Statistics for SPDEs

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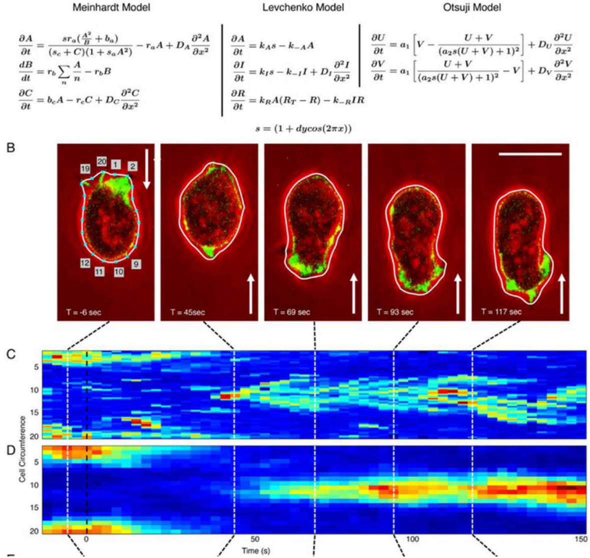
[1] joint with Randolf Altmeyer (Cambridge), Till Bretschneider (Warwick), Josef Janak (Pavia), Gregor Pasemann (Berlin)

Cell motility: repolarisation due to signal gradient



Lockley, Ladds, Bretschneider (2015) Image based validation of dynamical models for cell reorientation, *Cytometry A*

A stochastic Meinhardt model

Activator-inhibitor model $X = (A, I)$:

$$\begin{cases} \partial_t A(t,x) = D_A \partial_x^2 A(t,x) + f_A(X(t,x),x) + \sigma_A \dot{W}_A(t,x), \\ \partial_t I(t,x) = D_I \partial_x^2 I(t,x) + f_I(X(t,x),x) + \sigma_I \dot{W}_I(t,x), \end{cases}$$

on the $1D$-torus $\mathbb{R}/L\mathbb{Z}$ with *space-time white noises* $\dot{W}_A$, $\dot{W}_I$,

$$f_A(u,x) = r_A \frac{s(x)\left(b_A + u_1^2\right)}{(s_I + |u_2|)\left(1 + s_A u_1^2\right)} - r_A u_1, \quad f_I(u,x) = b_I u_1 - r_I u_2,$$

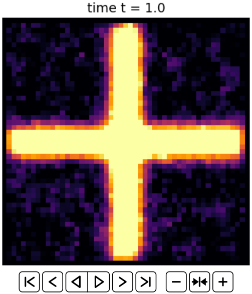
and extracellular signal $s(x)$.

Observation: $A(t, x_i)$ for $t \in [0, T]$, locations x_i
Goal: estimate unknown parameters like diffusivity D_A

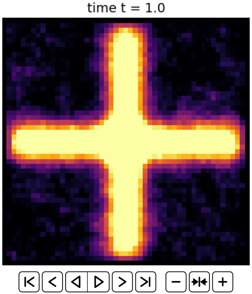
Altmeyer, Bretschneider, Janak, Reiß (2022): Parameter estimation in an SPDE-model for cell repolarisation

Stochastic heat equation: simulations

$$\dot{X}(t,x) = \vartheta \Delta X(t,x) + \dot{W}(t,x)$$



Left: $\vartheta = 4$



Right: $\vartheta = 8$

Outline

MLE for Ornstein-Uhlenbeck process

Observe $(X_t,\ t \in [0, T])$ continuously in time, where

$$dX(t) = \vartheta X(t)dt + \sigma dW(t)$$

Maximum-likelihood estimator (MLE):

$$\hat{\vartheta} := \frac{\int_0^T X(t)dX(t)}{\int_0^T X_t^2 dt}$$

$$= \frac{\int_0^T X(t)\,(\vartheta X(t)\,dt + \sigma\,dW(t))}{\int_0^T X(t)^2 dt} = \vartheta + \frac{\int_0^T X(t)\sigma\,dW(t)}{\int_0^T X(t)^2 dt}$$

