Mean field stochastic control under sublinear expectation

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Outline

1. Objective of the talk
2. Preliminaries
3. Derivative of a function of a law under G-expectation
4. Mean field stochastic control problems under G-expectation

Brief state of the art

Mean-field problems:

1) Mean-Field SDEs have been intensively studied for a longer time as limit equ. for systems with a large number of particles (propagation of chaos)(Bossy, Méléard, Sznitman, Talay,...);

2) Mean-Field Games and related topics, since 2006-2007 by J.M.Lasry and P.L.Lions, Huang-Caines-Malhamé (2006);

....

3) +) Mean-Field BSDEs/FBSDEs and associated nonlocal PDEs:

- Preliminary works in: Buckdahn, Dijehiche, L., Peng (2009, AOP),
  Buckdahn, L., Peng (2009, SPA);
- Classical solution of non-linear PDE related with the mean-field SDE:
  Buckdahn, L., Peng, Rainer (2017, AOP (2014, Arxiv));
- For the case with jumps: L., Hao (2016, NODEA);
- For the case with the mean-field forward and backward SDE jumps:
  L. (2017, SPA);
- For the case with continuous coefficients:
  L., Liang, Zhang (2018, JMAA)

Brief state of the art

+) Controlled mean-field forward and backward SDEs:

• For Pontryagin's maximum principle: Li (2012, Automatica);

+ with partial observations: Buckdahn, Li, Ma (2017, AAP);

$\rightsquigarrow$ Acciaio, Backhoff-Veraguas, Carmona (2019, SICON): Controlled mean-field stochastic system:

$$dX_t^v = b(t, P_{(X_t^v, v_t)}, X_t^v, v_t)dt + \sigma(t, P_{(X_t^v, v_t)}, X_t^v, v_t)dW_t, \ t \in [0, T]...$$

• For Peng's maximum principle: Buckdahn, Djehiche, L. (2011, AMO);

$\rightsquigarrow$Buckdahn, Li., Ma (2016, AMO): Controlled mean-field stochastic system:

$$dX_t^v = b(t, P_{X_t^v}, X_t^v, v_t)dt + \sigma(t, P_{X_t^v}, X_t^v, v_t)dW_t, \ t \in [0, T]...$$

$\rightsquigarrow$Buckdahn, Chen, Li (2021, SPA): Controlled mean-field stochastic system:

$$dX_t^v = b(t, P_{(X_t^v, v_t)}, X_t^v, v_t)dt + \sigma(t, P_{(X_t^v, v_t)}, X_t^v, v_t)dW_t, \ t \in [0, T]...$$

Nonlinear expectation:

1) Peng introduced a fully non linear expectation, called G-expectation $\hat{\mathbb{E}}[\cdot]$; he proved it well characterizes the Knightian uncertainty (2006).

2) G-SDE and G-BSDE:

- Peng (2010, Arxiv);
- Hu, Ji, Peng, Song (2014, SPA);

....

3) Stochastic maximum principle for optimal control problems (without mean-field term in the coefficients):

- Bagiani, Meeyeer-Brandis, Oksendal (2014, PUQR);
- Sun (2016, JCAM);
- Hu, Ji (2016, SICON);

....

1. Objective of the talk

Investigate Pontryagin's stochastic maximum principle
for a mean-field stochastic control problem under G-expectation.

The novelties in our work:

• The dynamic of the state process is given by a G-SDE of mean-field type. Also the cost functional is a G-expectation of mean-field type.

• Our coefficients depend not only on the controlled state process X_t^u and its control u_t but also on terms $\hat{\mathbb{E}}[\varphi(X_t^u)]$, and so their derivatives have to be considered. For this we develop a more general and more direct approach for these derivatives than Hu and Ji (2016);

• Particularly subtle is the study for the term $\hat{\mathbb{E}}[\varphi(X_t^u)]$ in the running cost coefficient. A measurable selection theorem for a set-valued mapping with values in the subspace of probability measures representing the G-expectation has to be proven. The case with a term $\hat{\mathbb{E}}[\varphi(X_T^u)]$ only in the terminal cost coefficient is easier to treat and can be handled with Sion's minimax theorem;

• The SMP we obtain is of a new form.

We review some notations and results in the G-expectation framework, immediately adapted to our framework.

Let $\Omega = C([0,T];\mathbb{R})$ and $\mathcal{H} = L_{ip}(\Omega)$.

Definition 2.1.

A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: For each $X, Y \in \mathcal{H}$,

(i) Monotonicity: $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$, if $X \geq Y$;

(ii) Constant preserving: $\hat{\mathbb{E}}[c] = c$, for $c \in \mathbb{R}$;

(iii) Sub-additivity: $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(iv) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, for all real $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2.

Two d-dimensional random vectors X_1 and X_2 defined respectively on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \quad \text{for every } \varphi \in C_{b.Lip}(\mathbb{R}^d).$$

Definition 2.3.

On the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, an n- dimensional random vector Y is said to be independent of a d-dimensional random vector X, denoted by $Y \perp X$, if

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}], \quad \text{for every } \varphi \in C_{b.Lip}(\mathbb{R}^{d+n}).$$

A d-dimensional random vector $\bar{X}$ is said to be an independent copy of X if $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp X$.

Definition 2.4.

A d-dimensional random vector X defined on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-normally distributed if for any $a, b \geq 0$,

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of X. Here the letter G denotes the function $G(A) := \frac{1}{2}\hat{\mathbb{E}}[\langle AX, X\rangle]$, for $A \in \mathbb{S}(d)$, where $\mathbb{S}(d)$ is the space of all $d \times d$ symmetric matrices.

Let $B_t(\omega) := \omega_t$, $\omega \in \Omega, t \in [0, T]$, be the coordinate process on Ω. We recall

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) : n \geq 1,$$
$$0 \leq t_1 < t_2 \cdots < t_n \leq T,\ \varphi \in C_{b.Lip}(\mathbb{R}^n)\}.$$

The G-expectation on $L_{ip}(\Omega)$ is defined by

$$\hat{\mathbb{E}}[X] := \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1}\xi_1, \sqrt{t_2 - t_1}\xi_2, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)],$$

for all
$X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}),\ n \geq 1,\ 0 \leq t_1 < \cdots < t_n \leq T$, where $\{\xi_i\}_{i=1}^n$ is a collection of n identically distributed random variables on a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$ such that, for all $1 \leq i \leq n$, ξ_i is G-normally distributed and independent of $(\xi_1, \ldots, \xi_{i-1})$.

Then under $\hat{\mathbb{E}}$, the coordinate process B_t is a G-Brownian motion defined by the following properties:

(a) $B_0 = 0$;

(b) For every t, $s \geq 0$, $B_{t+s} - B_t$ is independent of $(B_{t_1}, \ldots, B_{t_n})$, for all $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$;

(c) $B_{t+s} - B_t \overset{d}{=} \sqrt{s}\xi$, for $t, s \geq 0$, where ξ is G-normally distributed.

Remark 2.1.

(i) It is easy to check that the G-Brownian motion is symmetric, i.e., $(-B_t)_{t\geq 0}$ is also a G-Brownian motion.

(ii) If, in particular, $G(A) = \frac{1}{2}\operatorname{tr}(A)$, then the G-expectation is just a linear expectation with respect to the Wiener measure P, i.e., $\hat{\mathbb{E}} = E_P$, and the G-Brownian motion is a classical Brownian motion over $(\Omega, \mathcal{B}(\Omega), P)$.

The conditional G-expectation (knowing $\mathcal{F}_t = \sigma\{B_s,\ s \in [0,t]\}$) for $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ at $t = t_j, 1 \leq j \leq n$, is defined by

$$\hat{\mathbb{E}}_{t_j}[X] := \phi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_j} - B_{t_{j-1}}),$$

where $\phi(x_1, \ldots, x_j) = \hat{\mathbb{E}}[\varphi(x_1, \ldots, x_j, B_{t_{j+1}} - B_{t_j}, \ldots, B_{t_n} - B_{t_{n-1}})]$.

For $p \geq 1$, $L_G^p(\Omega)$ is the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. The conditional G-expectation $\hat{E}_t[\cdot]$ $(t \geq 0)$ can be extended continuously to $L_G^1(\Omega)$.

