BSDEs driven by G-Brownian motion under degenerate case and its application to the regularity of fully nonlinear PDEs

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G-expectation

- Let $T > 0$ be given and let $\Omega_T = C_0([0,T];\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]$ with $\omega_0 = 0$.
- Canonical process $B_t(\omega) := \omega_t$, for $\omega \in \Omega_T$ and $t \in [0,T]$.
- $$\begin{aligned} Lip(\Omega_t) := \quad & \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) : \\ & N \geq 1, t_1 < \cdots < t_N \leq t, \varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})\}, \end{aligned}$$

  where $C_{b.Lip}(\mathbb{R}^{d\times N})$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^{d\times N}$.

G-expectation

Let $G : \mathbb{S}_d \to \mathbb{R}$ be a given monotonic and sublinear function, where $\mathbb{S}_d$ denotes the set of $d \times d$ symmetric matrices. Then there exists a unique bounded, convex and closed set $\Sigma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Sigma} \operatorname{tr}[A\gamma] \text{ for } A \in \mathbb{S}_d,$$

where $\mathbb{S}_d^+$ denotes the set of $d \times d$ nonnegative matrices. If there exists a $\underline{\sigma}^2 > 0$ such that $\gamma \geq \underline{\sigma}^2 I_d$ for any $\gamma \in \Sigma$, G is called non-degenerate. Otherwise, G is called degenerate.

- If $d = 1$, then $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ for $a \in \mathbb{R}$. G is degenerate iff $\underline{\sigma}^2 = 0$.

G-expectation

Peng (2004-2008) constructed the G-expectation $\hat{\mathbb{E}} : Lip(\Omega_T) \to \mathbb{R}$ and the conditional G-expectation $\hat{\mathbb{E}}_t : Lip(\Omega_T) \to Lip(\Omega_t)$ as follows:

- For each $s_1 \leq s_2 \leq T$ and $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, define $\hat{\mathbb{E}}[\varphi(B_{s_2} - B_{s_1})] = u(s_2 - s_1, 0)$, where u is the viscosity solution of the following G-heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \;\; u(0,x) = \varphi(x).$$

- For each $X = \varphi_N(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) \in Lip(\Omega_T)$, define

$$\hat{\mathbb{E}}_{t_i}[X] = \varphi_i(B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}) \text{ and } \hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[\varphi_1(B_{t_1})],$$

where

$$\varphi_i(x_1, \ldots, x_i) := \hat{\mathbb{E}}[\varphi_{i+1}(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i})].$$

G-expectation

- G-expectation space $(\Omega_T, Lip(\Omega_T), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t\in[0,T]})$ is a consistent sublinear expectation space, $(B_t)_{t\in[0,T]}$ is called the G-Brownian motion under $\hat{\mathbb{E}}$.
- $L_G^p(\Omega_t)$ denotes the completion of $Lip(\Omega_t)$ under the norm $||X||_{L_G^p} := (\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $p \geq 1$. It is clear that $\hat{\mathbb{E}}_t$ can be continuously extended to $L_G^1(\Omega_T)$ under the norm $||\cdot||_{L_G^1}$.

- S. Peng, Filtration consistent nonlinear expectations and evaluations of contingent claims, Acta Math. Appl. Sin., 20(2) (2004), 1-24.
- S. Peng, Nonlinear expectations and nonlinear Markov chains, Chin. Ann. Math., 26B(2) (2005), 159-184.
- S. Peng, G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type, Stochastic analysis and applications, Abel Symp., Vol. 2, Springer, Berlin, 2007, 541-567.
- S. Peng, Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Process. Appl., 118 (2008), 2223-2253.
- S. Peng, Nonlinear Expectations and Stochastic Calculus under Uncertainty, Springer (2019).

Representation theorem of G-expectation

Theorem (Denis-Hu-Peng (2011), Hu-Peng (2009))

There exists a unique weakly compact and convex set of probability measures $\mathcal{P}$ on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for all } X \in L_G^1(\Omega_T),$$

where $\mathcal{B}(\Omega_T) = \sigma(B_s : s \leq T)$.

- L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, Potential Anal., 34 (2011), 139-161.
- M. Hu, S. Peng, On representation theorem of G-expectations and paths of G-Brownian motion, Acta Math. Appl. Sin. Engl. Ser., 25 (2009), 539-546.

Representation theorem of G-expectation

For this $\mathcal{P}$, define

$$\mathbb{L}^p(\Omega_t) := \left\{ X \in \mathcal{B}(\Omega_t) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\} \text{ for } p \geq 1.$$

It is easy to check that $L_G^p(\Omega_t) \subset \mathbb{L}^p(\Omega_t)$. For each $X \in \mathbb{L}^1(\Omega_T)$,

$$\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X]$$

is still called the G-expectation. The capacity associated to $\mathcal{P}$ is defined by

$$c(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for } A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is polar if $c(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. We do not distinguish two random variables X and Y if $X = Y$ q.s.