Asymptotic theory for $\vartheta < 0$:

$$\frac{\sqrt{T}}{\sqrt{2|\vartheta|}}\left(\hat{\vartheta} - \vartheta\right) \xrightarrow{T|\vartheta|\to\infty} \mathcal{N}(0, 1)$$

Stochastic heat equation with constant diffusivity

$$\boxed{\dot X(t,x) = \vartheta \Delta X(t,x) + \sigma \dot W(t,x)}$$

- Laplace operator $\Delta : H^2(D) \to L^2(D)$ with Dirichlet or Neumann boundary condition on $D \subseteq \mathbb{R}^d$
- diffusivity constant $\vartheta > 0$
- space-time white noise $\dot W$, $\sigma > 0$

Spectral decomposition:
(λ_k, e_k) eigensystem of Laplace Δ with ONB (e_k), $\lambda_k \sim -k^{2/d}$.

$$\boxed{X_k(t) := \langle X(t,\bullet), e_k\rangle_{L^2} \Rightarrow \dot X_k(t) = \vartheta \lambda_k X_k(t) + \sigma \dot W_k(t), \quad k \geqslant 1}$$

with independent Brownian motions $(W_k)_{k\geqslant 1}$.

$\rightsquigarrow$ sequence of independent Ornstein-Uhlenbeck processes!

Spectral estimator

Observe $(X_k(t),\ t \in [0,T])$ continuously in time, where

$$dX_k(t) = \vartheta \lambda_k X_k(t)dt + \sigma dW_k(t)$$

Maximum-likelihood estimator (at frequency k):

$$\hat{\vartheta}_k := \frac{\int_0^T X_k(t)dX_k(t)}{\lambda_k \int_0^T X_k(t)^2 dt} = \vartheta + \frac{\int_0^T X_k(t)\sigma dW_k(t)\,dt}{\lambda_k \int_0^T X_k(t)^2 dt}$$

Asymptotics:

$$\sqrt{T|\lambda_k|}\big(\hat{\vartheta}_k - \vartheta\big) \xrightarrow{T|\lambda_k|\to\infty} \mathcal{N}(0, 2\vartheta)$$

Consequence: (T fixed)
The drift parameter is identifiable via $k \to \infty$ when observing continuously $(X(t), t \in [0,T])$ (weak solution suffices!).
Equivalent: laws $\mathcal{L}_\vartheta(X(t), t \in [0,T])$ are singular for different ϑ.

Selective literature review

- M. Huebner and B. Rozovskii (1995) On asymptotic properties of MLEs for parabolic SPDEs, *PTRF*.
- I. Ibragimov and R. Khasminskii (1999-2001) Problems of estimating the coefficients of SPDEs, parts I-III, *TPA*.
- I. Cialenco and N. Glatt-Holtz (2011) Parameter estimation for the stochastically perturbed Navier-Stokes equations, *SPA*.
- S. Lototsky and B. Rozovskii (2017) *Stochastic partial differential equations*, Springer.
- **I. Cialenco (2018) Statistical inference for SPDEs: an overview, *SISP*.**
- P. Kriz and B. Maslowski (2019) Central limit theorems and minimum-contrast estimators for linear stochastic evolution equations, *Stoch*.
- C. Chong (2020) High-frequency analysis of parabolic SPDEs, *AOS*.
- R. Altmeyer and MR (2021) Nonparametric estimation for linear SPDEs from local measurements, *AAP*.
- O. Lang, P. van Leeuwen, D. Crisan, R. Potthast (2022) Bayesian inference for fluid dynamics: A case study, *FrontAMS*.
- O. Assaad, J. Gamain, C. Tudor (2022) Quadratic variation and drift parameter estimation for the stochastic wave equation, *StochDyn*.
- F. Hildebrandt, M. Trabs (2023) Nonparametric calibration for stochastic reaction-diffusion equations based on discrete observations, *SPA*.
- R. Altmeyer, A. Tiepner, M. Wahl (2024+) Optimal parameter estimation for linear SPDEs from multiple measurements, *AOS*.
- **stats4SPDEs:** `sites.google.com/view/stats4spdes/bibliography`

Local measurements

Heuristic explanation of identifiability:
The diffusivity in the drift grows with the frequency, while the white noise level remains constant.
$\rightsquigarrow$ signal-to-noise ratio grows in the frequency domain.