Theorem 2.1.[Peng, 2019, Springer]

Let

$\mathcal{P} = \{P$ probability on $(\Omega, \mathcal{B}(\Omega)) : E_P[X] \leq \hat{\mathbb{E}}[X]$, for all $X \in L_G^1(\Omega)\}$.

Then $\mathcal{P} \neq \emptyset$ is convex, weakly compact subset of the space $\mathcal{P}(\Omega)$ of all probability measures over $(\Omega, \mathcal{B}(\Omega))$ endowed with the topology of weak convergence, and

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \text{ for all } \xi \in L_G^1(\Omega).$$

The set $\mathcal{P}$ is said to represent $\hat{\mathbb{E}}$.

Definition 2.5.

Let $X = (X_1, \cdots, X_n)$ be a given n-dimensional random vector on a G-expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We define the functional $\mathbb{F}_X$ on the space of Lipschitz functions $C_{Lip}(\mathbb{R}^n)$ by putting

$$\mathbb{F}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)], \ \varphi \in C_{Lip}(\mathbb{R}^n).$$

The triple $(\mathbb{R}^n, C_{l.Lip}(\mathbb{R}^n), \mathbb{F}_X)$ forms a nonlinear expectation space, and $\mathbb{F}_X$ is called the distribution of X under $\hat{\mathbb{E}}$.

Spaces we work with:

- $M_G^{p,0}(0,T)$

$$:= \Big\{ \eta = \sum_{i=0}^{N-1} \xi_i I_{(s_i,s_{i+1}]} : N \geq 1,\ s_0 < \cdots < s_N, \xi_i \in L_G^p(\Omega)\ \mathcal{F}_{s_i}\text{-meas.} \Big\}.$$

- $M_G^p(0,T)$ - completion of $M_G^{p,0}(0,T)$ w.r.t. $\|\cdot\|_{M_G^p} := \left(\hat{\mathbb{E}}\Big[\int_0^T |\cdot|^p ds\Big]\right)^{\frac{1}{p}}$.
- $H_G^p(0,T)$ - completion of $M_G^{p,0}(0,T)$ w.r.t. $\|\cdot\|_{H_G^p} := \left(\hat{\mathbb{E}}\Big[\Big(\int_0^T |\cdot|^2 ds\Big)^{\frac{p}{2}}\Big]\right)^{\frac{1}{p}}$.
- $S_G^0(0,T) := \{\eta_s := h(s, B_{s_1 \wedge s}, \ldots, B_{s_n \wedge s}) : s_i \in [0,T],\ h \in C_{b,\,\mathrm{Lip}}(\mathbb{R}^{n+1})\}$.
- $S_G^p(0,T)$ - completion of $S_G^0(0,T)$ w.r.t. $\|\eta\|_{S_G^p} := \left(\hat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\eta|^p\Big]\right)^{\frac{1}{p}}$,

  for $\eta \in S_G^0(0,T)$, $p \geq 1$.

The stochastic integration under the G-expectation:

We define $\int_0^t \eta_s^n dB_s := \sum_{i=0}^{n-1} \xi_i^n (B_{t_{i+1}} - B_{t_i})$, for $\eta_t^n = \sum_{i=0}^{N_n - 1} \xi_i^n I_{(t_i, t_{i+1}]}(t) \in M_G^{2,0}(0,T)$, and for $\eta \in M_G^2(0,T)$ with $\|\eta^n - \eta\|_{M_G^2} \to 0$ $(n \to \infty)$, we define

$$\int_0^t \eta_s dB_s := L_G^2 - \lim_{n\to\infty} \int_0^t \eta_s^n dB_s,$$

where L_G^2 indicates the convergence in $L_G^2(\Omega)$:

$$\hat{\mathbb{E}}\Big[\Big|\int_0^t \eta_s^n dB_s - \int_0^t \eta_s dB_s\Big|^2\Big] \to 0,\ n \to \infty.$$

Similarly, we define $\int_0^t \xi_s d\langle B\rangle_s$ and $\int_0^t \xi_s ds$ for $\xi \in M_G^1(0,T)$, where $\langle B\rangle$ denotes the cross-variation process of B.

SDEs and BSDEs driven by G-Brownian motion:

For simplicity, we only consider the one-dimensional case $d = 1$, and so also the G-Brownian motion is supposed to be one-dimensional.

We consider the following G-SDE: For given $0 \leq t \leq T < \infty$,

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + h(s, X_s^{t,x})d\langle B\rangle_s + \sigma(s, X_s^{t,x})dB_s, \quad s \in [t, T], \\ X_t^{t,x} = x, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}$, and $b,\ h,\ \sigma : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions satisfying the following assumptions:

(H1) $b(\cdot, x),\ h(\cdot, x),\ \sigma(\cdot, x) \in M_G^p(0, T)$, $p \geq 2$, for all $x \in \mathbb{R}$;

(H2) There exists a constant $L > 0$ such that for all $x, x' \in \mathbb{R}$, $t \in [0, T]$,

$$|b(t, x) - b(t, x')| + |h(t, x) - h(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq L|x - x'|.$$

For simplicity, $X_s^{0,x}$ will be denoted by X_s^x, for $s \in [0, T]$, $x \in \mathbb{R}$.

Theorem 2.2. [Peng, 2010, Arxiv]

Assume that the conditions (H1) and (H2) hold. Then G-SDE (2.1) has a unique solution $(X_s^{t,x})_{s\in[t,T]} \in M_G^p(t,T)$. Moreover, there exists a constant $C \in \mathbb{R}$ depending on p, T, L and G such that, for all $x,\ y \in \mathbb{R},\ t,\ t' \in [0,T]$, we have

$$\text{i)}\quad \hat{\mathbb{E}}\Big[\sup_{s\in[0,t]} |X_s^x|^p\Big] \le C(1+|x|^p);$$

$$\text{ii)}\quad \hat{\mathbb{E}}[|X_t^x - X_{t'}^y|^p] \le C\left(|x-y|^p + (1+|x|^p)\left|t-t'\right|^{p/2}\right).$$

We also consider the following BSDE driven by a G-Brownian motion:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),\ 0 \leq t \leq T, \quad (2.2)$$

where the coefficient $f(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy the following conditions:

(H3) There exists some $\beta > 1$ such that, for all $y,\ z,\ f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H4) $|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|)$, $(t, \omega) \in [0, T] \times \Omega$, $y, z, y', z' \in \mathbb{R}$, for some constant $L > 0$.

For simplicity, we denote by $\mathfrak{S}_G^p(0, T)$ the collection of all processes (Y, Z, K) such that $Y \in S_G^p(0, T)$, $Z \in H_G^p(0, T)$, and K is a non increasing G-martingale with $K_0 = 0$ and $K_T \in L_G^p(\Omega_T)$.

Definition 2.6.[Hu, Ji, Peng, Song, 2014, SPA]

Let $\xi \in L_G^{\beta}(\Omega_T)$ and f satisfy (H3) and (H4) for $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of (2.2), if for some $1 < p \leq \beta$ the following properties hold:
(a) $(Y, Z, K) \in \mathfrak{S}_G^p(0, T)$;
(b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$, $t \in [0, T]$.

Theorem 2.3.[Hu, Ji, Peng, Song, 2014, SPA]

Assume that $\xi \in L_G^{\beta}(\Omega_T)$ and f satisfies $(\mathrm{H3})$ and $(\mathrm{H4})$ for $\beta > 1$. Then (2.2) has a unique solution (Y, Z, K).

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a sublinear expectation space, where $\Omega = C([0,T];\mathbb{R})$. Recall that, due to Theorem 2.1,

$$\mathcal{P} = \{P \text{ a probability on } (\Omega, \mathcal{B}(\Omega)) : E_P[X] \leq \hat{\mathbb{E}}[X], \text{ for } X \in \mathcal{H}\}$$

is a non empty convex, weakly compact subset of $\mathcal{P}(\mathbb{R})$ endowed with the topoplogy of weak convergence. Moreover,

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \text{ for all } \xi \in L_G^1(\Omega),$$

where the supremum is in fact a maximum: For all $\xi \in L_G^1(\Omega)$, there exists a $P \in \mathcal{P}$ such that $\hat{\mathbb{E}}[\xi] = E_P[\xi]$ (see [Peng, 2019, Springer]). Consequently, the set

$$\mathcal{P}_{\{\xi\}} := \{P \in \mathcal{P} : \hat{\mathbb{E}}[\xi] = E_P[\xi]\}$$

is nonempty.