Doob's inequality for G-martingale

Theorem (Soner-Touzi-Zhang (2011), Song (2011))

Let $1 \leq p < p'$ *and* $\xi \in L_G^{p'}(\Omega_T)$. *Then*

$$\left(\hat{\mathbb{E}}\left[\sup_{t\leq T}\left(\hat{\mathbb{E}}_t[|\xi|]\right)^p\right]\right)^{1/p} \leq \left(\hat{\mathbb{E}}\left[\sup_{t\leq T}\hat{\mathbb{E}}_t[|\xi|^p]\right]\right)^{1/p} \leq C\left(\hat{\mathbb{E}}[|\xi|^{p'}]\right)^{1/p'},$$

where

$$C = \left(1 + \frac{p}{p'-p}\right)^{1/p}.$$

- H. M. Soner, N. Touzi, J. Zhang, Martingale Representation Theorem under G-expectation, Stochastic Process. Appl., 121 (2011), 265-287.
- Y. Song, Some properties on G-evaluation and its applications to G-martingale decomposition, Sci. China Math., 54(2) (2011), 287-300.

Non-degenerate G-BSDE

Hu-Ji-Peng-Song (2014) studied the following BSDE driven by non-degenerate G-Brownian motion (G-BSDE)

$$\begin{aligned} Y_t = \quad & \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned}$$

We proved that the above G-BSDE has a unique solution (Y, Z, K), where K is a non-increasing G-martingale with $K_0 = 0$.

Soner-Touzi-Zhang (2012) studied a new type of fully nonlinear BSDE, called 2BSDE, by different formulation and method.

- M. Hu, S. Ji, S. Peng, Y. Song, Backward stochastic differential equations driven by G-Brownian motion, Stochastic Process. Appl., 124 (2014), 759-784.
- M. Hu, S. Ji, S. Peng, Y. Song, Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion, Stochastic Process. Appl., 124 (2014), 1170-1195.
- H. M. Soner, N. Touzi, J. Zhang, Wellposedness of Second Order Backward SDEs, Probab. Theory Related Fields, 153 (2012), 149-190.

Degenerate G-BSDE

For simplicity, we consider the following degenerate G-BSDE:

$$\begin{aligned} Y_t = \; & \xi + \int_t^T f(s, Y_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned}$$

where B is a 1-dimensional G-Brownian motion, $G(a) := \frac{1}{2}\bar{\sigma}^2 a^+$ for $a \in \mathbb{R}$ with $\bar{\sigma} > 0$.

- $0 \leq \langle B\rangle_{t+s} - \langle B\rangle_t \leq \bar{\sigma}^2 s$
- $\left(\hat{\mathbb{E}}\left[\left(\int_0^T |Z_t|^2 d\langle B\rangle_t\right)^{p/2}\right]\right)^{1/p}$

Notations

- $M^0(0,T) := \left\{ \eta_t = \sum_{k=0}^{N-1} \xi_k I_{[t_k, t_{k+1})}(t) : \xi_k \in Lip(\Omega_{t_k}) \right\}$;
- $||\eta||_{M_G^{p,\bar{p}}(0,T)} := \left(\hat{\mathbb{E}} \left[\left(\int_0^T |\eta_t|^p dt \right)^{\bar{p}/p} \right] \right)^{1/\bar{p}}$, p, $\bar{p} \geq 1$;
- $||\eta||_{H_G^{p,\bar{p}}(0,T;\langle B \rangle)} := \left(\hat{\mathbb{E}} \left[\left(\int_0^T |\eta_t|^p d\langle B \rangle_t \right)^{\bar{p}/p} \right] \right)^{1/\bar{p}}$;
- $M_G^{p,\bar{p}}(0,T)$ completion of $M^0(0,T)$ under the norm $|| \cdot ||_{M_G^{p,\bar{p}}(0,T)}$;
- $H_G^{p,\bar{p}}(0,T;\langle B \rangle)$ completion of $M^0(0,T)$ under the norm $|| \cdot ||_{H_G^{p,\bar{p}}(0,T;\langle B \rangle)}$;
- $M_G^p(0,T) := M_G^{p,p}(0,T)$, $H_G^p(0,T;\langle B \rangle) := H_G^{p,p}(0,T;\langle B \rangle)$;
- $S^0(0,T) := \{ (h(t, B_{t_1 \wedge t}, \ldots, B_{t_N \wedge t}))_{t \in [0,T]} : h \in C_{b.Lip}(\mathbb{R}^{N+1}) \}$;
- $||\eta||_{S_G^p(0,T)} := \left(\hat{\mathbb{E}} \left[\sup_{t \leq T} |\eta_t|^p \right] \right)^{1/p}$, $p \geq 1$;
- $S_G^p(0,T)$ completion of $S^0(0,T)$ under the norm $|| \cdot ||_{S_G^p(0,T)}$.

Assumptions

(H1) There exists a $\bar{p} > 1$ such that $\xi \in L_G^{\bar{p}}(\Omega_T)$, $f(\cdot, y) \in M_G^{1,\bar{p}}(0,T)$ and $g(\cdot, y, z) \in H_G^{1,\bar{p}}(0,T;\langle B\rangle)$ for any y, $z \in \mathbb{R}$;

(H2) There exists a constant $L > 0$ such that, for any $(t,\omega) \in [0,T]\times\ \Omega_T$, (y,z), $(\bar{y},\bar{z}) \in \mathbb{R} \times \mathbb{R}$,

$$
\begin{aligned}
&|f(t,\omega,y) - f(t,\omega,\bar{y})| + |g(t,\omega,y,z) - g(t,\omega,\bar{y},\bar{z})| \\
&\leq L(|y-\bar{y}| + |z-\bar{z}|).
\end{aligned}
$$

L^p-solution

We give the following L^p-solution of G-BSDE for $p \in (1, \bar{p})$.