Question: Identifiability in spatial domain?

Spatial resolution δ of measurement around $x_0 \in D$:

$$\boxed{X_\delta(t) := \int_D X(t,x)K_\delta(x-x_0)\,dx = (X(t,\bullet) * K_\delta)(x_0)}$$

$K_\delta(x) = \delta^{-d/2}K(x/\delta)$ for K with compact support, $\|K\|_{L^2} = 1$

$$\boxed{dX_\delta(t) = \vartheta(\Delta X(t,\bullet) * K_\delta)(x_0)dt + \sigma dW_\delta(t)}$$

scalar Brownian motion $W_\delta(t) = \int_D K_\delta(x-x_0)W(t,x)dx$.

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Augmented MLE from local measurements

By partial integration

$$\dot{X}_\delta(t) = \vartheta\big(X(t,\bullet) * \Delta K_\delta\big)(x_0) + \sigma \dot{W}_\delta(t)$$

Augmented local measurements: $(X_\delta(t), (X * \Delta K_\delta)(t, x_0))_{0 \leqslant t \leqslant T}$

Augmented MLE:

$$\hat{\vartheta}_\delta := \frac{\int_0^T (X * \Delta K_\delta)(t, x_0) dX_\delta(t)}{\int_0^T (X * \Delta K_\delta)^2(t, x_0)\, dt} = \vartheta + \frac{\int_0^T (X * \Delta K_\delta)(t, x_0) \sigma dW_\delta(t)}{\int_0^T (X * \Delta K_\delta)^2(t, x_0)\, dt}$$

Proposition.(Altmeyer, MR 2021)

$$\delta^{-1}\big(\hat{\vartheta}_\delta - \vartheta\big) \xrightarrow{\delta \to 0} \mathcal{N}\Big(0, \frac{2\vartheta}{T\|\nabla K\|_{L^2}^2}\Big)$$

ϑ is identifiable from local measurements $(X(t,x))_{0 \leqslant t \leqslant T, |x - x_0| \leqslant \delta}$.

Augmented MLE from local measurements

By partial integration

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ϑ is identifiable from local measurements $(X(t,x))_{0 \leqslant t \leqslant T, |x - x_0| \leqslant \delta}$.

Stochastic heat equation with varying diffusivity

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$

- $\Delta_\vartheta g(x) := \sum_{i=1}^d \partial_{x_i} \vartheta(x) \partial_{x_i} g(x)$
- spatially varying diffusivity $\vartheta \in C^1(D)$, $\vartheta(x) > 0$
- space-time white noise $\dot{W}$, $\sigma > 0$

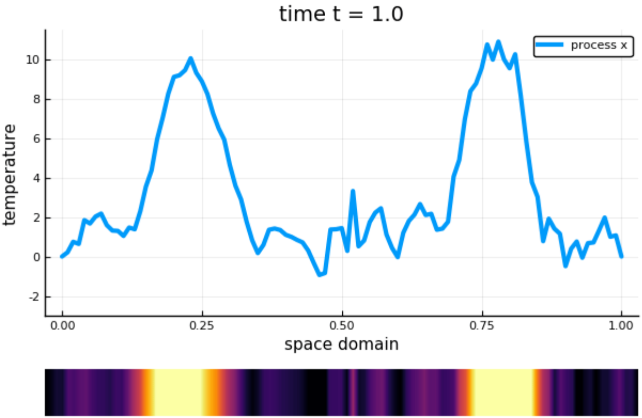
Spectral decomposition:

The eigenfunctions of Δ_ϑ depend on ϑ, i.e. are unknown.

