Martingale Benamou - Brenier

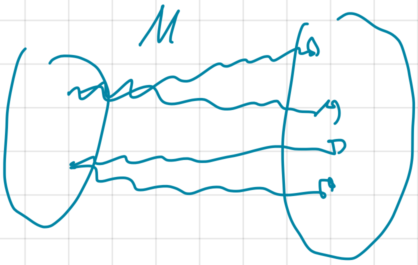
Backhoff - B - Huesmann - Källblad '20, Backhoff - B - Schachermayer Tschiderer '+24

also: Loeper, Conze - Henry-Labordere

$\mu, \nu \in \mathbb{R}$

Q:

$M_0 \sim \mu$



$\nu \sim M_1$

Strassen '65 $\exists (M_t)_{t \in [0,1]}$ $M_0 \sim \mu$, $M_1 \sim \nu \Leftrightarrow \mu \leq_c \nu$

$\Rightarrow$ ... Jensen $\quad \forall$ convex φ $\int \varphi \, d\mu \leq \int \varphi \, d\nu$

$\Leftarrow$ ... Hahn - Banach.

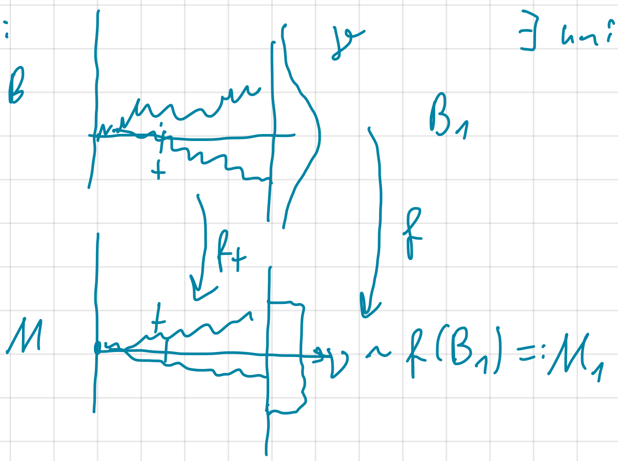
What is a natural Martingale

-) $\mu = \delta_0$, $\nu \sim \gamma \sim N(0,1)$, $M =$ B.M. (Brownian Motion)

-) $\mu = \alpha$, $\nu \sim \alpha * \gamma$, $M =$ B.M, $B_0 \sim \alpha$

-) $\mu = \delta_0$, ν on $\mathbb{R}$, centered.

Bass:



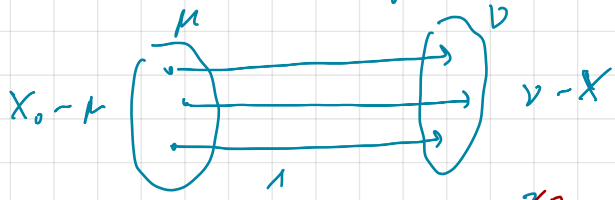
$\exists$ unique inc. f s.t. $f_\# \gamma = \nu$

$$M_t = \mathbb{E}[f(B_1) \mid \mathcal{F}_t] \text{ ... cont. Mart.}$$
$$= \mathbb{E}[f(B_t + \underbrace{(B_1 - B_t)}_{\sim \gamma_{1-t} \sim N(0,1-t)}) \mid \mathcal{F}_t]$$
$$= \underbrace{f * \gamma_{1-t}}_{=: f_t} (B_t)$$

extension: $B_0 \sim \alpha$ on $\mathbb{R}^d$

$$M_t = \mathbb{E}[\overset{\text{convex}}{\nabla V}(B_1) \mid \mathcal{F}_t]$$
$$= (\nabla V * \gamma_{1-t})(B_t)$$

Benamou-Brenier in optimal transport:

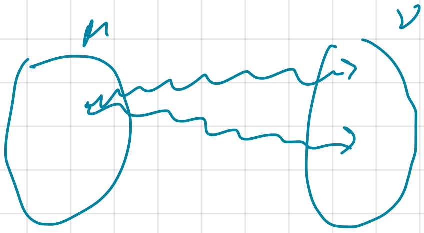


$$\min_{\substack{X_0 \sim \mu,\, X_1 \sim \nu \\ dX_t = v_t\, dt}} \mathbb{E} \int_0^1 \| v_t - \text{const} \|_2^{\cancel{2}\, p} \, dt$$

Huasmann-Trevison

—

Martingale version



$$\min_{\substack{M_0 \sim \mu,\, M_1 \sim \nu \\ dM_t = \sigma_t\, dB_t}} \underbrace{\mathbb{E} \int_0^1 \| \sigma - \mathrm{Id} \|_2^2 \, dt}_{M \text{ should be "close" to B.M.}} \qquad (MBB)$$

Theorem: μ, ν on $\mathbb{R}^d$, finite second moment, $\mu \leq_c \nu$.