Definition

(Y, Z, K) is called an L^p-solution of G-BSDE if the following properties hold:
(i) $Y \in S_G^p(0,T)$, $Z \in H_G^{2,p}(0,T;\langle B \rangle)$, K is a non-increasing G-martingale with $K_0 = 0$ and $K_T \in L_G^p(\Omega_T)$;
(ii)

$$\begin{aligned} Y_t = \quad & \xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t), \ t \leq T. \end{aligned}$$

d-dimensional G-Brownian motion

Let $B_t = (B_t^1, \ldots, B_t^d)^T$ be a d-dimensional G-Brownian motion satisfying

$$G(A) = G'(A') + \frac{1}{2}\sum_{i=d'+1}^{d} \bar{\sigma}_i^2 a_i^+,$$

where $d' < d$, $A' \in \mathbb{S}_{d'}$, $a_i \in \mathbb{R}$ for $d' < i \le d$,

$$A = \begin{pmatrix} A' & \cdots & \cdots & \cdots \\ \cdots & a_{d'+1} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & a_d \end{pmatrix} \in \mathbb{S}_d,$$

$G' : \mathbb{S}_{d'} \to \mathbb{R}$ is non-degenerate, $\bar{\sigma}_i > 0$ for $i = d'+1, \ldots, d$.

$$\begin{aligned} Y_t = \; & \xi + \int_t^T f(s, Y_s, Z_s')ds + \textstyle\sum_{i,j=1}^{d'} \int_t^T g_{ij}(s, Y_s, Z_s')d\langle B^i, B^j\rangle_s \\ & + \textstyle\sum_{l=d'+1}^{d} \int_t^T g_l(s, Y_s, Z_s', Z_s^l)d\langle B^l\rangle_s \\ & - \textstyle\sum_{k=1}^{d} \int_t^T Z_s^k dB_s^k - (K_T - K_t). \end{aligned}$$

Proposition

Suppose that ξ_i, f_i and g_i satisfy (H1) and (H2) for $i = 1, 2$. Let (Y^i, Z^i, K^i) be the L^p-solution of G-BSDE corresponding to ξ_i, f_i and g_i for some $p \in (1, \bar{p})$. Then there exists a positive constant C depending on p, $\bar{\sigma}$, L and T satisfying

$$|\hat{Y}_t|^p \leq C\hat{\mathbb{E}}_t \left[|\hat{\xi}|^p + \left(\int_t^T |\hat{f}_s| ds \right)^p + \left(\int_t^T |\hat{g}_s| d\langle B \rangle_s \right)^p \right],$$

$$|Y_t^i|^p \leq C\hat{\mathbb{E}}_t \left[|\xi_i|^p + \left(\int_t^T |f_i(s, 0)| ds \right)^p + \left(\int_t^T |g_i(s, 0, 0)| d\langle B \rangle_s \right)^p \right],$$

$$\hat{\mathbb{E}} \left[\left(\int_0^T |Z_s^i|^2 d\langle B \rangle_s \right)^{p/2} \right] + \hat{\mathbb{E}} \left[|K_T^i|^p \right] \leq C\Lambda_i,$$

Prior estimates of G-BSDE

Proposition

$$\hat{\mathbb{E}}\left[\left(\int_0^T |\hat{Z}_s|^2 d\langle B\rangle_s\right)^{p/2}\right]$$
$$\leq C\left\{\hat{\mathbb{E}}\left[\sup_{t\leq T}|\hat{Y}_t|^p\right] + (\Lambda_1+\Lambda_2)^{1/2}\left(\hat{\mathbb{E}}\left[\sup_{t\leq T}|\hat{Y}_t|^p\right]\right)^{1/2}\right\},$$

where

$$\begin{aligned}\Lambda_i = \quad & \hat{\mathbb{E}}\left[\sup_{t\leq T}|Y_t^i|^p\right] + \hat{\mathbb{E}}\left[\left(\int_0^T |f_i(s,0)|ds\right)^p\right] \\ & +\hat{\mathbb{E}}\left[\left(\int_0^T |g_i(s,0,0)|d\langle B\rangle_s\right)^p\right],\end{aligned}$$

$$\hat{Y}_t = Y_t^1 - Y_t^2,\ \hat{\xi} = \xi_1 - \xi_2,\ \hat{f}_s = f_1(s, Y_s^2) - f_2(s, Y_s^2),$$

$$\hat{g}_s = g_1(s, Y_s^2, Z_s^2) - g_2(s, Y_s^2, Z_s^2),\ \hat{Z}_t = Z_t^1 - Z_t^2.$$

Solution in the extended $\tilde{G}$-expectation space

Following Hu-Ji-Peng-Song (2014), the key point to obtain the solution of G-BSDE is to study the following type of G-BSDE:

$$Y_t = \varphi(B_T) + \int_t^T h(Y_s, Z_s) d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $h \in C_0^\infty(\mathbb{R}^2)$.