$\rightsquigarrow$ spectral approach is not feasible for nonparametrics.

1D-Simulation

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



1D-Simulation

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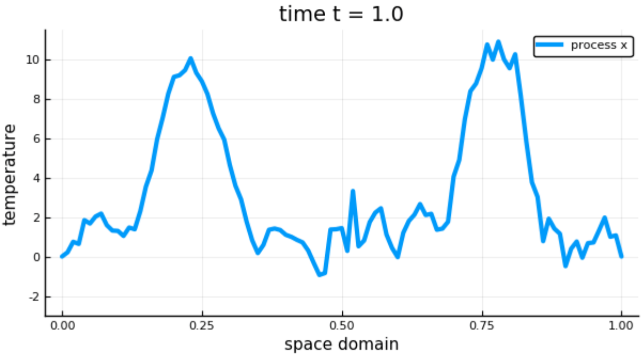
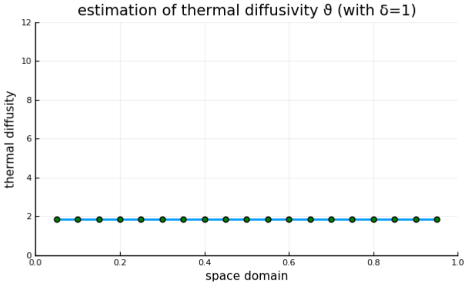


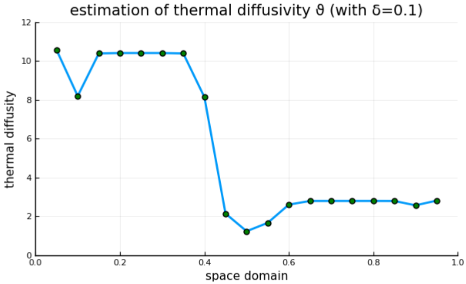
Estimator: $\delta = 1$

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



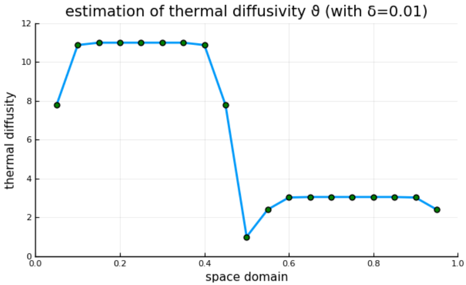
Estimator: $\delta = 0.1$

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$



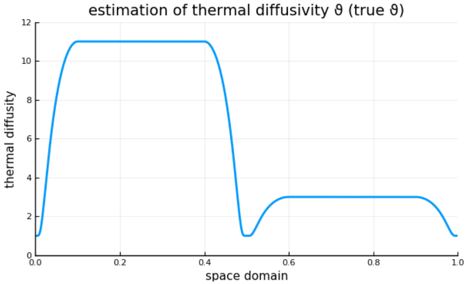
Estimator: $\delta = 0.001$

$$\boxed{\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)}$$



True function

$$\dot{X}(t,x) = \Delta_{\vartheta} X(t,x) + \sigma \dot{W}(t,x)$$



Pointwise estimate of diffusivity

$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$

Idea: Augmented MLE $\hat{\vartheta}_\delta$ estimates $\vartheta(x)$ locally around x_0.

$$\hat{\vartheta}_\delta - \vartheta(x_0) = \underbrace{\mathcal{O}_P\left(\frac{\int (\vartheta(x_0 + \delta \bullet) - \vartheta(x_0)) |\nabla K|^2}{\vartheta(x_0) \|\nabla K\|_{L^2}^2}\right)}_{\text{"BIAS"}} + \underbrace{\mathcal{O}_P\left(\frac{\sqrt{\vartheta(x_0)}\delta}{\sqrt{T}\|\nabla K\|_{L^2}}\right)}_{\sqrt{\text{"VARIANCE"}}}$$

Remarks:

- For $\vartheta \in C^1(D)$ the bias is $\mathcal{O}_P(\delta)$.
- For $\vartheta \in C^1(D)$ and K radial-symmetric the bias is $o_P(\delta)$.
- The total error rate is then $\mathcal{O}_P(\delta)$.