Let $\xi, \eta \in L_G^1(\Omega)$ and put $F(\lambda) := \hat{\mathbb{E}}[\xi + \lambda\eta],\ \lambda \in \mathbb{R}$. Now we study the differentiability of F. From the definition of the G-expectation $\hat{\mathbb{E}}$ we know that F is convex. Consequently, for all $\lambda \in \mathbb{R}$, there exists the right-derivative of F at λ

$$F_+^{'}(\lambda) = \lim_{0<\varepsilon\downarrow 0} \frac{F(\lambda+\varepsilon) - F(\lambda)}{\varepsilon}$$

and also the corresponding left-derivative

$$F_-^{'}(\lambda) = \lim_{0>\varepsilon\uparrow 0} \frac{F(\lambda+\varepsilon) - F(\lambda)}{\varepsilon},$$

and, for all $\lambda < \lambda'$, we have

$$F_-^{'}(\lambda) \le F_+^{'}(\lambda) \le \frac{F(\lambda') - F(\lambda)}{\lambda' - \lambda} \le F_-^{'}(\lambda').$$

Let us compute $F_+^{'}(0)$ with avoiding the G-martingale representation.

Lemma 3.1.

Let $\xi, \eta \in L_G^1(\Omega)$ and $0 < \varepsilon_l \downarrow 0$ $(l \to \infty)$, and let $P_l \in \mathcal{P}_{\{\xi+\varepsilon_l \eta\}}$, $l \geq 1$. Then we have

i) There exists a subsequence of (P_l), denoted by (P_{l_k}), and $P \in \mathcal{P}$, such that $P_{l_k} \rightharpoonup P$ (weak convergence of probability measures);
ii) If $P_l \rightharpoonup P$, as $\varepsilon_l \downarrow 0$ $(l \to \infty)$, for some $P \in \mathcal{P}$, then $P \in \mathcal{P}_{\{\xi\}}$.

We recall that the set $\mathcal{P}$ endowed with the weak convergence of probability measures is a compact metrisable space. Let $d(\cdot,\cdot)$ be a metric on $\mathcal{P}$ which is compatible with the weak convergence, e.g., we can choose the Lévy-Prokhorov metric (see Theorem 11.3-3, [Dudley, 2002]):

$$d(P,Q) := \sup\Big\{ \int_\Omega f d(P-Q),\ |f|_{BL} \le 1 \Big\},$$

where $|f|_{BL} = \sup\limits_{\omega\in\Omega} |f(\omega)| + \sup\limits_{\omega\neq\omega'} \dfrac{|f(\omega)-f(\omega')|}{|\omega-\omega'|_{C([0,T])}}$.

As $(\mathcal{P},d)$ is a compact metric space, it is, in particular, also separable.

For $A, B \subset \mathcal{P}$, we put

$$d(P, B) := \mathrm{dist}_B(P) = \inf\{d(P, Q) : Q \in B\}, \text{ for } P \in \mathcal{P},$$

and

$$\Gamma(A, B) := \sup_{P \in A} d(P, B).$$

Remark 3.1.

$\Gamma(A, B)$ is the maximal distance from B of the probabilities in A. In particular, $\Gamma(A, B) = 0$, if $A \subset B$. Of course, $\Gamma(\cdot, \cdot)$ is not symmetric, its symmetrisation is just the Hausdorff distance

$$d_H(A, B) = \max\{\Gamma(A, B), \Gamma(B, A)\}, \ A, B \subset \mathcal{P}.$$

Proposition 3.1.

We have $\Gamma\left(\mathcal{P}_{\{\xi+\varepsilon\eta\}}, \mathcal{P}_{\{\xi\}}\right) \to 0$, as $\varepsilon \downarrow 0$.

Let us now come to the computation of the right-derivative $F'_+(0)$ of $F(\lambda) = \hat{\mathbb{E}}[\xi + \lambda\eta]$ ($\xi, \eta \in L^1_G(\Omega)$) at $\lambda = 0$. For this we let $0 < \varepsilon_l \downarrow 0$, $P_l \in \mathcal{P}_{\{\xi+\varepsilon_l\eta\}}$ and $P \in \mathcal{P}_{\{\xi\}}$ be such that $P_l \rightharpoonup P$ (Due to Lemma 3.1 this choice is possible). Then, thanks to the domination of $(P_l)_{l\geq 1}$ by $\hat{\mathbb{E}}[\cdot]$, for all $\zeta \in L^1_G(\Omega)$, $E_{P_l}[\zeta] \to E_P[\zeta]$, as $l \to \infty$. Then, as $P_l \in \mathcal{P}_{\{\xi+\varepsilon_l\eta\}}$,

$$F'_+(0) = \lim_{\varepsilon\downarrow 0} \frac{\hat{\mathbb{E}}[\xi + \varepsilon\eta] - \hat{\mathbb{E}}[\xi]}{\varepsilon} \leq \lim_{l\to\infty} \frac{E_{P_l}[\xi + \varepsilon_l\eta] - E_{P_l}[\xi]}{\varepsilon_l} = E_P[\eta],$$

i.e., $F'_+(0) \leq E_P[\eta]$. On the other hand, for all $Q \in \mathcal{P}_{\{\xi\}}$,

$$F'_+(0) = \lim_{\varepsilon\downarrow 0} \frac{\hat{\mathbb{E}}[\xi + \varepsilon\eta] - \hat{\mathbb{E}}[\xi]}{\varepsilon} \geq \lim_{\varepsilon\downarrow 0} \frac{E_Q[\xi + \varepsilon\eta] - E_Q[\xi]}{\varepsilon} = E_Q[\eta].$$

Lemma 3.2.

For $\xi, \eta \in L_G^1(\Omega)$ and $F(\lambda) := \hat{\mathbb{E}}[\xi + \lambda\eta]$, we have

$$F_+^{'}(0) = \max_{P \in \mathcal{P}_{\{\xi\}}} E_P[\eta] = \hat{\mathbb{E}}_{\{\xi\}}[\eta],$$

where $\hat{\mathbb{E}}_{\{\xi\}}[\eta] := \sup_{P \in \mathcal{P}_{\{\xi\}}} E_P[\eta]$, $\eta \in L_G^1(\Omega)$, is a new sublinear expectation, and $\hat{\mathbb{E}}_{\{\xi\}}$ is dominated by $\hat{\mathbb{E}}$, i.e., $\hat{\mathbb{E}}_{\{\xi\}}[\,\cdot\,] \leq \hat{\mathbb{E}}[\,\cdot\,]$.

Remark 3.2.

From the above lemma it follows that

$$F'_-(0)=\lim_{0<\varepsilon\downarrow 0}\frac{\hat{\mathbb{E}}[\xi-\varepsilon\eta]-\hat{\mathbb{E}}[\xi]}{-\varepsilon}=-\lim_{0<\varepsilon\downarrow 0}\frac{\hat{\mathbb{E}}[\xi+\varepsilon(-\eta)]-\hat{\mathbb{E}}[\xi]}{\varepsilon}=-\hat{\mathbb{E}}_{\{\xi\}}[-\eta].$$

This shows in particular that $F(\lambda)=\hat{\mathbb{E}}[\xi+\lambda\eta]$ is differentiable at $\lambda=0$ if and only if $\hat{\mathbb{E}}_{\{\xi\}}[\eta]=-\hat{\mathbb{E}}_{\{\xi\}}[-\eta]$.
We also observe that, for all $\lambda\in\mathbb{R}$,

$$F'_+(\lambda)=\lim_{0<\varepsilon\downarrow 0}\frac{\hat{\mathbb{E}}[(\xi+\lambda\eta)+\varepsilon\eta]-\hat{\mathbb{E}}[\xi+\lambda\eta]}{\varepsilon}=\hat{\mathbb{E}}_{\{\xi+\lambda\eta\}}[\eta],$$

$$F'_-(\lambda)=-\lim_{0<\varepsilon\downarrow 0}\frac{\hat{\mathbb{E}}[(\xi+\lambda\eta)+\varepsilon(-\eta)]-\hat{\mathbb{E}}[\xi+\lambda\eta]}{\varepsilon}=-\hat{\mathbb{E}}_{\{\xi+\lambda\eta\}}[-\eta].$$

Corollary 3.1.