Set $\tilde{\Omega}_T = C_0([0,T];\mathbb{R}^2)$ and the canonical process is denoted by $(B, \tilde{B})$.

For each a_{11}, a_{12}, $a_{22} \in \mathbb{R}$, define

$$\tilde{G}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\right) = G(a_{11}) + \frac{1}{2}a_{22},$$

we have $\langle B, \tilde{B}\rangle_t = 0$ and $\langle \tilde{B}\rangle_t = t$. The $\tilde{G}$-expectation is denoted by $\tilde{\mathbb{E}}$, and the related spaces are denoted by

$$Lip(\tilde{\Omega}_t),\ L^p_{\tilde{G}}(\tilde{\Omega}_t),\ \tilde{M}^0(0,T),\ M^{p,\bar{p}}_{\tilde{G}}(0,T),\ H^{p,\bar{p}}_{\tilde{G}}(0,T;\langle B\rangle),\ S^p_{\tilde{G}}(0,T).$$

Solution in the extended $\tilde{G}$-expectation space

Lemma

Let $\varphi \in C_0^\infty(\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R}^2)$. Then, for each given $p > 1$, G-BSDE has a unique L^p-solution (Y, Z, K) in the extended $\tilde{G}$-expectation space such that $Y \in S_G^p(0, T)$, $Z \in H_{\tilde{G}}^{2,p}(0, T; \langle B \rangle)$ and $K_T \in L_{\tilde{G}}^p(\tilde{\Omega}_T)$.

Solution in the extended $\tilde{G}$-expectation space

Key point of proof. For each fixed $\varepsilon \in (0, \bar{\sigma})$, define

$$B_t^\varepsilon = B_t + \varepsilon \tilde{B}_t \text{ for } t \in [0, T].$$

Then $(B_t^\varepsilon)_{t\in[0,T]}$ is the G_ε-Brownian motion under $\tilde{\mathbb{E}}$, where

$$G_\varepsilon(a) = \frac{1}{2}[(\bar{\sigma}^2 + \varepsilon^2)a^+ - \varepsilon^2 a^-] \text{ for } a \in \mathbb{R}.$$

Let u_ε be the viscosity solution of the following PDE

$$\partial_t u + G_\varepsilon(\partial_{xx}^2 u + 2h(u, \partial_x u)) = 0, \ u(T, x) = \varphi(x).$$

By Krylov's regularity estimate, there exists a constant $\alpha \in (0, 1)$ such that $u_\varepsilon \in C^{1+\alpha/2, 2+\alpha}([0, T - \delta] \times \mathbb{R})$ for any $\delta > 0$.

Solution in the extended $\tilde{G}$-expectation space

Applying Itô's formula to $u_\varepsilon(t, B_t^\varepsilon)$, we obtain

$$Y_t^\varepsilon = \varphi(B_T^\varepsilon) + \int_t^T h(Y_s^\varepsilon, Z_s^\varepsilon) d\langle B^\varepsilon \rangle_s - \int_t^T Z_s^\varepsilon dB_s^\varepsilon - (K_T^\varepsilon - K_t^\varepsilon),$$

where $Y_t^\varepsilon = u_\varepsilon(t, B_t^\varepsilon)$, $Z_t^\varepsilon = \partial_x u_\varepsilon(t, B_t^\varepsilon)$ and

$$\begin{aligned} K_t^\varepsilon = & \int_0^t \frac{1}{2} \left[\partial_{xx}^2 u_\varepsilon(s, B_s^\varepsilon) + 2h(Y_s^\varepsilon, Z_s^\varepsilon) \right] d\langle B^\varepsilon \rangle_s \\ & - \int_0^t G_\varepsilon \left(\partial_{xx}^2 u_\varepsilon(s, B_s^\varepsilon) + 2h(Y_s^\varepsilon, Z_s^\varepsilon) \right) ds. \end{aligned}$$

Taking $\varepsilon \to 0$, we can prove the result.

Estimates of partial derivatives of u_ε

Let $\hat{\mathbb{E}}^\varepsilon$ be the G_ε-expectation on $(\Omega_T, Lip(\Omega_T))$. The canonical process $(B_t)_{t\in[0,T]}$ is the 1-dimensional G_ε-Brownian motion under $\hat{\mathbb{E}}^\varepsilon$. For each given $(t,x) \in [0,T) \times \mathbb{R}$, denote

$$B_s^{t,x} = x + B_s - B_t \text{ for } s \in [t,T].$$

Applying Itô's formula to $u_\varepsilon(s, B_s^{t,x})$ under $\hat{\mathbb{E}}^\varepsilon$, we obtain that the following G_ε-BSDE

$$Y_s^{t,x} = \varphi(B_T^{t,x}) + \int_s^T h(Y_r^{t,x}, Z_r^{t,x}) d\langle B\rangle_r - \int_s^T Z_r^{t,x} dB_r - (K_T^{t,x} - K_s^{t,x})$$

has a unique solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ satisfying $Y_s^{t,x} = u_\varepsilon(s, B_s^{t,x})$, $Z_s^{t,x} = \partial_x u_\varepsilon(t, B_s^{t,x})$ and $K_t^{t,x} = 0$.

