Integration along stochastic processes

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based on joint works with Oleg Butkovsky, Konstantinos Dareiotis, Lucio Galeati, Khoa Lê

Averaging of distributions

Some well-known objects in stochastic analysis are of the form

$$\mathcal{L}_t = \int_0^t f(X_s)\, ds,$$

where f is a distribution and X is a stochastic process.

Example: $f = \delta$, $X =$ Brownian motion $\rightsquigarrow \mathcal{L} =$ local time at 0.

Integrand is not well-defined, but the integral is!

We are interested in general conditions on f and X that allow one to define such integrals.

Main applications in mind concern SDEs with irregular drift b

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Construction 1: explicit X

If $X \equiv B \in \{\text{fractional Brownian motion}, \text{stable Lévy process}\}$, then

$$\mathbb{E} f(B_t) = \mathcal{P}_t^H f(0),$$

where $\mathcal{P}_t^H$ is a regularising kernel, e.g.:

$$\|\nabla \mathcal{P}_t^H f\|_\infty \lesssim t^{-H} \|f\|_\infty,$$

and H is the self-similatity exponent (either the Hurst parameter or the inverse of the stability index). $H \gg 1$ is allowed!

More generally, with $\mathbb{E}_s(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_s)$,

$$\mathbb{E}_s f(B_t) = \mathcal{P}_{t-s}^H f(\mathbb{E}_s B_t),$$

$$\|\mathcal{P}_t^H f\|_{C^{\beta+\gamma}} \lesssim t^{-\gamma H} \|f\|_{C^\beta},$$

where $\gamma \geq 0$, $C^\beta = B^\beta_{\infty,\infty}$.

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Construction 1: explicit X

This already gives a lot:

$$\int_0^1 f(B_s)\,ds = \lim_{n\to\infty}\sum_{i=0}^{2^n-1}\mathbb{E}_{i2^{-n}}\int_{i2^{-n}}^{(i+1)2^{-n}} f(B_s)\,ds$$

$$= \lim_{n\to\infty}\sum_{i=0}^{2^n-1}\int_{i2^{-n}}^{(i+1)2^{-n}} (\mathcal{P}^H_{s-i2^{-n}}f)(\mathbb{E}_{i2^{-n}}B_{(i+1)2^{-n}})\,ds$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{2^n-1}\underbrace{\int_{(i+1/2)2^{-n}}^{(i+1)2^{-n}} \big(\mathbb{E}_{(i+1/2)2^{-n}} - \mathbb{E}_{i2^{-n}}\big)f(B_s)\,ds}_{\text{martingale!}}$$

Everything converges if $f \in C^\beta$, $1 + \beta H > 1/2$.

A systematic generalisation of this argument can be provided by the stochastic sewing lemma [Lê '18].

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$$\begin{aligned}\int_0^1 f(B_s)\,ds &= \lim_{n\to\infty}\sum_{i=0}^{2^n-1}\mathbb{E}_{i2^{-n}}\int_{i2^{-n}}^{(i+1)2^{-n}} f(B_s)\,ds\\ &= \lim_{n\to\infty}\sum_{i=0}^{2^n-1}\int_{i2^{-n}}^{(i+1)2^{-n}} (\mathcal{P}^H_{s-i2^{-n}}f)(\mathbb{E}_{i2^{-n}}B_{(i+1)2^{-n}})\,ds\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{2^n-1}\underbrace{\int_{(i+1/2)2^{-n}}^{(i+1)2^{-n}}\big(\mathbb{E}_{(i+1/2)2^{-n}}-\mathbb{E}_{i2^{-n}}\big)f(B_s)\,ds}_{\text{martingale!}}\end{aligned}$$

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Construction 2: perturbed X

More often the process X is not known explicitly, but it admits a "fast-slow" decomposition:

$$X = \varphi + B$$

where B is as before and φ is "slower": for some $\gamma > H$, some constant C, for all s, t, one has the bound almost surely:

$$\mathbb{E}_s |\varphi_t - \mathbb{E}_s \varphi_t| \leq C|t-s|^\gamma. \qquad \text{(SMO)}$$

Remark: This is a natural analogue of a BMO condition, but different from the existing BMO process concept (see [Lê '22] for an overview and new results).

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Construction 2: perturbed X

Theorem (G '20, Butkovsky-Dareiotis-G '22, Galeati-G '22)

Let $b \in C^{\alpha}$, $\alpha \in (0,1]$, $\alpha > 1 - 1/H$, and $X = \varphi + B$ satisfy

$$dX_t = b(X_t)\,dt + dB_t.$$

Then φ satisfies (SMO) with

- $\gamma = 1 + \alpha H$ *(fBM);*
- $\gamma = 2 \wedge (1 + \alpha H)$ *(stable Lévy).*

Remark: So do approximate solutions, e.g. Picard iterates, Euler scheme.