Let $\varphi \in C^1(\mathbb{R})$ with a bounded Lipschitz derivative $\partial\varphi : \mathbb{R} \to \mathbb{R}$, and let $\xi, \eta \in L_G^1(\Omega)$. Then, for $H(\lambda) := \hat{\mathbb{E}}[\varphi(\xi + \lambda\eta)]$, $\lambda \in \mathbb{R}$, we have

$$\text{i) } H_+^{'}(0) = \hat{\mathbb{E}}_{\{\varphi(\xi)\}}[\partial\varphi(\xi)\eta]; \qquad \text{ii) } H_-^{'}(0) = -\hat{\mathbb{E}}_{\{\varphi(\xi)\}}[-\partial\varphi(\xi)\eta].$$

The control state space U: a non-empty, closed and convex bounded subset of $\mathbb{R}^d$. A process $u : [0,T] \times \Omega \to U$ is said to be an admissible control, if $u \in \mathcal{U} := M_G^2(0,T;U)$. For $u \in \mathcal{U}$:

The dynamics of the controlled stochastic system:

$$\begin{cases} dX_t^u = \sigma(X_t^u, \hat{\mathbb{E}}[\varphi_1(X_t^u)], u_t)dB_t + b(X_t^u, \hat{\mathbb{E}}[\varphi_2(X_t^u)], u_t)dt \\ \qquad\qquad + \beta(X_t^u, \hat{\mathbb{E}}[\varphi_3(X_t^u)], u_t)d\langle B\rangle_t, \ t \in [0,T], \\ X_0^u = x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $b, \ \beta : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$, and $\varphi_1, \ \varphi_2, \ \varphi_3 : \mathbb{R} \longrightarrow \mathbb{R}$.

The cost functional:

$$J(u) := \hat{\mathbb{E}}[\Phi(X_T^u, \hat{\mathbb{E}}[\varphi_4(X_T^u)]) + \int_0^T l(t, X_t^u, \hat{\mathbb{E}}[\varphi_5(X_t^u)], u_t)dt], \tag{4.2}$$

where $\Phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $l : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$, and $\varphi_4, \ \varphi_5 : \mathbb{R} \longrightarrow \mathbb{R}$.

Control problem: A control $\hat{u} \in \mathcal{U}$ satisfying

$$J(\hat{u}) = \inf_{u \in \mathcal{U}} J(u)$$

is said to be optimal.

Objective: Characterise the optimal control with the help of Pontryagin's stochastic maximum principle.

We shall make the following standard assumptions.

Assumption

(**A.1**) The functions $\varphi_i, 1 \leq i \leq 5$, are continuously differentiable, Φ and l are continuously differentiable w.r.t. (x, y), and $b,\ \sigma,\ \beta$ are continuously differentiable w.r.t. (x, y, v).

(**A.2**) All the derivatives in (**A.1**) are Lipschitz continuous and bounded.

Remark 4.1.

For given $u(\cdot) \in \mathcal{U}$, X^u is called a solution of the above mean-field G-SDE if $X^u \in M_G^2(0, T; \mathbb{R}^n)$ satisfies (4.1). Under the above assumptions, due to Theorem 2.2, G-SDE (2.1) has a unique solution.

To simplify the dynamics and the related computations, we put $\beta = 0$, and so SDE (4.1) becomes

$$\begin{cases} dx_t^u = b(x_t^u, \hat{\mathbb{E}}[\varphi_1(x_t^u)], u_t)dt + \sigma(x_t^u, \hat{\mathbb{E}}[\varphi_2(x_t^u)], u_t)dB_t, \ t \in [0, T], \\ x_0^u = x \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

The cost functional is still given by (4.2),

$$J(u) := \hat{\mathbb{E}}[\Phi(x_T^u, \hat{\mathbb{E}}[\varphi_4(x_T^u)]) + \int_0^T l(t, x_t^u, \hat{\mathbb{E}}[\varphi_5(x_t^u)], u_t)dt]. \tag{4.4}$$

We suppose that there exists an optimal control $\hat{u} \in \mathcal{U}$, that is, $J(\hat{u}) \leq J(u')$, for all $u' \in \mathcal{U}$. Let us denote $\hat{x}_t := x_t^{\hat{u}}$, $t \in [0, T]$.

Let $u \in \mathcal{U}$ be an arbitrary admissible control process.

Convex variational method.By x_t^θ we denote the state process defined by SDE (4.3) with the control process u_t^θ defined as convex perturbation of $\hat{u}_t$:

$$u_t^\theta = \hat{u}_t + \theta(u_t - \hat{u}_t),\ t \in [0, T],\ \theta \in [0, 1].$$

We put $v_t = u_t - \hat{u}_t$, $t \in [0,T]$, and introduce the following notations:

$$
\begin{array}{ll}
\hat{b}(t) = b(t, \hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_1(t)], \hat{u}_t), & \hat{b}_x(t) = b_x(t, \hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_1(t)], \hat{u}_t), \\
\hat{\sigma}(t) = \sigma(t, \hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_2(t)], \hat{u}_t), & \hat{\sigma}_x(t) = \sigma_x(t, \hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_2(t)], \hat{u}_t), \\
\hat{\Phi}(t) = \Phi(\hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_4(t)]), & \hat{\Phi}_x(t) = \Phi_x(\hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_4(t)]), \\
\hat{l}(t) = l(t, \hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_5(t)], \hat{u}_t), & \hat{l}_x(t) = l_x(t, \hat{x}_t, \hat{\mathbb{E}}[\hat{\varphi}_5(t)], \hat{u}_t),
\end{array} \tag{4.5}
$$

with $\hat{\varphi}_i(t) = \varphi_i(\hat{x}_t)$, $\hat{\varphi}'_i(t) = \varphi'_i(\hat{x}_t), i = 1, 2, 4, 5$, and similarly are defined $\hat{b}_y(t)$, $\hat{b}_v(t)$, $\hat{\sigma}_y(t)$, $\hat{\sigma}_v(t)$, $\hat{l}_y(t)$ and $\hat{l}_v(t)$.

Lemma 4.1.

Let $z=(z_t)\in M_G^2(0,T;\mathbb{R})$ be the unique solution of the following SDE

$$\begin{cases} \mathrm{d}z_t = \Big(\hat{b}_x(t)z_t + \hat{b}_y(t)\hat{\mathbb{E}}_{\{\varphi_1(\hat{x}_t)\}}[\hat{\varphi}_1'(t)z_t] + \hat{b}_v(t)v_t\Big)\mathrm{d}t \\ \qquad\qquad + \Big(\hat{\sigma}_x(t)z_t + \hat{\sigma}_y(t)\hat{\mathbb{E}}_{\{\varphi_2(\hat{x}_t)\}}[\hat{\varphi}_2'(t)z_t] + \hat{\sigma}_v(t)v_t\Big)\mathrm{d}B_t,\ t\in[0,T], \\ z_0 = 0. \end{cases} \tag{4.6}$$

Then, it holds that

$$\lim_{\theta\to 0}\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\frac{x_t^\theta-\hat{x}_t}{\theta}-z_t|^2]=0.$$

Lemma 4.2.

