p–Brownian motion and the p–Laplacian

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Barbu/Rehmeier/R: arXiv: 2409.18744v1
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Core example: Heat equation on $\mathbb{R}^d$:

$$\frac{\partial}{\partial t} u(t,x,y) = \Delta_x u(t,x,y),\ (t,x) \in (0,\infty) \times \mathbb{R}^d,$$
$$u(0,x,y) = \delta_y(x)\ (= \text{Dirac measure in } y \in \mathbb{R}^d).$$

Solution: Classical **heat kernel**

$$u(t,x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\, e^{-\frac{1}{4t}|x-y|^2},\ (t,x) \in (0,\infty) \times \mathbb{R}^d.$$

GENERAL **Linear** Parabolic PDE (more precisely: **linear** Fokker-Planck equation)

↕

P R O B A B I L I T Y

Wiener measure $\mathbb{W}_y$ on $C([0,\infty);\mathbb{R}^d)_y$ [Wiener 1923]
For $W(t) : C([0,\infty);\mathbb{R}^d)_y \to \mathbb{R}^d$,
$W(t)(w) := w(t),\ t \geq 0,$

$$\underbrace{(W(t))_*(\mathbb{W}_y)}_{\text{“push forward”}}(\mathrm{d}x) = u(t,x,y)\mathrm{d}x,\ t > 0,$$

$$(W(0))_*(\mathbb{W}_y) = \delta_y$$

$(W(t))_{t\geq 0}, \mathbb{W}_y)_{y\in\mathbb{R}^d}$ “Brownian motion”

Markov process!

linear Markov process (described by SDE)

0. Motivation and longterm programme: Nonlinear case

A N A L Y S I S

Core example: parabolic p–Laplace equation on $\mathbb{R}^d$ with $p > 2$:

$$\frac{\partial}{\partial t} u(t,x,y) = \operatorname{div}(|\nabla u|^{p-2} \nabla u)(t,x,y),\ (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

$$u(0,x,y) = \delta_y(x) \ (= \text{Dirac measure in } y \in \mathbb{R}^d).$$

Solution: **Barenblatt solution**

$$u(t,x,y) = t^{-k}\left(C_1 - q t^{-\frac{kp}{d(p-1)}} |x-y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}},$$

$(t,x) \in (0,\infty) \times \mathbb{R}^d$, where $k := \left(p - 2 + \frac{p}{d}\right)^{-1}$,

$q := \frac{p-2}{p} \left(\frac{k}{d}\right)^{\frac{1}{p-1}}$ and $C_1 > 0$ s.th. $\int_{\mathbb{R}^d} u(t,x,y)\,\mathrm{d}x = 1$.

P R O B A B I L I T Y

Our result: $\exists$ prob. measure P_y on $C([0,\infty);\mathbb{R}^d)_y$ s. th.

$$\underbrace{(X(t))_*(P_y)(\mathrm{d}x)}_{\text{"push forward"}} = u(t,x,y)\,\mathrm{d}x,\ \ t > 0,\ \ (\text{McKean!})$$

where $X = \big(X(t)\big)_{t \geq 0}$ is the solution of

$$\mathbf{d}\boldsymbol{X(t)} = \boldsymbol{\nabla(|\nabla u(t, X(t), y)|^{p-2})}\mathbf{d}\boldsymbol{t}$$

$$+ \boldsymbol{|\nabla u(t, X(t), y)|^{\frac{p-2}{2}}}\mathbf{d}\boldsymbol{W(t)},\ t > 0,\ (X(0))_*(P_y) = \delta_y.$$

$((X(t))_{t \geq 0}, P_y)_{y \in \mathbb{R}^d}$ "p–Brownian motion"

Nonlinear Markov process!

GENERAL **Nonlinear** Parabolic PDE (more precisely: **nonlinear** Fokker-Planck equation)

$\updownarrow$

nonlinear (time-inhomogeneous) Markov process (described by MVSDE)

1. Introduction

Since fundamental work by Einstein [Einstein: Ann.Phys.1905], [von Smoluchowski: Ann.Phys.1906], and [Wiener: J.Math.Phys.1923] the close relationship between **Brownian motion** and the Laplace operator, more precisely the **classical heat equation**,

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{HE}$$

with

$$\Delta u := \operatorname{div} \nabla u$$

is well known.

Open: Is the same true for the p–Laplace operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) \ ?$$

More precisely, is there a "***p*-Brownian motion**" related to the **parabolic *p*-Laplace equation**

$$\frac{\partial u(t,x)}{\partial t} = \Delta_p u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \tag{p–LE}$$

in an analogous way (at least if $p > 2$)?

Here, $\Delta, \operatorname{div}, \nabla$ are spatial differential operators in $x \in \mathbb{R}^d$ and $|\cdot| := |\cdot|_{\mathbb{R}^d}$.

Δ_p extensively studied e.g. in

PDE, see e.g. monographs (and papers): [Ladyzhenskaya/Sollonikov/Uralceva 1988], [Kamin/Vázquez 1991], [DiBenedetto 1993], [Lindqvist 2006], [Lindqvist 2019], ... and the references therein.

Nonlinear Funct. Analysis, see **e.g.** monographs: [J.L. Lions 1969], [Brezis 1983], ...

