcal{O}}_N$.

Remark. Burzoni et al. [2020] consider coefficients $(b,\sigma)(t,\mu,v)$, $(t,\mu,v) \in [0,T] \times \mathcal{P}(\mathbb{R}) \times U$. While they use only deterministic control processes, we overcome this difficulty by considering our stochastic control in the law.

<u>**Sketch of Proof.**</u> Fix $N \in \mathbb{N}$ and let $c_j := c_j\left(Ne^{K^*T}\right)$. Then, for all $(t, \mu_1, \mu_2) \in \overline{\mathcal{O}}_N$, $\mu_1, \mu_2 \in \mathcal{M}_{Ne^{K^*t}} \subset \mathcal{M}_{Ne^{K^*T}}$, it follows from (5.1) that

$$\sup_{(t,\mu_1,\mu_2)\in\overline{\mathcal{O}}_N} \sum_{j=1}^{\infty} c_j \langle \mu_i, f_j \rangle^2 \leq 1, \quad i = 1, 2.$$

We suppose that

$$\sup_{\overline{\mathcal{O}}_N}(u - v) > 0,$$

and we prove that this leads to a contradiction.

Since $u - v$ is continuous and $\overline{\mathcal{O}}_N$ is compact, the maximum

$$\ell := \max_{(t,\mu_1,\mu_2)\in\overline{\mathcal{O}}_N} \big((u - v)(t, \mu_1, \mu_2) - 2\eta(T - t)\big)$$

can be achieved and there exists sufficiently small η_0, such that, for all $\eta \in (0, \eta_0]$, we have $\ell > 0$.

Now we use the standard argument of doubling variables to construct test functions for u and v.

Step 1. Doubling of variables. For $\varepsilon > 0$ and $\eta \in (0, \eta_0]$, we define

$$\begin{aligned}\phi_\varepsilon(t,\mu_1,\mu_2,s,\nu_1,\nu_2) &:= u\left(t,\mu_1,\mu_2\right) - v\left(s,\nu_1,\nu_2\right) \\ &\quad - \frac{1}{\varepsilon}\left((t-s)^2 + d(\mu_1,\nu_1) + d(\mu_2,\nu_2)\right) - \eta(T-t+T-s),\end{aligned} \tag{5.4}$$

where d is a distance-like function defined for $\mu, \nu \in \mathcal{M}_{Ne^{K^*T}}$ by the relation

$$d(\mu,\nu) := \sum_{j=1}^{\infty} c_j \langle \mu - \nu, f_j \rangle^2; \tag{5.5}$$

recall that $\{f_j\}_{j=1}^{\infty} = \Theta = \bigcup_{j=1}^{\infty} \mathcal{X}\left(x^j\right)$ (see Lemma 5.1). We observe that $d(\cdot,\cdot)$ is compatible with the weak convergence in $\mathcal{M}_{Ne^{K^*T}}$: Both generate the same topology.

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The following corollary is a straightforward conclusion of Theorems 5.1 and 5.2.

Corollary 5.1.

Let Assumptions 2.2 and 5.1(N) hold, for all $N \geq 1$. Under the assumption that

$$b_j(t,(y,v),\gamma) = b_j(t,\gamma), \quad \sigma_j(t,(y,v),\gamma) = \sigma(t,\gamma), \tag{5.6}$$

$(t,(y,v),\gamma) \in [0,T] \times (\mathbb{R} \times U) \times \mathcal{P}_2(\mathbb{R} \times U)$, $j = 1,2$, are independent of (y,v), the value function ϑ is the unique viscosity solution to HJB equation (4.2) on $\mathcal{O}$.

Thank you very much for your attention!