The directional derivative of the cost functional J is given by

$$\lim_{\theta\downarrow 0}\frac{J(\hat{u}+\theta v)-J(\hat{u})}{\theta}$$
$$=\hat{\mathbb{E}}_{\{\psi(\hat{u})\}}\Big[\hat{\Phi}_x(T)z_T+\hat{\Phi}_y(T)\hat{\mathbb{E}}_{\{\varphi_4(\hat{x}_T)\}}[\varphi_4'(\hat{x}_T)z_T]$$
$$+\int_0^T\Big(\hat{l}_x(t)z_t+\hat{l}_y(t)\hat{\mathbb{E}}_{\{\varphi_5(\hat{x}_t)\}}[\varphi_5'(\hat{x}_t)z_t]+\hat{l}_v(t)v_t\Big)dt\Big],$$

where

$$\psi(\hat{u})=\hat{\Phi}(T)+\int_0^T\hat{l}(t)dt=\Phi(\hat{x}_T,\hat{\mathbb{E}}[\varphi_4(\hat{x}_T)])+\int_0^T l(t,\hat{x}_t,\hat{\mathbb{E}}[\varphi_5(\hat{x}_t)],\hat{u}_t)dt;$$

for the other abbreviating notations, see (4.5).

Now, we consider the special case where σ and b are independent of y, and we still put $\beta = 0$. More general cases can be studied with the same approach as that we develop here, but, of course, this is related with more involved computations. In the case we study here (4.1) becomes

$$\begin{cases} dx_t^u = \sigma(x_t^u, u_t)dB_t + b(x_t^u, u_t)dt, \; t \in [0, T], \\ x_0^u = x \in \mathbb{R}^n. \end{cases} \tag{4.7}$$

Concerning the cost functional, we make the following assumption
(A.3) $\hat{\Phi}_y(T) \geq 0$, $\hat{l}_y(t) \geq 0, \; t \in [0, T]$, quasi-surely.
Of course, this assumption is, in particular, satisfied, if the partial derivates $\partial_y \Phi(.,.)$ and $\partial_y l(.,.,.,.)$ are everywhere non negative.

Then from the optimality of $\hat{u}$, thanks to Lemma 4.2, we have

$$
\begin{aligned}
0 \leq \lim_{\theta \downarrow 0} \frac{J(\hat{u}+\theta v) - J(\hat{u})}{\theta}
&= \hat{\mathbb{E}}_{\{\psi(\hat{u})\}}\Big[\hat{\Phi}_x(T) z_T + \hat{\Phi}_y(T) \hat{\mathbb{E}}_{\{\varphi_4(\hat{x}_T)\}}[\varphi_4'(\hat{x}_T) z_T] \\
&\quad + \int_0^T \Big(\hat{l}_x(t) z_t + \hat{l}_y(t) \hat{\mathbb{E}}_{\{\varphi_5(\hat{x}_t)\}}[\varphi_5'(\hat{x}_t) z_t] + \hat{l}_v(t) v_t\Big) dt\Big] \\
&= \sup_{P^1 \in \mathcal{P}_{\{\psi(\hat{u})\}}} \Big\{ E_{P^1}\Big[\hat{\Phi}_x(T) z_T + \int_0^T \Big(\hat{l}_x(t) z_t + \hat{l}_v(t) v_t\Big) dt\Big] \\
&\quad + E_{P^1}[\hat{\Phi}_y(T)] \sup_{P^2 \in \mathcal{P}_{\{\varphi_4(\hat{x}_T)\}}} E_{P^2}[\varphi_4'(\hat{x}_T) z_T] \\
&\quad + \int_0^T E_{P^1}[\hat{l}_y(t)] \sup_{P^3 \in \mathcal{P}_{\{\varphi_5(\hat{x}_t)\}}} E_{P^3}[\varphi_5'(\hat{x}_t) z_t] dt \Big\}.
\end{aligned} \tag{4.8}
$$

A measurable selection theorem.

Let $\xi = (\xi_t)$ and $\eta = (\eta_t)$ be in $M_G^2(0,T;\mathbb{R})$ such that the following assumptions are satisfied:

(B.1) $\hat{\mathbb{E}}\big[|\xi_t - \xi_s|^2 + |\eta_t - \eta_s|^2\big] \le C|t-s|,\ t,s \in [0,T],\ C \in \mathbb{R}.$

Remark 4.2.

Recall from Theorem 2.2 that, for $\phi, \psi : \mathbb{R} \to \mathbb{R}$ Lipschitz functions, the processes $\xi = (\xi_t = \phi(\hat{x}_t))$ and $\eta = (\eta_t = \psi(\hat{x}_t))$ satisfy assumption (B.1). We also observe that, for all $\xi = (\xi_t),\ \eta = (\eta_t) \in M_G^2(0,T;\mathbb{R})$ satisfying (B.1), the function $t \mapsto \hat{\mathbb{E}}_{\{\xi_t\}}[\eta_t]$ is Borel measurable.

Indeed, from (B.1) it follows that, for all $\varepsilon > 0$, the function $t \mapsto \hat{\mathbb{E}}[\xi_t + \varepsilon\eta_t] - \hat{\mathbb{E}}[\xi_t]$ is continuous and, hence, Borel measurable. Consequently, Lemma 3.2 shows that also

$$t \mapsto \hat{\mathbb{E}}_{\{\xi_t\}}[\eta_t] = \lim_{0<\varepsilon\downarrow 0} \frac{1}{\varepsilon}\big(\hat{\mathbb{E}}[\xi_t + \varepsilon\eta_t] - \hat{\mathbb{E}}[\xi_t]\big), \quad t \in [0,T]. \tag{4.9}$$

is a Borel function.

Theorem 4.1.

Assume that $\xi = (\xi_t),\ \eta = (\eta_t) \in M_G^2(0,T;\mathbb{R})$ satisfy (B.1). Then the mapping

$$[0,T] \ni t \mapsto \mathcal{P}_{\{\xi_t|\eta_t\}} := \left\{ R \in \mathcal{P}_{\{\xi_t\}} \ :\ E_R[\eta_t] = \hat{\mathbb{E}}_{\{\xi_t\}}[\eta_t] \right\} \subset \mathcal{P} \quad (4.10)$$

is a weakly measurable set-valued function with non empty values which are compact subsets of $(\mathcal{P}, d)$ (Recall that d is the Lévy-Prokhorov metric on $\mathcal{P}$).

Remark 4.3.

Recall that, if $(X, \mathcal{G})$ is a measurable space and Y a topological space, a set-valued function $G : X \ni x \mapsto G(x) \subset Y$ for which the values $G(x)$ are non empty, closed subsets of Y, is called *weakly measurable* if, for all open subset $\mathcal{O}$ of Y, it holds $\{x \in X\ :\ G(x) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{G}$.

Theorem 4.2.

Assume that $\xi = (\xi_t),\ \eta = (\eta_t) \in M_G^2(0,T;\mathbb{R})$ satisfy (B.1). Then the mapping $[0,T] \ni t \mapsto \mathcal{P}_{\{\xi_t|\eta_t\}} \subset \mathcal{P}$ admits a $\mathcal{B}([0,T]) - \mathcal{B}(\mathcal{P})$- measurable selection ($\mathcal{B}([0,T])$ and $\mathcal{B}(\mathcal{P})$ is the Borel σ-field over $[0,T]$ and $(\mathcal{P}, d)$, respectively), i.e., there is a selection $R_t \in \mathcal{P}_{\{\xi_t|\eta_t\}}$, $t \in [0,T]$, such that the mapping $t \mapsto R_t$ is $\mathcal{B}([0,T]) - \mathcal{B}(\mathcal{P})$- measurable.

The proof of this theorem is an immediate consequence of Theorem 4.1 and the Kuratowski and Ryll-Nardzewski measurable selection theorem (cf. [Kuratowski, Ryll-Nardzewski, 1965]).