General linear result I

$$\dot{X}(t,x) = A_\vartheta X(t,x) + \sigma(x)\dot{W}(t,x)$$

with second order operator $A_\vartheta = \Delta_\vartheta + \sum_i a_i(x)\partial_{x_i} + b(x)$.

Augmented MLE: $\hat{\vartheta}_\delta := \frac{\int_0^T (X*\Delta K_\delta)(t,x_0)dX_\delta(t)}{\int_0^T (X*\Delta K_\delta)^2(t,x_0)\,dt}$

Theorem. (Altmeyer, MR 2021)
Under mild regularity conditions we have

$$\delta^{-1}\big(\hat{\vartheta}_\delta(x_0) - \vartheta(x_0)\big) \xrightarrow{\delta\to 0} \mathcal{N}\Big(\frac{\int \langle \nabla\vartheta(x_0), x\rangle |\nabla K(x)|^2 dx}{\|\nabla K\|_{L^2}^2}, \frac{2\|K\|_{L^2}^2}{T\|\nabla K\|_{L^2}^2}\Big)$$

General linear result II

Consequences and remarks:

- Robust to lower order drift coefficients $a(x), b(x)$ and noise level $\sigma(x)$
- $\vartheta(x_0)$ is nonparametrically identifiable from local observations ($\rightsquigarrow$ local singularity of laws)
- Estimator can be applied at different locations x_i separately
- Confidence intervals with Gaussian quantiles
- Convergence rate δ is minimax-optimal for any estimator

Proof ingredients:

- Scaling via $X(\delta^2 t, \delta x)$, $t \in [0, \delta^{-2} T], x \in \delta^{-1} D$
- Heat kernel and semigroup bounds (via Feynman-Kac)
- Martingale CLT

General linear result II

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Semi-linear SPDEs

Stochastic reaction-diffusion equation:

$$\dot{X}(t,x) = \vartheta\Delta X(t,x) + f(X(t,x)) + B\dot{W}(t,x)$$

with $f \in C_b^\infty(\mathbb{R})$ or $f(x) = \sum_{i\leqslant m} a_i x^i$ and $a_m < 0$.
Assume $X \in C([0,T], W^{s,p}(D))$, $s > d/p$.

Stochastic Burgers equation:

$$\dot{X}(t,x) = \vartheta\partial_x^2 X(t,x) - X(t,x)\partial_x X(t,x) + B\dot{W}(t,x)$$

with $d = 1$, $B = (-\Delta)^{-\gamma}$ for $\gamma > 1/4$.

Theorem. (Altmeyer, Cialenco, Pasemann 2020)
The *same* estimator, applied to these semilinear SPDEs, has the *same* asymptotic behaviour as in the linear case.

Idea of proof:

- Splitting technique $\dot{\bar{X}} = \vartheta\Delta\bar{X} + B\dot{W}$, $\tilde{X} = X - \bar{X}$
- $|\langle\tilde{X}(t), K_\delta\rangle| \leqslant \|\tilde{X}(t)\|_{W^{s,p}}\|K_\delta\|_{W^{-s,q}} \lesssim \delta^{s+d/q-d/2}$ small

Semi-linear SPDEs

Stochastic reaction-diffusion equation:

$$\dot{X}(t,x) = \vartheta \Delta X(t,x) + f(X(t,x)) + B\dot{W}(t,x)$$

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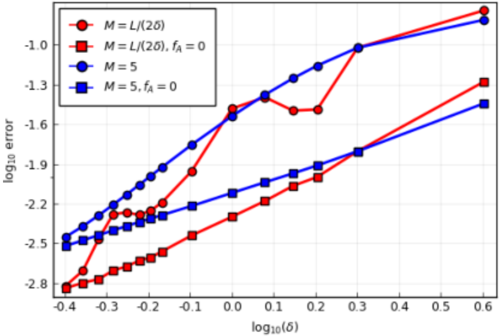
Cell repolarisation

Convergence of estimator for synthetic data



Diffusivity estimation from experimental data

$$D_A = 1.605 \times 10^{-2} \ \pm 0.022 \times 10^{-2}$$

Robust for different cell data; physically meaningful.

