Generalized couplings and SFDEs

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International Seminar on SDEs and Related Topics,
November 26, 2021

Couplings

Definition

P a Markov kernel on $(E, \mathcal{E})$. $\mathbb{P}_x$ law of chain starting at $x \in E$.

$$C(\mathbb{P}_x, \mathbb{P}_y) := \{\xi \in \mathcal{M}_1(E^{\mathbb{N}_0} \times E^{\mathbb{N}_0}) : \xi_1 = \mathbb{P}_x, \xi_2 = \mathbb{P}_y\}$$

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Proposition

If P admits an ipm μ and, for each $x, y \in E$ there is some $\xi \in C(\mathbb{P}_x, \mathbb{P}_y)$ s.t. $\xi\big((\zeta, \eta) : \zeta_i = \eta_i\big) \to 1$, then μ is unique and $P_n(x, .) \to \mu$ in TV.

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- Sometimes, such couplings don't exist even if μ is unique. Then try to find *asymptotic coupling*.
- Even asymptotic couplings are often hard to find. Try generalized (as.) couplings.

Generalized couplings

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$$\hat{C}(\mathbb{P}_x, \mathbb{P}_y) := \{\xi \in \mathcal{M}_1(E^{\mathbb{N}_0} \times E^{\mathbb{N}_0}) : \xi_1 \ll \mathbb{P}_x, \xi_2 \ll \mathbb{P}_y\}$$

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$$\mathrm{d}X(t) = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W(t),$$

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$$\mathrm{d}Y(t) = f(Y_t)\,\mathrm{d}t + g(Y_t)\,\mathrm{d}W(t) + \lambda(X(t) - Y(t))\mathrm{d}t, \quad Y_0 = y$$

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Under suitable conditions, for λ large, this defines a gen. coupling and $|X(t) - Y(t)| \to 0$ a.s. and uniqueness of an ipm follows from next theorem.

Main result

Theorem (S. 2020)

Let (E, d) be a metric space and P a Markov kernel on E. Assume that for every $x, y \in E$ there exists $\alpha_{x,y} > 0$ s.t. for every $\varepsilon > 0$ there exists some $\xi^{\varepsilon}_{x,y} \in \hat{C}(\mathbb{P}_x, \mathbb{P}_y)$ s.t.

$$\xi^{\varepsilon}_{x,y}\Big((\zeta,\eta) \in E^{\mathbb{N}_0} \times E^{\mathbb{N}_0} : \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{[0,\varepsilon]}\big(\rho(\zeta_i,\eta_i)\big) \geq \alpha_{x,y}\Big) > 0,$$

where either $\rho = d$, or ρ is lsc and positive definite and E is Polish. Then P admits at most one ipm.

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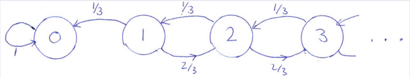

Idea of proof when (E, d) is Polish

Assume there exist at least two different ipm

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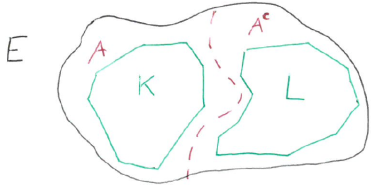
Assume there exist at least two different ipm

- Then, by ergodic decomposition theorem, there are two different ergodic ipm. These are mutually singular.
- Let μ, ν be mutually singular ipm concentrated on A and A^c respectively. By inner regularity, for $\varepsilon > 0$ there exist compact sets $K \subset A$ and $L \subset A^c$ s.t. $\mu(K^c) < \varepsilon$, $\nu(L^c) < \varepsilon$. Then $d(K, L) > 0$.
- Apply Birkhoff's ergodic theorem to get a contradiction.