Estimates of partial derivatives of u_ε

Let $\mathcal{P}^\varepsilon$ be a weakly compact and convex set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that

$$\hat{\mathbb{E}}^\varepsilon[X] = \sup_{P \in \mathcal{P}^\varepsilon} E_P[X] \text{ for all } X \in L^1_{G_\varepsilon}(\Omega_T).$$

For each given $(t, x) \in [0, T) \times \mathbb{R}$, denote

$$\mathcal{P}^\varepsilon_{t,x} = \{P \in \mathcal{P}^\varepsilon : E_P[K_T^{t,x}] = 0\}.$$

Estimates of partial derivatives of u_ε

The following estimates for G_ε-BSDE are useful.

Proposition

Suppose $\varphi \in C_0^\infty(\mathbb{R})$ *and* $h \in C_0^\infty(\mathbb{R}^2)$. *For* $(t, x, \Delta) \in [0, T) \times \mathbb{R}\times\mathbb{R}$, *let* $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ *and* $(Y_s^{t,x+\Delta}, Z_s^{t,x+\Delta}, K_s^{t,x+\Delta})_{s\in[t,T]}$ *be two solutions of* G_ε*-BSDE. Then, for* $p > 1$, $P \in \mathcal{P}_{t,x}^\varepsilon$ *and* $P^\Delta \in \mathcal{P}_{t,x+\Delta}^\varepsilon$,

$$\sup_{s\in[t,T]} \left|Y_s^{t,x+\Delta} - Y_s^{t,x}\right|^p \leq C|\Delta|^p,$$

$$E_P\left[\left(\int_t^T \left|Z_s^{t,x+\Delta} - Z_s^{t,x}\right|^2 d\langle B\rangle_s\right)^{p/2} + \left|K_T^{t,x+\Delta}\right|^p\right] \leq C|\Delta|^p,$$

$$E_{P^\Delta}\left[\left(\int_t^T \left|Z_s^{t,x+\Delta} - Z_s^{t,x}\right|^2 d\langle B\rangle_s\right)^{p/2} + \left|K_T^{t,x}\right|^p\right] \leq C|\Delta|^p,$$

where the constant $C > 0$ *depends on* p, $\bar{\sigma}$, φ, h *and* T.

In the following theorem, we obtain the formula of $\partial_x u_\varepsilon$ based on $u_\varepsilon(t,x) = Y_t^{t,x}$.

Theorem

Suppose that $\varphi \in C_0^\infty(\mathbb{R})$ *and* $h \in C_0^\infty(\mathbb{R}^2)$. *Then, for each* $(t,x) \in [0,T) \times \mathbb{R}$, *we have*

$$\partial_x u_\varepsilon(t,x) = E_P\left[\Gamma_T^{t,x}\varphi'(B_T^{t,x})\right] \text{ for any } P \in \mathcal{P}_{t,x}^\varepsilon,$$

where $(\Gamma_s^{t,x})_{s\in[t,T]}$ *is the solution of the following* G*-SDE:*

$$d\Gamma_s^{t,x} = h_y'(Y_s^{t,x}, Z_s^{t,x})\Gamma_s^{t,x} d\langle B\rangle_s + h_z'(Y_s^{t,x}, Z_s^{t,x})\Gamma_s^{t,x} dB_s,\ \Gamma_t^{t,x} = 1.$$

Estimates of partial derivatives of u_ε

Now we give the estimate for $\partial^2_{xx} u_\varepsilon$.

Theorem

Suppose that $\varphi \in C_0^\infty(\mathbb{R})$ *and* $h \in C_0^\infty(\mathbb{R}^2)$. *Then*

$$\partial^2_{xx} u_\varepsilon(t,x) \geq -C \text{ for } (t,x) \in [0,T) \times \mathbb{R},$$

where the constant $C > 0$ *depends on* $\bar{\sigma}$, φ, h *and* T.

Remark

The constant C *in the above theorem is independent of* $\varepsilon \in (0, \bar{\sigma})$.

Key point of proof. Set $\hat{Y}^\Delta = Y^{t,x+\Delta} - Y^{t,x}$ and $\hat{Z}^\Delta = Z^{t,x+\Delta} - Z^{t,x}$. For any given $P \in \mathcal{P}^\varepsilon_{t,x}$, we obtain

Estimates of partial derivatives of u_ε

$$\hat{Y}_t^{\Delta} = E_P\left[\hat{Y}_T^{\Delta}\Gamma_T^{t,x} + \int_t^T \Gamma_r^{t,x} I_r^{\Delta} d\langle B\rangle_r - \int_t^T \Gamma_r^{t,x} dK_r^{t,x+\Delta}\right].$$

Since $\Gamma_r^{t,x} > 0$ and $dK_r^{t,x+\Delta} \leq 0$, we get

$$\hat{Y}_t^{\Delta} \geq E_P\left[\hat{Y}_T^{\Delta}\Gamma_T^{t,x} + \int_t^T \Gamma_r^{t,x} I_r^{\Delta} d\langle B\rangle_r\right].$$

Noting that $|\hat{Y}_T^{\Delta} - \varphi'(B_T^{t,x})\Delta| \leq C\Delta^2$ and $|I_r^{\Delta}| \leq C(|\hat{Y}_r^{\Delta}|^2 + |\hat{Z}_r^{\Delta}|^2)$, we obtain

$$\hat{Y}_t^{\Delta} \geq E_P\left[\Gamma_T^{t,x}\varphi'(B_T^{t,x})\right]\Delta - C\Delta^2.$$

Similarly, for any given $P^{\Delta} \in \mathcal{P}_{t,x+\Delta}^{\varepsilon}$, we can get

$$\hat{Y}_t^{\Delta} \leq E_{P^{\Delta}}\left[\Gamma_T^{t,x+\Delta}\varphi'(B_T^{t,x+\Delta})\right]\Delta + C\Delta^2,$$

where $C > 0$ depends on $\bar{\sigma}$, φ, h and T.

