What is non-trace class noise?

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§1 Introduction

- $\mathcal{O} \subseteq \mathbb{R}^d$ open
- $(b_n)_{n\geq 1}$ O.N.B. $L^2(\mathcal{O})$
- $(W_n)_{n\geq 1}$ i.i.d. Brownian motions
- $(\mu_n)_{n\geq 1}$ scalars.

Consider the noise "$W(t,x) = \sum_{n\geq 1} \mu_n b_n(x) W_n(t)$"

Ex (1) $\mu_n = 1$ W is a space-time white noise

(2) $\mu \in \ell^2$ W is a trace class noise

space regularity: $L^2(\mathcal{O})$ (even $L^q(\mathcal{O})$ if $\|b_n\|_\infty \leq C$)

Q What is in between cases (1) and (2)?

§2 Motivation from SPDEs

Often noise as rough as possible

too rough $\longrightarrow$ renormalization as in Hairer

Ex (nonlinear stochastic heat eq)

$$(*)\quad \begin{cases} du - \Delta u\, dt + \cancel{\psi(u)}\, dt = \phi(u)\, dW \\ u(0) = u_0, \text{ bdr cond.} \end{cases}$$

polynomial growth, loc lipschitz

ψ, ϕ in theory of (Agresti – V. 2022)

recording EPFL workshop (Survey 2025)

In the theory for local well-posed we need estimates for: $\mathbb{E}\| \underbrace{\phi(u)}_{g} \sum_{n\geq 1} \gamma_n \mu_n b_n \|^2_{H^{-s,q}} \leq C \| \underbrace{\phi(u)}_{g} \|^{2}_{L^\eta}$

$(\gamma_n)_{n\geq 1}$ i.i.d $N(0,1)$

Q For which μ, b_n, η, s, q does this hold?

$s \in (0,1)$ smoothness, q in L^q, $H^{-s,q}(\mathcal{O})$ Bessel potential space

η ela! take in $(1,\infty)$ as small as possible

$$\|\phi(u)\|_{L^\eta} \overset{\phi(u)=u^3}{=} \|u\|^3_{L^{3\eta}} \overset{\text{Sobolev emb.}}{\leq} \|u\|^3_{D((-\Delta)^{\cdots})}$$

Previous results: suboptimal for $\mu \in \ell^2$.

Important contributions of many authors
Cerrai, Da Prato, Zabczyk, ...

Usually formulated with

$$\underset{\substack{\uparrow\\ \text{regularizes}}}{\mathbb{E}\|e^{t\Delta}}\Big(g\sum_{n\geq 1}\gamma_n\mu_n e_n\Big)\Big\|^2_{L^q} \leq \underset{\substack{\uparrow\\ \text{price.}}}{C\, t^{-\frac{5}{2}}} \cdots$$

Usually assumed (e_n) eigen system of Δ

→ forces O bdd.

→ $\|e_n\|_\infty$ unbounded in n.

usually $q=2$, $\eta=2$

We found that it is better to stay away from $q=\eta=2$.

§ 3 Main result: Sobolev embedding for Gaussian series

Thm Suppose $(f_n)_{n\geq 1}$ O.N.B. for $L^2(\mathcal{O})$ with $\sup_{n\geq 1} \|f_n\|_\infty \leq C$ (can be weakened).

Suppose that $q \in (1,\infty)$, $\eta \in (1,q)$, $\zeta \in [2,\infty)$, $s \in (0,d)$

zeta

$$\boxed{(a)\ \frac{s}{d} + \frac{1}{q} \geq \frac{1}{\eta} + \frac{1}{2} - \frac{1}{\zeta}} \qquad \boxed{(b)\ \frac{1}{\eta} - \frac{1}{\zeta} < \frac{1}{2}} \to \text{innocent in appl.}$$

Then $\left(\mathbb{E}\left\| g \sum_{n\geq 1} \gamma_n \mu_n f_n \right\|^2_{H^{-s,q}(\mathcal{O})}\right)^{\frac{1}{2}} \leq \tilde{C} \|\mu\|_{\ell^\zeta} \|g\|_{L^\eta(\mathcal{O})}$

observations ① (a) is sharp even in the periodic setting

② Scaling show that (a) is the correct cond.

③ $\eta = 2$ forces $q > 2$ or nonoptimal results.

④ General case $f_n \in L^\infty$, without unif bound.
Replace cond on μ by $\left(\sum_{n\geq 1} |\mu_n|^\zeta \|f_n\|^2_\infty\right)^{\frac{1}{\zeta}} < \infty$

Δ_D on a ball, $\|f_n\|_\infty$ grow polynomially in n

⑤ Open problem if (b) is necessary
However not so relevant for SPDE of 2^{nd} order if $d \geq 2$.

⑥ (f_n) don't need to be an eigensystem of Δ
$\mathcal{O}$ can be unbdd.

⑦ $\|M_g T_\mu\|_{\gamma(L^2, H^{-s,q})}$ as LHS $T_\mu = \sum_{n\geq 1} \mu_n e_n \otimes f_n$
(e_n) O.N.B. for $L^2(\mathcal{O})$

$$\|R\|_{\gamma(H,X)} = \left(\mathbb{E}\left\|\sum_{n\geq 1} \gamma_n R h_n\right\|^2\right)^{\frac{1}{2}}$$

§ 4 Case $q=2$, $p=\infty$

Then $\mu_n = 1$ wlog.

$$\left(E \left\| g \sum_{n\geq 1} \gamma_n e_n \right\|^2_{H^{-s,2}}\right)^{\frac{1}{2}} = \|(1-\Delta)^{-\frac{s}{2}} M_g\|_{HS(L^2)}$$

$$= \|M_{k_s * g}\|_{HS(L^2)}$$

$$\overset{\text{Simon '76}}{=} \underbrace{\|k_s\|_{L^2}}_{<\infty \iff s>\frac{d}{2}} \|g\|_{L^2}$$

/ $d=1$ space time white noise

\ $d=2$ "almost!!"

Conclusion $q=2$ leads to suboptimal results

Simon '76, Cwikel '77 $S^r(L^2)$ instead of HS.

§ 5 Ingredients of the proof

Case $p=\infty$, (b_n) O.N.B. general
(difficult) (q,s,n)

Case $p=2$, $s=0$ classical
(q,s,n)

Multilinear complex interpolation.

Nasty puzzle.

§6 Application to the stochastic heat eq

$$\begin{cases} du - \Delta u \, dt = \underbrace{g \, dW}_{H^{-s,q}} & \text{on } \mathbb{T}^d \\ u_0 = 0 \end{cases}$$

Thm: same conditions on $(s, q, \eta, \wp)$. Let $p > 2$.

$\mu \in \ell^{\wp}$, $g \in L^p_{\mathcal{F}}((0,T) \times \Omega ; L^{\eta})$

$\Rightarrow$ $u \in L^p(\Omega ; E)$ for each of the following

$E = L^p(0,T ; H^{1-s,q})$

$E = H^{\theta,p}(0,T ; H^{1-s-2\theta,q})$ $\theta \in [0, \frac{1}{2})$ + estimates.

$E = C([0,T] ; B^{1-s-\frac{2}{p}})$ trace embedding [AV] survey

proof: Combine the main results with SMR

$M_g T_\mu \in L^p(\Omega \times (0,T) ; \gamma(L^2, H^{-s,q}))$