General case

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The assumption that (E, d) is Polish is used twice:

- Ergodic decomposition theorem
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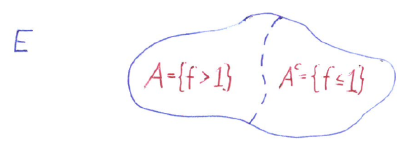
Part of the proof

Assume that μ, ν are different ipm. Assume they are equivalent and $f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.



$$\int_A P(y, A^c)\mathrm{d}\nu(y) = \int_{A^c} P(y, A)\mathrm{d}\nu(y)$$

$$\int_A P(y, A^c)f(y)\mathrm{d}\nu(y) = \int_{A^c} P(y, A)f(y)\mathrm{d}\nu(y),$$

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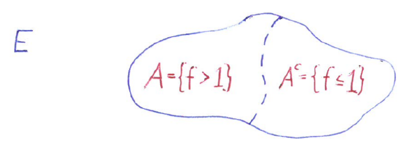

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hence $P(y, A^c) = 0$ for μ-a.a. $y \in A$ and vice versa, so $\frac{1}{\mu(A)}\mu|_A$ and $\frac{1}{\mu(A^c)}\mu|_{A^c}$ are mutually singular ipm.

General case

Proposition (Replacement for inner regularity)

Let $\mu \perp \nu$ be pm on (E, d) and $A \in \mathcal{B}(E)$ s.t. $\mu(A) = 1$, $\nu(A) = 0$. For $\varepsilon > 0$ there exist closed sets $K \subset A$ and $L \subset A^c$ s.t. $\mu(K) > 1 - \varepsilon$ and $\nu(L) > 1 - \varepsilon$ and $d(K, L) > 0$.

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Idea of proof

Fix $\varepsilon > 0$. Pick *closed* sets $K \subset A$, $B \subset A^c$ s.t. $\mu(K), \nu(B) > 1 - \varepsilon$. Note:

$$B \cap \left(K_{1/n}\right)^c \uparrow B.$$

Then, for n large and $L := B \cap \left(K_{1/n}\right)^c$ the claim holds.

A counterexample

If P is a Markov kernel on a separable metric space (E, d), $\rho : E \times E \to [0, \infty)$ is continuous and pos. definite such that for every pair $x, y \in E$ there exists a coupling of $\mathbb{P}_x$ and $\mathbb{P}_y$ s.t. $\rho(X_n, Y_n) \to 0$ a.s., then uniqueness of an ipm does *not* follow.

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Example

Let $I \subset [0, 1]$ s.t. $\lambda^*(I) = 1$ and $\lambda_*(I) = 0$ and $J := [0, 1] \backslash I$.

$$E_1 := \{(z, 1) : z \in I\}, \ E_2 := \{(z, 2) : z \in J\}, \ E := E_1 \cup E_2.$$

$$d(x, y) = \begin{cases} |x - y|, & \text{if } (x, y) \in E_1 \times E_1 \text{ or } (x, y) \in E_2 \times E_2, \\ 1 & \text{if } (x, y) \in E_1 \times E_2 \text{ or } (x, y) \in E_2 \times E_1. \end{cases}$$

(E, d) is separable metric.

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(E, d) is separable metric. $\rho\big((v, i), (w, j)\big) := |v - w|$ is continuous and pos. definite.

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(E, d) is separable metric. $\rho\big((v, i), (w, j)\big) := |v - w|$ is continuous and pos. definite.
We construct a Markov kernel P with at least 2 ipm and, for each $x, y \in E$, a coupling s.t. $\rho(X_n, Y_n) \to 0$ a.s.