Existence and uniqueness

Lemma

Let $\varphi \in C_0^\infty(\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R}^2)$. Then, for each given $p > 1$, G-BSDE has a unique L^p-solution (Y, Z, K) in the G-expectation space.

Key point of proof. (Y, Z, K) is the L^p-solution in the extended $\tilde{G}$-expectation space

$$Y_t = \varphi(B_T) + \int_t^T h(Y_s, Z_s) d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

Applying Itô's formula to $u_\varepsilon(t, B_t)$, we get

$$\begin{aligned} \tilde{Y}_t^\varepsilon = \quad & \varphi(B_T) + \int_t^T h(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) d\langle B\rangle_s \\ & - \int_t^T \tfrac{1}{2}\varepsilon^2 \left(\partial_{xx}^2 u_\varepsilon(s, B_s) + 2h(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon)\right)^- ds \\ & - \int_t^T \tilde{Z}_s^\varepsilon dB_s - (L_T^\varepsilon - L_t^\varepsilon), \end{aligned}$$

Existence and uniqueness

where $\tilde{Y}_t^\varepsilon = u_\varepsilon(t, B_t)$, $\tilde{Z}_t^\varepsilon = \partial_x u_\varepsilon(t, B_t)$ and

$$\begin{aligned} L_t^\varepsilon = \quad & \int_0^t \tfrac{1}{2} \left[\partial_{xx}^2 u_\varepsilon(s, B_s) + 2h(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \right] d\langle B \rangle_s \\ & - \int_0^t G\left(\partial_{xx}^2 u_\varepsilon(s, B_s) + 2h(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \right) ds \\ & - \int_0^t \tfrac{1}{2}\varepsilon^2 \left(\partial_{xx}^2 u_\varepsilon(s, B_s) + 2h(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \right)^+ ds. \end{aligned}$$

L^ε is non-increasing with $L_0^\varepsilon = 0$ under $\tilde{\mathbb{E}}$ and

$$\left(\partial_{xx}^2 u_\varepsilon(s, B_s) + 2h(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) \right)^- \leq C \text{ for } \varepsilon \in (0, \bar{\sigma}).$$

Applying Itô's formula to $|\tilde{Y}_t^\varepsilon - Y_t|^2$ on $[0, T]$, we obtain

$$\lim_{\varepsilon \downarrow 0} \tilde{\mathbb{E}} \left[\left(\int_0^T |\tilde{Z}_t^\varepsilon - Z_t|^2 d\langle B \rangle_t \right)^{p/2} \right] = 0.$$

Theorem

Suppose that ξ, f and g satisfy (H1) and (H2). Then G-BSDE has a unique L^p-solution (Y, Z, K) for each given $p \in (1, \bar{p})$.

Existence and uniqueness

The following example shows that f can not contain z in G-BSDE.

Example

Let B be a 1-dimensional G-Brownian motion with $G(a) := \frac{1}{2}\bar{\sigma}^2 a^+$ for $a \in \mathbb{R}$. we can prove $(((\langle B\rangle_s)^{-1/5})_{s\in[0,T]} \in H_G^{2,p}(0,T;\langle B\rangle)$ for each $p > 1$, which implies $\int_0^T (\langle B\rangle_s)^{-1/5} dB_s \in L_G^p(\Omega_T)$ for each $p > 1$. Consider the following linear G-BSDE:

$$Y_t = \int_0^T (\langle B\rangle_s)^{-1/5} dB_s + \int_t^T Z_s ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

If the above G-BSDE has an L^p-solution (Y, Z, K), then we can deduce

$$Y_0 \geq \frac{5}{4} T^{4/5} \varepsilon^{-2/5} \text{ for each } \varepsilon > 0,$$

which contradicts to $Y_0 \in \mathbb{R}$. Thus, for each given $p > 1$, the above G-BSDE has no L^p-solution (Y, Z, K).

For simplicity, we only consider 1-dimensional G-Brownian motion with $G(a) = \frac{1}{2}\bar{\sigma}^2 a^+$. For each fixed $t \in [0, T]$ and $x \in \mathbb{R}$, consider the following G-FBSDE:

$$\begin{aligned} dX_s^{t,x} &= b(s, X_s^{t,x})ds + h(s, X_s^{t,x})d\langle B\rangle_s + \sigma(s, X_s^{t,x})dB_s, \\ dY_s^{t,x} &= -f(s, X_s^{t,x}, Y_s^{t,x})ds - g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})d\langle B\rangle_s \\ &\quad +Z_s^{t,x}dB_s + dK_s^{t,x}, \\ X_t^{t,x} &= x,\ Y_T^{t,x} = \varphi(X_T^{t,x}). \end{aligned}$$

We need the following assumptions:

(A1) b, h, σ, f, g are continuous in (s, x, y, z).