Construction 2: perturbed X

Theorem (G '20, Butkovsky-Dareiotis-G '22, Galeati-G '22)

If φ satisfies (SMO), $f \in C^{\beta}$, and

$$(\beta - 1)H + \gamma > 0, \qquad 1 + \beta H > 1/2$$

then

$$\mathcal{L}_t^{X,f} = \int_0^t f(X_s)\, ds$$

is well-defined and belongs to $C^{1+\beta H}(L^p(\Omega))$, for all $p < \infty$.

In many situations, $f = \nabla b$, so $\beta = \alpha - 1$. This leads to the conditions $(\alpha - 1)H + 1 + \alpha H > 0$, $1 + (\alpha - 1)H > 1/2$

$$\Rightarrow \alpha > 1 - 1/(2H).$$

Remark: With buckling (choosing $f = b$), this allows to extend the previous theorem to certain ranges of negative α.

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Application 1 - Stability ($\Rightarrow$ existence & uniqueness)

Take X, $\tilde{X}$ to be solutions of

$$dX_t = b(X_t)\,dt + dB_t, \qquad d\tilde{X}_t = \tilde{b}(\tilde{X}_t)\,dt + dB_t,$$

where $b \in C^\alpha$, $\alpha < 1$, and $B =$ fractional Brownian motion.
Formally,

$$X_t - \tilde{X}_t = \int_0^t (X_s - \tilde{X}_s) \underbrace{\int_0^1 \nabla b(\theta \tilde{X}_s + (1-\theta)X_s)\,d\theta\,ds}_{=:d\mathcal{L}_s^{X;\tilde{X},\nabla b}}$$

$$+ \int_0^t (b - \tilde{b})(\tilde{X}_s)\,ds.$$

Theorem (Galeati-G '22)

If $\alpha > 1 - 1/(2H)$ and $\varphi, \tilde{\varphi}$ satisfy (SMO) with $\gamma = 1 + \alpha H$, then $\|\mathcal{L}^{X;\tilde{X},\nabla b}\|_{C^{1+(\alpha-1)H}}$ has Gaussian moments. As a consequence,

$$\Big\| \sup_{t\in[0,1]} |X - \tilde{X}| \Big\|_{L^p(\Omega)} \lesssim \|X_0 - \tilde{X}_0\|_{L^p(\Omega)} + \|b - \tilde{b}\|_{C^{\alpha-1}}.$$

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Application 2 - Malliavin regularity

Let X, B be as before. Take directional derivative in the noise in the direction of $h \in C(0,1)$:

$$d\partial_h X_t = \nabla b(X_t)\partial_h X_t \, dt + h_t.$$

Based on the above, we can rewrite this as

$$d\partial_h X_t = \partial_h X_t \, d\mathcal{L}_t^{X,\nabla b} + h_t.$$

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Let q be such that $q < \underbrace{((1-\alpha)H))^{-1}}_{>2}$. Then

$$\Big\| \sup_{t\in[0,1]} \|h \mapsto \partial_h X_t\|_{L(C^{q-var},\mathbb{R}^d)} \Big\|_{L^p(\Omega)} \lesssim 1.$$

Recalling that the Cameron-Martin space of B is embedded in $C^{2-\mathrm{var}}$, this bounds the Malliavin derivative of X.

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Application 3 - CLT

Let B be a standard BM and consider a multiplicative SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$$

[Kurtz-Protter '91]: If $b, \sigma \in C^1$, the Euler-Maruyama scheme satisfies a CLT : $V^n = \sqrt{n}(X - X^n)$ converges in law to V, where

$$dV_t = \nabla b(X_t) V_t\,dt + \nabla\sigma(X_t) V_t\,dB_t + \frac{1}{\sqrt{2}}(\nabla\sigma\sigma)(X_t)\,dW_t.$$

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Theorem (Dareiotis-G-Lê '23+)

Suppose $\alpha > 0$ and $b \in C^\alpha$, $\sigma \in C^2$ and nondegenerate. Then equation (1) is well-defined, well-posed, and V^n converges in law to its solution V.

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Remarks/Outlook

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- In (SMO) one can replace $|t - s|$ by a control $w(s, t)$ (but w can not be completely arbitrary)
- ... which allows for extensions to time-dependent b
- ... which is relevant, for example, in applications to McKean-Vlasov equations

Questions:

- For Lévy-driven SDEs we get optimal condition on α only if $1/H \geq 2/3$. Is there a better substitute for (SMO)?
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Thank you for the attention!