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Construction of P

$$\pi_1(A) := \{z \in I : (z, 1) \in A\},$$
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$$\mu(A) = \lambda^*(\pi_1(A)), \nu(A) = \lambda^*(\pi_2(A)), A \in \mathcal{B}(E)$$

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E_2

E

E_1

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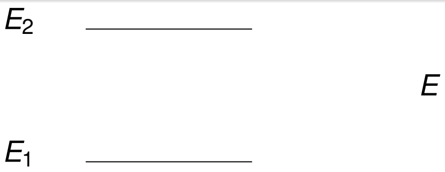

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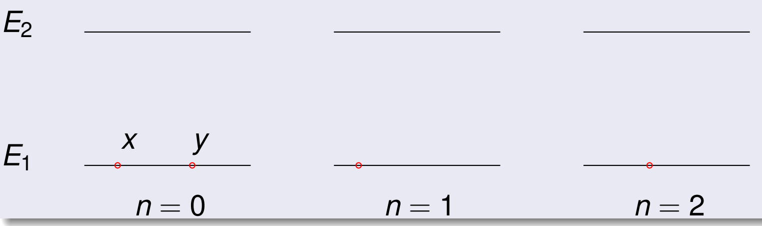
NB: μ and ν are mutually singular ipm. For $x, y \in E$ there is a coupling such that $\rho(X_n, Y_n) \to 0$.

Construction of the coupling

First case: $x, y \in E_1$



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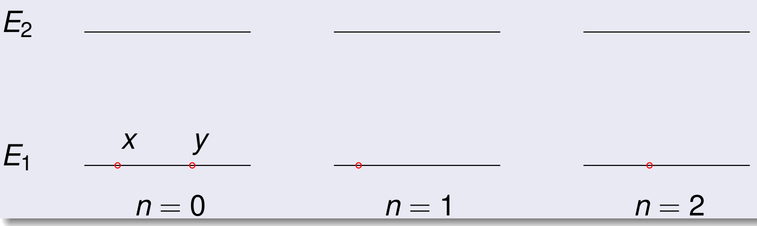
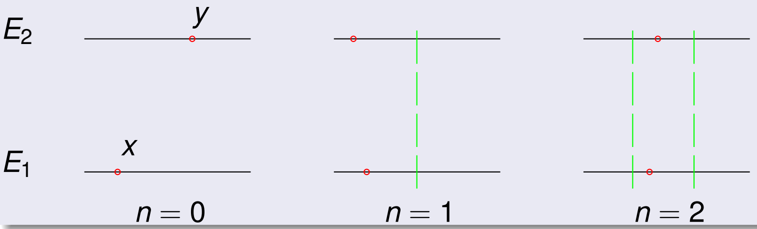


Second case: $x \in E_1$, $y \in E_2$



Uniqueness of weak solution via generalized couplings

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with Hölder cont. f, g and nondegenerate g.

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$$\mathrm{d}Y^\varepsilon(t) = f^\varepsilon(Y_t^\varepsilon)\mathrm{d}t + g^\varepsilon(Y_t^\varepsilon)\mathrm{d}W(t), \quad Y_0^\varepsilon = x$$
$$\mathrm{d}Z^\varepsilon(t) = f^\varepsilon(Z_t^\varepsilon)\mathrm{d}t + g^\varepsilon(Z_t^\varepsilon)\mathrm{d}W(t) + \kappa_\varepsilon\big(X(t) - Z^\varepsilon(t)\big)\mathrm{d}t,\ Z_0^\varepsilon = x.$$

Then show for $T > 0$

- $\sup_{0\le t\le T} |X(t) - Z^\varepsilon(t)| \to 0$ in probability,
- $\mathrm{d}_{TV}\big(\mathcal{L}(Z^\varepsilon), \mathcal{L}(Y^\varepsilon)\big) \to 0.$

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$$dX(t) = f(X_t)\,dt + g(X_t)\,dW(t), \quad X_0 = x \in \mathcal{C} = \mathcal{C}([-1,0],\mathbb{R}^d)$$

with Hölder cont. f, g and nondegenerate g. Let

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- $\sup_{0\le t\le T} |X(t) - Z^\varepsilon(t)| \to 0$ in probability,
- $d_{TV}\big(\mathcal{L}(Z^\varepsilon), \mathcal{L}(Y^\varepsilon)\big) \to 0$.

This implies weak uniqueness.

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Thanks for your attention !