SPDEs with multiplicative noise

Stochastic heat equation with multiplicative noise:

$$\dot{X}(t,x) = \vartheta \Delta X(t,x) + \sigma(X(t))\dot{W}(t,x)$$

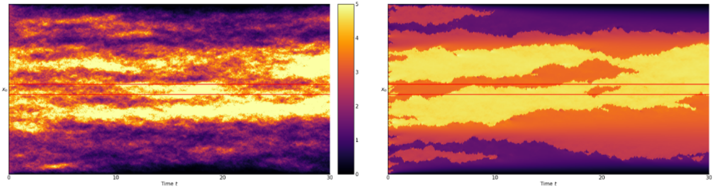


Figure 1: Realisation of the stochastic heat equation with multiplicative noise $\sigma_2(x) = (0.20 \times |x|^{0.8} + 0.01) \wedge 5$ (left) and $\sigma_3(x) = 10e^{-10|x-2|} + 10e^{-10|x-4|}$ (right). The horizontal lines indicate the support of the kernel K_{δ,x_0}.

SPDEs with multiplicative noise

Stochastic heat equation with multiplicative noise:

$$\dot{X}(t,x) = \vartheta \Delta X(t,x) + \sigma(X(t))\dot{W}(t,x)$$

Assumptions:

$\sigma \in C(\mathbb{R})$, X is a continuous weak solution (pathwise).

Theorem. (Janak, MR 2023)

The estimator $\hat{\vartheta}_\delta$ from before satisfies with *stable convergence in law*

$$\delta^{-1}(\hat{\vartheta}_\delta - \vartheta) \xrightarrow{\delta \to 0} \mathcal{N}\Big(0, \frac{2\vartheta \|K\|_{L^2}^2}{\|\nabla K\|_{L^2}^2} \bullet \frac{\int_0^T \sigma^4(X(t,x_0))\,dt}{(\int_0^T \sigma^2(X(t,x_0))\,dt)^2}\Big)$$

Estimating the quadratic variation, an improved estimator ϑ_δ^* satisfies

$$\delta^{-1}(\vartheta_\delta^* - \vartheta) \xrightarrow{\delta \to 0} \mathcal{N}\Big(0, \frac{2\vartheta \|K\|_{L^2}^2}{\|\nabla K\|_{L^2}^2}\Big)$$

Dynamic versus static noise in SPDEs

SPDE observations under measurement errors: Observe Y, the SPDE solution X corrupted by space-time white noise $\dot{V}$:

$$Y(t,x) = X(t,x) + \varepsilon \dot{V}(t,x), \quad t \in [0,T], x \in D,$$
$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$

with *static noise level* $\varepsilon > 0$ and *dynamic noise level* $\sigma > 0$.

Estimator for $\vartheta(x_0)$:
Preaverage Y by convolution with $K_\varepsilon(t,x) = K(\varepsilon t, \varepsilon^{1/2} x)$ in time and space (parabolic scaling).
Plugin into $\hat{\vartheta}$ instead of X, preserve martingale property by time shift $t \rightsquigarrow t+\varepsilon$ in the integrator.

$$\hat{\vartheta}_\varepsilon(x) = \frac{\int_0^{T-\varepsilon} (Y * \Delta K_\varepsilon)(t,x)\, d(Y * K_\varepsilon)(t+\varepsilon, x)}{\int_0^{T-\varepsilon} (Y * \Delta K_\varepsilon)(t,x_0)\,(Y * \Delta K_\varepsilon)(t+\varepsilon,x)\, dt}$$