To come back to our studies we define now

$$\mathcal{R}_{\{\varphi_5(\hat{x})\}} := \left\{R = (R_t) : [0,T] \to \mathcal{P} \text{ Borel meas.} : R_t \in \mathcal{P}_{\{\varphi_5(\hat{x}_t)\}},\ \forall t\right\},$$
$$\mathcal{R}_{\{\varphi_5(\hat{x})|\varphi_5'(\hat{x})z\}} := \left\{R \in \mathcal{R}_{\{\varphi_5(\hat{x})\}} : R_t \in \mathcal{P}_{\{\varphi_5(\hat{x}_t)|\varphi_5'(\hat{x}_t)z_t\}}, \forall t\right\}, \text{ where}$$
$$\mathcal{P}_{\{\varphi_5(\hat{x}_t)|\varphi_5'(\hat{x}_t)z_t\}} := \left\{R_t \in \mathcal{P}_{\{\varphi_5(\hat{x}_t)\}} : E_R[\varphi_5'(\hat{x}_t)z_t] = \hat{\mathbb{E}}_{\{\varphi_5(\hat{x}_t)\}}[\varphi_5'(\hat{x}_t)z_t]\right\}.$$

From Theorem 4.2 we know that $\mathcal{R}_{\{\varphi_5(\hat{x})|\varphi_5'(\hat{x})z\}} \neq \emptyset$, and so $\mathcal{R}_{\{\varphi_5(\hat{x})\}} \supset \mathcal{R}_{\{\varphi_5(\hat{x})|\varphi_5'(\hat{x})z\}} \neq \emptyset$.
Moreover, we observe that, for all $R = (R_t) \in \mathcal{R}_{\{\varphi_5(\hat{x})\}}$,

$$\int_0^T E_{P^1}[\hat{l}_y(t)] E_{R_t}[\varphi_5'(\hat{x}_t) z_t] dt \leq \int_0^T E_{P^1}[\hat{l}_y(t)] \sup_{P^3 \in \mathcal{P}_{\{\varphi_5(\hat{x}_t)\}}} E_{P^3}[\varphi_5'(\hat{x}_t) z_t] dt, \tag{4.11}$$

and, if $R = (R_t) \in \mathcal{R}_{\{\varphi_5(\hat{x})|\varphi_5'(\hat{x})z\}}$, we have equality. Consequently,

$$\begin{aligned} \sup_{R \in \mathcal{R}_{\{\varphi_5(\hat{x})\}}} & \int_0^T E_{P^1}[\hat{l}_y(t)] E_{R_t}[\varphi_5'(\hat{x}_t) z_t] dt \\ &= \int_0^T E_{P^1}[\hat{l}_y(t)] \sup_{P^3 \in \mathcal{P}_{\{\varphi_5(\hat{x}_t)\}}} E_{P^3}[\varphi_5'(\hat{x}_t) z_t] dt, \end{aligned}$$

and since $E_{P^1}[\hat{\Phi}_y(T)] \geq 0$ and $E_{P^1}[\hat{l}_y(t)] \geq 0$, $t \in [0,T]$, using the notation

$$\mathcal{P}\{\hat{u}\} := \mathcal{P}_{\{\psi(\hat{u})\}} \times \mathcal{P}_{\{\varphi_4(\hat{x}_T)\}} \times \mathcal{R}_{\{\varphi_5(\hat{x})\}},$$

(Observe that this set does not depend on the perturbing control $u = (u_t)$) we obtain from (4.8)

$$\begin{aligned} 0 &\leq \lim_{\theta \downarrow 0} \frac{J(\hat{u}+\theta v) - J(\hat{u})}{\theta} \\ &= \sup_{(P,Q,R) \in \mathcal{P}_{\{\hat{u}\}}} \Big\{ E_P\Big[\hat{\Phi}_x(T) z_T + \int_0^T \Big(\hat{l}_x(t) z_t + \hat{l}_v(t) v_t\Big) dt\Big] \\ &\qquad + E_P[\hat{\Phi}_y(T)] E_Q[\varphi_4'(\hat{x}_T) z_T] + \int_0^T E_P[\hat{l}_y(t)] E_{R_t}[\varphi_5'(\hat{x}_t) z_t] dt \Big\}. \end{aligned} \tag{4.12}$$

Relation (4.12) brings us to introduce the following (classical) adjoint BSDEs (under linear expectation):

1) Under $P \in \mathcal{P}_{\{\psi(\hat{u})\}}$,

$$\begin{cases} dp_s(P) = -\big(\hat{b}_x(s)p_s(P) + \hat{l}_x(s)\big)ds - \hat{\sigma}_x(s)q_s(P)d\langle B\rangle_s + q_s(P)dB_s + dN_s(P), \\ p_T(P) = \hat{\Phi}_x(T), \quad s \in [0,T], \\ \qquad N(P) \in \mathcal{M}_P^{2,\perp}(0,T) \text{ with } N_0(P) = 0; \end{cases} \tag{4.13}$$

2) Under $Q \in \mathcal{P}_{\{\varphi_4(\hat{x}_T)\}}$,

$$\begin{cases} d\tilde{p}_s(Q) = -\hat{b}_x(s)\tilde{p}_s(Q)ds - \hat{\sigma}_x(s)\tilde{q}_s(Q)d\langle B\rangle_s + \tilde{q}_s(Q)dB_s + d\tilde{N}_s(Q), \\ \tilde{p}_T(Q) = \varphi_4'(\hat{x}_T), \quad s \in [0,T], \\ \qquad \tilde{N}(Q) \in \mathcal{M}_Q^{2,\perp}(0,T) \text{ with } \tilde{N}_0(Q) = 0; \end{cases} \tag{4.14}$$

3) Under R_t, $t \in [0,T)$, for $R = (R_t) \in \mathcal{R}_{\{\varphi_5(\hat{x})\}}$,

$$\begin{cases} dp_s(t,R_t) = -\hat{b}_x(s)p_s(t,R_t)ds - \hat{\sigma}_x(s)q_s(t,R_t)d\langle B\rangle_s + q_s(t,R_t)dB_s + dN_s(t,R_t) \\ p_t(t,R_t) = \varphi_5'(\hat{x}_t), \quad s \in [0,t], \\ \qquad N(t,R_t) \in \mathcal{M}_R^{2,\perp}(0,t) \text{ with } N_0(t,R_t) = 0. \end{cases} \tag{4.15}$$

Remark 4.4.

1) For the above BSDEs we consider $(\Omega, \mathcal{B}(\Omega))$ endowed with the filtration $\mathbb{F}^B = (\mathcal{F}_s)$ generated by the G-BM B (=coordinate process on Ω). For a given probability measure P over $(\Omega, \mathcal{B}(\Omega))$ the associated filtration is the one augmented by all P-null sets: $\mathbb{F}^P = \mathbb{F}^B \vee \mathcal{N}_P$.

2) Note: Under any $P \in \mathcal{P}$, the G-BM B is only a continuous square integrable martingale, and so the martingale representation may not hold for $(B, \mathbb{F}^P)$. So it is necessary to introduce the second square integrable P-martingale N with $N_0 = 0$ and joint quadratic variation $\langle B, N\rangle^P \big(= (\langle B, N\rangle_s^P)\big) = 0$ (We write $N \in \mathcal{M}_P^{2,\perp}(0,T)$).

(Continued)

3) Recall that $\langle B\rangle$ is the quadratic variation process of the G-BM B under $\hat{\mathbb{E}}$: For all $\pi_t^N=\{0=t_0^N<t_1^N<\cdots<t_N^N=t\},\ N\geq 1$, sequence of partitions of $[0,t]$ with mesh $|\pi_t^N|=\max\limits_{0\leq j\leq N-1}(t_{j+1}^N-t_j^N)\to 0\ (N\to\infty)$,

$$\hat{\mathbb{E}}\left[\left|\sum_{j=0}^{N-1}(B_{t_{j+1}^N}-B_{t_j^N})^2-\langle B\rangle_t\right|^2\right]\to 0\ (N\to\infty).$$

And so, for all $P\in\mathcal{P}$, $\langle B\rangle$ coincides P-a.s. with the quadratic variation process $\langle B\rangle^P$ of B as P-martingale, $\langle B\rangle_t^P=\langle B\rangle_t,\ t\in[0,T],\ P\text{-}a.s.$ Also recall that, under the G-expectation the increments of $\langle B\rangle$ are independent and stationary, and $\underline{\sigma}^2ds\leq d\langle B\rangle_s\leq\overline{\sigma}^2ds$, ds-a.e., quasi-surely.

Following El Karoui and Huang [1997] and Buckdahn et al. [2010], we see for all $P \in \mathcal{P}$ that there exists a unique triplet of processes

$$(p(P), q(P), N(P)) \in M_P^2(0,T) \times M_P^2(0,T) \times \mathcal{M}_P^{2,\perp}(0,T)$$

which solves the adjoint equation (4.13) and (4.14) (equation (4.14) with Q instead of P), respectively. The same we also have for the BSDEs (4.15), only that here this BSDE is considered over the time interval $[0,t]$, so that the unique solution triplet $(p(t,R_t), q(t,R_t), N(t,R_t))$ belongs to $M_R^2(0,t) \times M_R^2(0,t) \times \mathcal{M}_R^{2,\perp}(0,t)$, $t \in [0,T)$.