-) $\exists$ unique Minimizer M^*

-) M^* is a strong Markov-Martingale

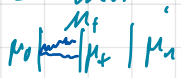
-) M solves (MBB) $\Leftrightarrow$ M is a Bass-mart.

Actually, here the implication "$\Leftarrow$" is true if (μ,ν) is irreducible.

$$M_t = \mathbb{E}[\nabla V(B_1) \mid \mathcal{F}_t]$$

$\Big($ Brenier: $\inf_{\substack{q \in \mathrm{cpl}(\gamma, \nu) \\ T: \gamma \to \nu}} \int |T(x) - x|^2 \, d\mu(x)$ -- optimizer $T = \nabla V$ $\Big)$

Corollary. For any $\mu \leq \nu$ $\exists$ unique Bass-mart. $M_0 \sim \mu$, $M_1 \sim \nu$.

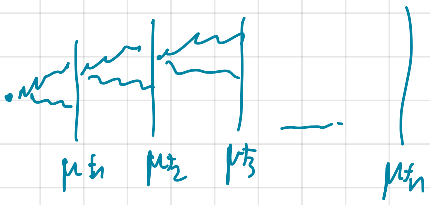


$\mu_t := \mathrm{law}(M_t)$

Remarks: -) time consistent interpolation

-) solves: $\min_{M_0 \sim \mu,\, M_1 \sim \nu}$ adapted Wasserstein $(M, \text{B.M.})$

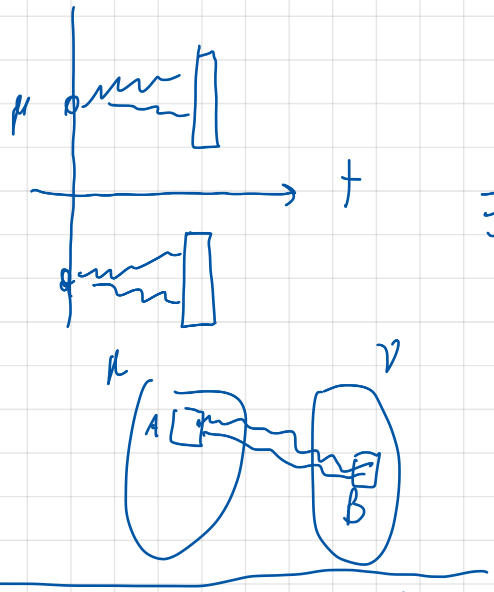
-) Bass-local-volatility model.



-) Kellerer's theorem '72

In $\mu = \delta_0$ Proof ... like Brenier's thm between μ, ν

Example:



Irreducibility of (μ, ν)

$\forall A,\ \mu(A) > 0,\ \forall B,\ \nu(B) > 0,$
$\exists$ Mart M s.t.

$$\mathbb{P}(M_0 \in A, M_1 \in B) > 0$$

MBB $\iff$ $\sup_{\substack{M_0 \sim \mu,\, M_1 \sim \nu \\ dM_t = \sigma_t dB_t}} \mathbb{E}\int_0^1 \mathrm{tr}\,\sigma\, dt = \inf_{\psi \text{ convex}} \int \psi\, d\nu - \int (\psi^* * \gamma)^* d\mu$

Legendre

convolution

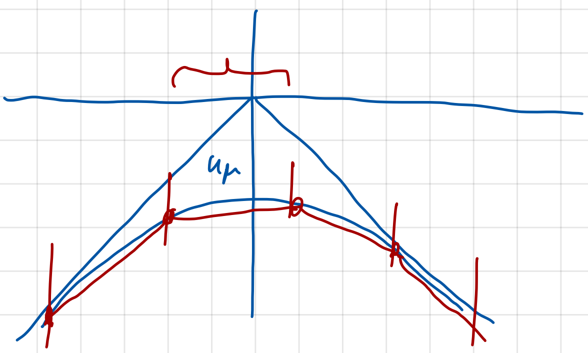
$\exists\ \psi_{opt} \iff$ irred.

$\Rightarrow \nu = \psi^*_{opt}$

$$M_t = \mathbb{E}[\nabla \psi^*_{opt}(B_1) \mid \mathcal{F}_t]$$

$$\psi^*(y) = \sup_{x \cdot y} \langle x, y \rangle - \psi(x)$$

$u_\mu(k) := -\int |x - k|\, d\mu(x)$ $u_\mu \geq u_\nu$



$\boxed{\text{de March - Touzi}}$

Ghoussoub-Kim-Lim

$\Uparrow$

finer