Dynamic versus static noise in SPDEs

SPDE observations under measurement errors:

$$Y(t,x) = X(t,x) + \varepsilon \dot{V}(t,x), \quad t \in [0,1],\ x \in D,$$
$$\dot{X}(t,x) = \Delta_\vartheta X(t,x) + \sigma \dot{W}(t,x)$$

Theorem. (Pasemann, MR 2024+)
Assume $\vartheta \in C^\beta(\bar{D})$, $\inf_x \vartheta(x) > 0$, $X_0 \in L^p(D)$ for $p = p(d,\beta)$. Averaging preaverage-plugin estimators $\hat{\vartheta}_\varepsilon(x)$ over x in a neighbourhood of x_0 yields an estimator with

$$|\hat{\vartheta}(x_0) - \vartheta(x_0)| \lesssim_{\mathbb{P}} \begin{cases} \left(\frac{\varepsilon}{\sigma}\right)^{1/2}, & \text{if } d = 1,\ \beta = \frac{3}{2} \text{ or } d = 2,\ \beta = 1 \\ \left(\frac{\varepsilon}{\sigma}\right)^{\beta(d+2)/(4\beta+2d)}, & \text{if } d \geqslant 3,\ 1 \leqslant \beta < \frac{2d+6}{d+4} \end{cases}$$

Different noise impact:

The dynamic noise $\sigma \dot{W}$ excites X (energy input) and facilitates estimation of ϑ, the static noise $\varepsilon \dot{V}$ blurs the data and allows access to X only at resolutions larger than $(\varepsilon, \varepsilon^{1/2})$ in (t,x).

Covariance for stochastic evolution equations

Stochastic evolution equation under measurement errors:

$$Y(t) = X(t) + \varepsilon \dot{V}(t), \quad t \in [0, T],$$
$$\dot{X}(t) = A_\vartheta X(t) + \sigma \dot{W}(t), \quad X_0 = 0$$

with normal generators $A_\vartheta : \mathrm{dom}(A_\vartheta) \subseteq H \to H$ on a Hilbert space H and $\mathrm{dom}(A_\vartheta) = \mathrm{dom}(A_{\vartheta'})$.

Proposition. $Y \sim \mathcal{N}_{cyl}(0, Q_\vartheta)$ on $L^2([0, T]; H)$ where

$$Q_\vartheta = \varepsilon^2 \,\mathrm{Id} + \sigma^2 S_\vartheta S_\vartheta^* \quad \text{with} \quad S_\vartheta f(t) = \int_0^t e^{A_\vartheta (t-s)} f(s)\, ds$$

The Hellinger distance between the observation laws satisfies

$$\boxed{H(\mathcal{L}_\vartheta(Y), \mathcal{L}_{\vartheta'}(Y)) \leqslant \tfrac{1}{2} \| Q_\vartheta^{-1/2} Q_{\vartheta'}^{1/2} - (Q_{\vartheta'}^{-1/2} Q_\vartheta^{1/2})^* \|_{HS}}$$

Hellinger bound for stochastic evolution equations

Stochastic evolution equation under measurement errors:

$$Y(t) = X(t) + \varepsilon \dot{V}(t), \quad t \in [0, T],$$
$$\dot{X}(t) = A_\vartheta X(t) + \sigma \dot{W}(t), \quad X_0 = 0$$

Theorem. If the spectra of A_ϑ, $A_{\vartheta'}$ have negative real part, then the Hellinger distance satisfies