Moreover, standard BSDE estimates show that, for all $p \geq 1$, there is some constant $C_p \in \mathbb{R}_+$ (independent of the underlying probability measure $P \in \mathcal{P}$) s.t.

$$E_P\big[\sup_{s\in[0,T]} |p_s(P)|^p + \big(\int_0^T |q_s(P)|^2 d\langle B\rangle_s + \langle N(P)\rangle_T\big)^{p/2}\big] \leq C_p. \quad (4.16)$$

Similar estimates we have for the solution $(p(t,R_t), q(t,R_t), N(t,R_t))$ of BSDE (4.15), for all $t \in [0,T]$, only that unlike in (4.16), here T has to be replaced by t.

Applying now Itô's formula to $p_s(P)z_s$, we have

$$p_T(P)z_T = \int_0^T \big(p_s(P)\hat{b}_v(s)v_s - \hat{l}_x(s)z_s\big)ds + \int_0^T \zeta_s(P)dB_s + \int_0^T q_s(P)\hat{\sigma}_v(s)v_s d\langle B\rangle_s + \int_0^T z_s dN_s(P), \tag{4.17}$$

where $\zeta_s(P) := p_s(P)(\hat{\sigma}_x(s)z_s + \hat{\sigma}_v(s)v_s) + z_s q_s(P)$, and $\int_0^{\cdot} \zeta_s(P)dB_s$ and $\int_0^{\cdot} z_s dN_s(P)$ are P-martingales.

Thus, recallig that $p_T(P) = \hat{\Phi}_x(T)$, we have

$$\begin{aligned} &E_P\big[\hat{\Phi}_x(T)z_T + \int_0^T \big(\hat{l}_x(s)z_s + \hat{l}_v(s)v_s\big)ds\big] \\ &= E_P\Big[\int_0^T v_s\Big(\big(p_s(P)\hat{b}_v(s) + \hat{l}_v(s)\big)ds + q_s(P)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big]. \end{aligned} \tag{4.18}$$

An analogous argument using now the solution of BSDE (4.14) yields, for $Q \in \mathcal{P}$,

$$\begin{aligned}&E_Q[\varphi_4'(\hat{x}_T)z_T] = E_Q[\tilde{p}_T(Q)z_T]\\&= E_Q\Big[\int_0^T v_s\Big(\tilde{p}_s(Q)\hat{b}_v(s)ds + \tilde{q}_s(Q)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big].\end{aligned} \tag{4.19}$$

Finally, making use in the same way of the solution $(p(t,R_t), q(t,R_t), N(t,R_t)) \in M_R^2(0,t) \times M_R^2(0,t) \times \mathcal{M}_R^{2,\perp}(0,t)$ of BSDE (4.15), we obtain, for $t \in [0,T]$,

$$\begin{aligned}&E_{R_t}[\varphi_5'(\hat{x}_t)z_t] = E_{R_t}[p_t(t,R_t)z_t]\\&= E_{R_t}\Big[\int_0^t v_s\Big(p_s(t,R_t)\hat{b}_v(s)ds + q_s(t,R_t)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big].\end{aligned} \tag{4.20}$$

Let us introduce now

$$\begin{aligned}
& \Theta[P,Q,R](v) \\
= \;& E_P\big[\hat{\Phi}_x(T)z_T + \int_0^T \big(\hat{l}_x(s)z_s + \hat{l}_v(s)v_s\big)ds\big] \\
& + E_P\big[\hat{\Phi}_y(T)\big]E_Q\big[\varphi_4'(\hat{x}_T)z_T\big] + \int_0^T E_P\big[\hat{l}_y(t)\big]E_{R_t}\big[\varphi_5'(\hat{x}_t)z_t\big],
\end{aligned} \tag{4.21}$$

and from the above computation we see that

$$\begin{aligned}
& \Theta[P,Q,R](v) \\
= \;& E_P\Big[\int_0^T v_s\Big(\big(p_s(P)\hat{b}_v(s) + \hat{l}_v(s)\big)ds + q_s(P)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big] \\
& + E_P\Big[\hat{\Phi}_y(T)\Big]E_Q\Big[\int_0^T v_s\Big(\tilde{p}_s(Q)\hat{b}_v(s)ds + \tilde{q}_s(Q)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big] \\
& + \int_0^T E_P\Big[\hat{l}_y(t)\Big]E_{R_t}\Big[\int_0^t v_s\Big(p_s(t,R_t)\hat{b}_v(s)ds + q_s(t,R_t)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big]dt.
\end{aligned} \tag{4.22}$$

Suppose $p_s(t,R_t) := 0$, $q_s(t,R_t) := 0$, for $t < s \le T$, and define the probability measure $\widetilde{R} := \int_0^T \frac{1}{T}dt \cdot \big(\delta_t \otimes R_t\big)$ over the space $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F})$. Then, with $(t,\omega)\mapsto p_s(t,R_t)(\omega), q_s(t,R_t)(\omega)$ and $t\mapsto E_P\Big[\hat{l}_y(t)\Big]$ interpreted as random variables over $[0,T]\times\Omega$, we have

$$\begin{aligned} & \int_0^T E_P\Big[\hat{l}_y(t)\Big]E_{R_t}\Big[\int_0^t v_s\Big(p_s(t,R_t)\hat{b}_v(s)ds + q_s(t,R_t)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big]dt \\ = \;& TE_{\widetilde{R}}\Big[\int_0^T v_s\Big(E_P[\hat{l}_y]\Big(p_s(\cdot,R_\cdot)\hat{b}_v(s)ds + q_s(\cdot,R_\cdot)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)\Big)\Big]. \end{aligned} \tag{4.23}$$

Let us define $\Omega_{\{T\}} := [0,T]\times\Omega$ and embed the probabilites P and Q in the space of probabilities over $(\Omega_{\{T\}}, \mathcal{B}([0,T])\otimes\mathcal{F})$ in a canonical way by making the identification $P := \delta_T\otimes P$ and $Q := \delta_T\otimes Q$. Then, thanks to (4.22) and (4.23),

$$
\begin{aligned}
&\Theta[P,Q,R](v)\\
=&\int_{\Omega_{\{T\}}}\int_0^T v_s\Big\{\Big(\big(p_s(P)\hat{b}_v(s)+\hat{l}_v(s)\big)ds+q_s(P)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)dP\\
&\qquad\qquad +E_P[\hat{\Phi}_y(T)]\Big(\tilde{p}_s(Q)\hat{b}_v(s)ds+\tilde{q}_s(Q)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)dQ\\
&\qquad\qquad +TE_P[\hat{l}_y(\cdot)]\Big(p_s(\cdot,R_\cdot)\hat{b}_v(s)ds+q_s(\cdot,R_\cdot)\hat{\sigma}_v(s)d\langle B\rangle_s\Big)d\tilde{R}\Big\}\\
=&\int_{\Omega_{\{T\}}}\int_0^T v_s\{\hat{b}_v(s)dsdp_s(P,Q,R)+\hat{l}_v(s)dsdP+\hat{\sigma}_v(s)d\langle B\rangle_s dq_s(P,Q,R)\},
\end{aligned}
\tag{4.24}
$$

where

$$
\begin{aligned}
dp_s(P,Q,R):&=p_s(P)dP+E_P[\hat{\Phi}_y(T)]\tilde{p}_s(Q)dQ+TE_p[\hat{l}_y(\cdot)]p_s(\cdot,R_\cdot)d\tilde{R}\\
dq_s(P,Q,R):&=q_s(P)dP+E_P[\hat{\Phi}_y(T)]\tilde{q}_s(Q)dQ+TE_P[\hat{l}_y(\cdot)]q_s(\cdot,R_\cdot)d\tilde{R}.
\end{aligned}
$$