(A2) There exist a constant $L_1 > 0$ and a positive integer m such that for any $s \in [0, T]$, x, x', y, y', z, $z' \in \mathbb{R}$,

$$\begin{aligned}
&|b(s,x) - b(s,x')| + |h(s,x) - h(s,x')| \\
&+|\sigma(s,x) - \sigma(s,x')| \leq L_1|x - x'|, \\
&|\varphi(x) - \varphi(x')| \leq L_1(1 + |x|^m + |x'|^m)|x - x'|, \\
&|f(s,x,y) - f(s,x',y')| + |g(s,x,y,z) - g(s,x',y',z')| \\
&\leq L_1[(1 + |x|^m + |x'|^m)|x - x'| + |y - y'| + |z - z'|].
\end{aligned}$$

Define

$$u(t,x) = Y_t^{t,x} \text{ for } (t,x) \in [0,T] \times \mathbb{R}.$$

Proposition

Suppose that (A1) and (A2) hold. Then
(i) For each $(t,x) \in [0,T) \times \mathbb{R}$, *we have* $Y_s^{t,x} = u(s, X_s^{t,x})$ *for* $s \in [t,T]$.
(ii) $u(\cdot,\cdot)$ *is the unique viscosity solution of the following fully nonlinear PDE:*

$$\begin{cases} \partial_t u + G(\sigma^2(t,x)\partial_{xx}^2 u + 2h(t,x)\partial_x u + 2g(t,x,u,\sigma(t,x)\partial_x u)) \\ +b(t,x)\partial_x u + f(t,x,u) = 0, \\ u(T,x) = \varphi(x). \end{cases}$$

For each $(t,x) \in [0,T) \times \mathbb{R}$, set

$$\mathcal{P}_{t,x} = \{P \in \mathcal{P} : E_P[K_T^{t,x}] = 0\}.$$

In order to obtain $\partial_x u(t,x)$, we need the following assumption.

(A3) b'_x, h'_x, σ'_x, φ', f'_x, f'_y, g'_x, g'_y, g'_z are continuous in (s,x,y,z).

Notation: $g'_x(s) = g'_x(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$, similar for $g'_y(s)$, $g'_z(s)$, $f'_x(s)$ and $f'_y(s)$.

Regularity of PDEs

Theorem

Suppose that (A1)-(A3) hold. Then, for each $(t,x)\in[0,T)\times\mathbb{R}$, *we have*

$$\begin{aligned}\partial_{x+}u(t,x) = \ & \sup_{P\in\mathcal{P}_{t,x}} E_P\Big[\varphi'(X_T^{t,x})\hat{X}_T^{t,x}\Gamma_T^{t,x} + \int_t^T f_x'(s)\hat{X}_s^{t,x}\Gamma_s^{t,x}ds \\ & + \int_t^T g_x'(s)\hat{X}_s^{t,x}\Gamma_s^{t,x}d\langle B\rangle_s\Big],\end{aligned}$$

$$\begin{aligned}\partial_{x-}u(t,x) = \ & \inf_{P\in\mathcal{P}_{t,x}} E_P\Big[\varphi'(X_T^{t,x})\hat{X}_T^{t,x}\Gamma_T^{t,x} + \int_t^T f_x'(s)\hat{X}_s^{t,x}\Gamma_s^{t,x}ds \\ & + \int_t^T g_x'(s)\hat{X}_s^{t,x}\Gamma_s^{t,x}d\langle B\rangle_s\Big],\end{aligned}$$

where $(\hat{X}_s^{t,x})_{s\in[t,T]}$ *and* $(\Gamma_s^{t,x})_{s\in[t,T]}$ *satisfy the following* G*-SDEs:*

$$\begin{aligned}& d\hat{X}_s^{t,x} = b_x'(s,X_s^{t,x})\hat{X}_s^{t,x}ds + h_x'(s,X_s^{t,x})\hat{X}_s^{t,x}d\langle B\rangle_s + \sigma_x'(s,X_s^{t,x})\hat{X}_s^{t,x}dB_s, \\ & d\Gamma_s^{t,x} = f_y'(s)\Gamma_s^{t,x}ds + g_y'(s)\Gamma_s^{t,x}d\langle B\rangle_s + g_z'(s)\Gamma_s^{t,x}dB_s, \\ & \hat{X}_t^{t,x} = 1,\ \Gamma_t^{t,x} = 1.\end{aligned}$$

Regularity of PDEs

In order to obtain $\partial_t u(t,x)$, we need the following assumption.