$$H(\mathcal{L}_\vartheta(Y), \mathcal{L}_{\vartheta'}(Y)) \leqslant$$
$$T\Big(\|(\tfrac{\varepsilon^2}{\sigma^2} R_{\vartheta'}^2 + \mathrm{Id})^{-1/2}(A_{\vartheta'} - A_\vartheta)(\mathrm{Id} - 2TR_\vartheta)^{-1/2}(\tfrac{\varepsilon^2}{\sigma^2} R_\vartheta^2 + \mathrm{Id})^{-1/2}\|_{HS(H)}$$
$$+ \|(\tfrac{\varepsilon^2}{\sigma^2} R_\vartheta^2 + \mathrm{Id})^{-1/2}(A_{\vartheta'} - A_\vartheta)(\mathrm{Id} - 2TR_{\vartheta'})^{-1/2}(\tfrac{\varepsilon^2}{\sigma^2} R_{\vartheta'}^2 + \mathrm{Id})^{-1/2}\|_{HS(H)} \Big)$$

with real part operators $R_\vartheta = (A_\vartheta + A_\vartheta^*)/2$.

Parametric lower bounds

Minimax lower bounds for parameter $\vartheta > 0$:

process	estimation rate
$dX_t = -\vartheta X_t dt + dW_t$	$T^{-1/2}(\varepsilon^2\vartheta^2 + 1)\vartheta^{1/2}$
$\dot{X}(t,x) = \vartheta \Delta X(t,x) + \dot{W}(t,x)$	$T^{-1/2}\varepsilon^{(d+2)/4}$
$\dot{X}(t,x) = \nu \Delta X(t,x) + \vartheta \partial_\xi X(t,x) + \dot{W}(t,x)$	$T^{-1/2}\nu^{(d+2)/4}\varepsilon^{d/4}$
$\dot{X}(t,x) = \nu \Delta X(t,x) - \vartheta X(t,x) + \dot{W}(t,x)$	$T^{-1/2}\nu^{d/4}\varepsilon^{(d-2)_+/4}$

Remarks:

- OU process: no impact of ε for ϑ fixed.
- Diffusivity: dim. $d \leqslant 5$; identifiable for $T \to \infty$ or $\varepsilon \to 0$
- Transport: $d \leqslant 7$; ident.: $T \to \infty$ or $\nu \to 0$ or $\varepsilon \to 0$
- Reaction: $d \leqslant 9$ ($d = 2$ has factor $(\log \varepsilon^{-1})^{-1/2}$); identifiable for $T \to \infty$ or $\nu \to 0$ or (if $d \geqslant 2$) $\varepsilon \to 0$

Nonparametric lower bounds

Minimax lower bounds for $\vartheta(x_0) > 0$, $\vartheta \in C^\beta(D)$:

process	rate
$\dot X(t,x) = \Delta_\vartheta X(t,x) + \dot W(t,x)$	$T^{-\frac{\beta}{2\beta+d}} \varepsilon^{\frac{\beta(d+2)}{4\beta+2d}}$
$\dot X(t,x) = \Delta X(t,x) + \operatorname{div}(\vartheta(x) X(t,x)) + \dot W(t,x)$	$T^{-\frac{\beta}{2\beta+d}} \varepsilon^{\frac{\beta d}{4\beta+2d}}$
$\dot X(t,x) = \Delta X(t,x) - \vartheta(x) X(t,x) + \dot W(t,x)$	$T^{-\frac{\beta}{2\beta+d}} \varepsilon^{\frac{\beta(d-2)_+}{4\beta+2d}}$

Restrictions:

- Diffusivity: $d \leqslant 5$; $T \leqslant \varepsilon^{1-\beta}$
- Transport: $d \leqslant 7$; $T \leqslant \varepsilon^{-\beta}$
- Source: $d \leqslant 9$; $T \leqslant \varepsilon^{-\beta-(d\wedge 2)/2}$.

Remarks:

- Diffusivity for T fixed: same rate as in upper bound.
- *Different* rate for T larger than restriction: diffusivity case and T fixed $\rightsquigarrow \beta = 1$ critical.

Summary

- Spectral estimator for SPDEs.
- Local measurements and nonparametric estimation.
- Optimal convergence rate δ, robust to lower order operators and multiplicative noise.
- Static versus dynamic noise.
- Hellinger bounds for stochastic evolution equations yield information structure for coefficients in SPDEs.

Thanks a lot for your attention!