We remark that $dsdp_s(P,Q,R)$ and $d\langle B\rangle_s dq_s(P,Q,R)$ are signed measures on $\Omega_{\{T\}} \times [0,T]$ not depending on v and so neither on the perturbing control u. Then, from (4.12), (4.22) and (4.24), and with the Hamiltonian measure

$$dH_v(s,P,Q,R){:=}\hat{b}_v(s)dsdp_s(P,Q,R)+\hat{l}_v(s)dsdP+\hat{\sigma}_v(s)d\langle B\rangle_s dq_s(P,Q,R)$$

we have, for all $u \in \mathcal{U}$ (Recalling that $v = u - \hat{u}$) that

$$\begin{aligned} 0 &\leq \sup_{(P,Q,R)\in\mathcal{P}\{\hat{u}\}} \Theta[P,Q,R](u-\hat{u}) \\ &= \sup_{(P,Q,R)\in\mathcal{P}\{\hat{u}\}} \int_{\Omega_{\{T\}}} \int_0^T (u_s - \hat{u}_s) dH_v(s,P,Q,R). \end{aligned} \tag{4.25}$$

Observe that (4.25) gives a necessary condition for the optimality of the control $\hat{u} \in \mathcal{U}$. We resume our main result:

Theorem 4.3. SMP

Suppose $(\mathrm{A}.1)$-$(\mathrm{A}.3)$ where b, σ are independent of y, and let $\hat{u}$ be an optimal control with state trajectory $\hat{x} = (\hat{x}_t)$. Then (4.25) gives a necessary optimality condition satisfied by all $u \in \mathcal{U}$.

In the particular case when $\hat{l}_y(t) = 0$, quasi-surely, dt-a.s., and $\hat{\Phi}_y(T)$ is deterministic, by using an argument by Hu and Ji based on Sion's minimax theorem, we can simplify the necessary optimality condition (4.25). Indeed, let us suppose

Assumption

(A.3') $l(t,x,y,u) = l(t,x,u)$, $\Phi(x,y) = \Phi_1(x) + \Phi_2(y)$, $(t,x,y,u) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times U$.

We observe that under Assumption (A.3') $\hat{l}_y(t) = 0$ everywhere on $[0,T] \times \Omega$ and $\hat{\Phi}_y(T) = (\Phi_2)_y(\hat{\mathbb{E}}[\varphi_4(T)])$ is deterministic.

Theorem 4.4.

Suppose $(\mathrm{A}.1)$, $(\mathrm{A}.2)$ and $(\mathrm{A}.3')$ where b are σ do not depend on y, and let $\hat{u}$ be an optimal control with the associated state trajectory $\hat{x} = (\hat{x}_t)$. Then there exists $(P^*, Q^*) \in \mathcal{P}_{\{\psi(\hat{u})\}} \times \mathcal{P}_{\{\varphi_4(\hat{x}_T)\}}$ such that, all $u \in \mathcal{U}$,

$$0 \leq (u_s - \hat{u}_s)\Big\{\hat{b}_v(s)ds dp_s(P^*, Q^*) + \hat{l}_v(s)ds dP + \hat{\sigma}_v(s)d\langle B\rangle_s dq_s(P^*, Q^*)\Big\}. \tag{4.26}$$

where

$$\begin{aligned} p_s(P^*, Q^*) : \quad &= \quad p_s(P^*)dP^* + \hat{\Phi}_y(T)\tilde{p}_s(Q^*)dQ^*, \\ dq_s(P^*, Q^*) : \quad &= \quad q_s(P^*)dP^* + \hat{\Phi}_y(T)\tilde{q}_s(Q^*)dQ^*. \end{aligned}$$

Sufficient conditions for optimality:

We define the Hamiltonian random field

$$dH(t,x,u,p,q) := H_1(x,u,p)dt + H_2(x,u,q)d\langle B\rangle_t,$$

with $H_1(x,u,p) := b(x,u)p$ and $H_2(x,u,q) := \sigma(x,u)q$, we and make the following additional assumptions:

Assumptions

(A.4) The function Φ is convex in (x,y); the functions φ_4 and φ_5 are convex; the $dH(t,x,u,p,q)$ is convex in (x,u) (defined by the convexity of $H_1(\cdot,\cdot,p)$ and that of $H_2(\cdot,\cdot,q)$).

(A.5) $\Phi_y(x,y) := \partial_y\Phi(x,y) \geq 0,\ l_y(t,x,y,u) \geq 0,\ x,y \in \mathbb{R},\ t \in [0,T],\ u \in U.$

Theorem 4.5.

Assume the conditions $(\mathrm{A}.1)$-$(\mathrm{A}.2)$ and $(\mathrm{A}.4)$-$(\mathrm{A}.5)$ are satisfied and let $\hat{u} \in \mathcal{U}$ be a control process with associated state process $\hat{x} = (\hat{x}_t)$, and let $(p(P), q(P), N(P))$, $(\tilde{p}(Q), \tilde{q}(Q), \tilde{N}(Q))$ and $(p(t, R_t), q(t, R_t), N(t, R_t)$, $t \in [0, T]$, $(P, Q, R) \in \mathcal{P}\{\hat{u}\}$, be the solutions of (4.13), (4.14) and (4.15), respectively. If (4.25) holds for all $u \in \mathcal{U}$, then $\hat{u}$ is an optimal control.

Example 4.1.

We consider the following linear-quadratic control problem. The state equation is given by

$$\begin{cases} dx_t^u = (Ax_t^u + Bu_t)dt + (Cx_t^u + Du_t)dB_t, \\ x(0) = x \in \mathbb{R}, \end{cases}$$

where $u \in \mathcal{U}$ and A, B, C, D are constants. We associate the cost functional

$$J(u) = \frac{1}{2}\hat{\mathbb{E}}\left[\int_0^T ((x_t^u)^2 + u_t^2)dt + (x_T^u)^2 + \hat{\mathbb{E}}[(x_T^u)^2]\right], \ u \in \mathcal{U}.$$

The stochastic optimal control problem consists in minimizing the cost functional over $\mathcal{U}$.

The adjoint BSDEs (4.13) and (4.14) take the form

$$\begin{aligned} dp_s(P) &= (-Ap_s(P)_s + \hat{x}_s)ds - Cq_s(P)d\langle B\rangle_s + q_s(P)dB_s + dN_s(P), \\ d\tilde{p}_s(Q) &= -A\tilde{p}_s(Q)_s ds - C\tilde{q}_s(Q)d\langle B\rangle_s + \tilde{q}_s(Q)dB_s + d\tilde{N}_s(Q), \\ p_T(P) &= \hat{x}_T,\ \tilde{p}_T(Q) = \hat{x}_T, \end{aligned}$$

Note: $(p(t,R_t), q(t,R_t), N(t,R_t)) = 0$, for all $t \in [0,T]$, since the running cost l only depends on (x^u, u). So, putting

$\psi(\hat{u}) := \frac{1}{2}\Big(\int_0^T ((\hat{x}_t)^2 + \hat{u}_t^2)dt + (\hat{x}_T)^2 + \hat{\mathbb{E}}[(\hat{x}_T)^2]\Big)$, from Theorem 4.4:

Optimality condition

There exists $(P^*, Q^*) \in \mathcal{P}_{\{\psi(\hat{u})\}} \times \mathcal{P}_{\{(\hat{x}_T)^2\}}$ s.t., for all $u \in \mathcal{U}$,

$$\begin{aligned} 0 &\le (u_s - \hat{u}_s)\Big(Bdsdp_s(P^*, Q^*) + \hat{u}_s dsdP^* + Dd\langle B\rangle_s dq_s(P^*, Q^*)\Big) \\ &= (u_s - \hat{u}_s)\Big(Bds(p_s(P^*)dP^* + \tilde{p}_s(Q)dQ^*) + \hat{u}_s dsdP^* \\ &\qquad + Dd\langle B\rangle_s(q_s(P^*)dP^* + \tilde{q}_s(Q^*)dQ^*\Big). \end{aligned} \tag{4.27}$$

On the other hand, we see that our example also satisfies the assumptions (A.4)-(A.5).

Lemma 4.3.

For our linear-quadratic control problem of Example 4.1 the condition (4.27) is a necessary but also sufficient optimality condition for an admissible control $\hat{u}$.

Thank you very much
for your attention!