(A4) b'_t, h'_t, σ'_t, f'_t, g'_t are continuous in (s,x,y,z), and there exist a constant $L_2 > 0$ and a positive integer m_1 such that for any $s \in [0,T]$, x, y, $z \in \mathbb{R}$,

$$\begin{aligned}&|b'_t(s,x)| + |h'_t(s,x)| + |\sigma'_t(s,x)| + |f'_t(s,x,y)| + |g'_t(s,x,y,z)| \\ &\leq L_2(1 + |x|^{m_1} + |y|^{m_1} + |z|^2).\end{aligned}$$

Regularity of PDEs

Theorem

Suppose that (A1)-(A4) hold. Then, for each $(t,x) \in (0,T) \times \mathbb{R}$, *we have*

$$\begin{aligned}\partial_{t+}u(t,x) = \quad & \sup_{P \in \mathcal{P}_{t,x}} E_P \Big[\varphi'(X_T^{t,x})\bar{X}_T^{t,x}\Gamma_T^{t,x} + \int_t^T \Big(f'_x(s)\bar{X}_s^{t,x} \\ & + \tfrac{T-s}{T-t} f'_t(s) - \tfrac{1}{T-t} f(s) \Big) \Gamma_s^{t,x} ds + \int_t^T \Big(\tfrac{g'_z(s)Z_s^{t,x}}{2(T-t)} \\ & + g'_x(s)\bar{X}_s^{t,x} + \tfrac{T-s}{T-t} g'_t(s) - \tfrac{1}{T-t} g(s) \Big) \Gamma_s^{t,x} d\langle B \rangle_s \Big],\end{aligned}$$

$$\begin{aligned}\partial_{t-}u(t,x) = \quad & \inf_{P \in \mathcal{P}_{t,x}} E_P \Big[\varphi'(X_T^{t,x})\bar{X}_T^{t,x}\Gamma_T^{t,x} + \int_t^T \Big(f'_x(s)\bar{X}_s^{t,x} \\ & + \tfrac{T-s}{T-t} f'_t(s) - \tfrac{1}{T-t} f(s) \Big) \Gamma_s^{t,x} ds + \int_t^T \Big(\tfrac{g'_z(s)Z_s^{t,x}}{2(T-t)} \\ & + g'_x(s)\bar{X}_s^{t,x} + \tfrac{T-s}{T-t} g'_t(s) - \tfrac{1}{T-t} g(s) \Big) \Gamma_s^{t,x} d\langle B \rangle_s \Big],\end{aligned}$$

Regularity of PDEs

Theorem

where $f'_t(s) = f'_t(s, X_s^{t,x}, Y_s^{t,x})$*, similar for* $f(s)$*,* $f'_x(s)$*,* $g(s)$*,* $g'_x(s)$*,* $g'_z(s)$ *and* $g'_t(s)$*,* $(\bar{X}_s^{t,x})_{s\in[t,T]}$ *satisfies the following* G*-SDE:*

$$\begin{aligned}
\bar{X}_s^{t,x} = \ & \int_t^s \left[b'_x(r, X_r^{t,x})\bar{X}_r^{t,x} + \tfrac{T-r}{T-t} b'_t(r, X_r^{t,x}) - \tfrac{1}{T-t} b(r, X_r^{t,x}) \right] dr \\
& + \int_t^s \left[h'_x(r, X_r^{t,x})\bar{X}_r^{t,x} + \tfrac{T-r}{T-t} h'_t(r, X_r^{t,x}) - \tfrac{1}{T-t} h(r, X_r^{t,x}) \right] d\langle B\rangle_r \\
& + \int_t^s \left[\sigma'_x(r, X_r^{t,x})\bar{X}_r^{t,x} + \tfrac{T-r}{T-t} \sigma'_t(r, X_r^{t,x}) - \tfrac{1}{2(T-t)} \sigma(r, X_r^{t,x}) \right] dB_r.
\end{aligned}$$

The following theorem gives the condition for $\partial_{x+}u(t,x) = \partial_{x-}u(t,x)$.

Theorem

Suppose that (A1)-(A4) hold. If $\sigma(t,x) \neq 0$ *for some* $(t,x) \in (0,T) \times \mathbb{R}$*, then* $\partial_{x+}u(t,x) = \partial_{x-}u(t,x)$.

Finally, we study $\partial^2_{xx} u(t,x)$. We need the following assumption.

(A5) b''_{xx}, h''_{xx}, σ''_{xx}, f''_{xx}, f''_{xy}, f''_{yy}, g''_{xx}, g''_{xy}, g''_{xz}, g''_{yy}, g''_{yz}, g''_{zz} are continuous in (s,x,y,z) and bounded by a constant $L_3 > 0$.

Regularity of PDEs

Theorem

Suppose that (A1)-(A3) and (A5) hold. Then, for each $(t,x) \in [0,T) \times \mathbb{R}$*, we have*

$$\Delta^{-1}\left[\partial_{x-}u(t,x+\Delta) - \partial_{x+}u(t,x)\right] \geq -C(1+|x|^{2m}) \text{ for } \Delta \in (0,1],$$

$$\Delta^{-1}\left[\partial_{x+}u(t,x+\Delta) - \partial_{x-}u(t,x)\right] \geq -C(1+|x|^{2m}) \text{ for } \Delta \in [-1,0),$$

where the constant $C > 0$ *depends on* L_1*,* L_3*,* $\bar{\sigma}$ *and* T*.*

- Mingshang Hu, Shaolin Ji, Xiaojuan Li, BSDEs driven by G-Brownian motion under degenerate case and its application to the regularity of fully nonlinear PDEs, Preprint Transactions of the American Mathematical Society.

Thank you!