Nonlinear Potential Theory, see **e.g.** monographs: [Adams/Hedberg 1996], [Heinonen/Kilpeläinen/Martio 1993], [Mingione 2018], ... and the references therein

Applications (Physics, Cimatology), see **e.g.** [Ladyzhenskaya 1967], [Pelissier 1975], etc. ...

Note: **Linear Potential Theory** has played a crucial role in developing and exploiting the relation between the (2-) Laplacian and Brownian motion and, more generally, between large classes of linear partial (and pseudo) differential operators and their associated Markov processes for more than 60 years, see **e.g.** monographs:

[Dynkin 1965], [Stroock/Varadhan 1967], [Blumenthal/Getoor 1968], [Bliedtner/Hansen 1986], [Ethier/Kurtz 1986], [Sharpe 1988], [Freidlin 1996], [Rogers/Williams 2000], [Doob 2001], [Liggett 2010], [Fukushima/Oshima/Takeda 2011], [Kolokoltsov 2011], [Stroock 2014], etc...

So, **non-linear Potential Theory** should play a key rôle to find the p-Brownian motion. First approach in this direction in fundamental papers:

[Peres/Sheffield 2008],
[Peres/Schramm/Sheffield/Wilson 2009],

where a deep relation of a stochastic game, the "tug-of-war" game with noise, was exploited to find a beautiful probabilistic description of the p–harmonic function solving the Dirichlet problem for the p–Laplacian on a bounded domain in $\mathbb{R}^d$.

In arXiv: 2409.18744v1 we propose a different approach, namely to construct the desired probabilistic counterpart to the p–Laplacian as a Markov process which is related to the parabolic p–Laplace equation (p–LE) in the same way as Brownian motion is to the classical heat equation (HE), and which then may be called a "p–Brownian motion".

2. Nonlinear Fokker–Planck equations (FPE) and McKean–Vlasov stochastic differential equations (SDE)

Let $\mathcal{P}(\mathbb{R}^d)$ denote the space of all Borel probability measures on $\mathbb{R}^d$, and for $1 \leq i, j \leq d$ consider measurable maps

$$b_i, a_{ij} : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$$

such that the matrix $(a_{ij})_{i,j}$ is pointwise symmetric and nonnegative definite. Then, a **nonlinear FPE** is an equation of type

$$\frac{\partial}{\partial t}\mu_t = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(a_{ij}(t,x,\mu_t)\mu_t) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i(t,x,\mu_t)\mu_t), \ (t,x) \in (0,\infty) \times \mathbb{R}^d, \ \text{(FPE)}$$

where the solution $[0, \infty) \ni t \mapsto \mu_t$ is a weakly continuous curve in $\mathcal{P}(\mathbb{R}^d)$ with some specified initial condition μ_0.
(FPE) is meant in the weak sense of Schwartz distributions. More precisely,

Definition 1

A **distributional solution** *to* (FPE) *with initial condition* ν *is a weakly continuous curve* $(\mu_t)_{t\geq 0}$ *of signed Borel measures on* $\mathbb{R}^d$ *of bounded variation such that* $(t,x)\mapsto a_{ij}(t,x,\mu_t)$ *and* $(t,x)\mapsto b_i(t,x,\mu_t)$ *are measurable on* $(0,\infty)\times\mathbb{R}^d$,

$$\int_0^T\int_{\mathbb{R}^d}\Big(|a_{ij}(t,x,\mu_t)|+|b_i(t,x,\mu_t)|\Big)\mu_t(dx)dt<\infty,\ \forall T>0,$$

and $\forall t\geq 0$

$$\int_{\mathbb{R}^d}\varphi\,d\mu_t-\int_{\mathbb{R}^d}\varphi\,d\nu=\int_0^t\int_{\mathbb{R}^d}\left(\sum_{i,j=1}^d a_{ij}(s,x,\mu_s)\,\frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_j}\,\varphi(x)+\sum_{i=1}^d b_i(s,x,\mu_s)\,\frac{\partial}{\partial x_i}\,\varphi(x)\right)\mu_s(dx)ds,$$

for all $\varphi\in C_0^\infty(\mathbb{R}^d)$ *and it is called* **probability solution**, *if, in addition,* ν *and each* μ_t, $t\geq 0$, *are in* $\mathcal{P}(\mathbb{R}^d)$.

The (in space) dual operator to the operator on the right hand side of (FPE) is called the corresponding Kolmogorov operator L_μ, i.e. its action on test functions $\varphi \in C_0^\infty(\mathbb{R}^d)$ is given as

$$L_{\mu_t}\varphi(t,x) = \sum_{i,j=1}^{d} a_{ij}(t,x,\mu_t)\frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_j}\,\varphi(x) + \sum_{i=1}^{d} b_i(t,x,\mu_t)\frac{\partial}{\partial x_i}\,\varphi(x), \qquad \text{(K)}$$

where $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

In turn, this operator determines the corresponding McKean–Vlasov SDE

$$dX(t) = b(t, X(t), \mu_t)dt + \sigma(t, X(t), \mu_t)dW(t),\ \ t > 0, \qquad \text{(MVSDEa)}$$

$$\mathcal{L}_{X(t)} = \mu_t,\ \ t \geq 0, \qquad \text{(MVSDEb)}$$

where $\sigma = (\sigma_{ij})_{ij}$ with $\sigma\sigma^\top = (a_{ij})_{ij}$, $b = (b_1, \ldots, b_d)$, $W(t)$, $t \geq 0$, is a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the maps $X(t) : \Omega \to \mathbb{R}^d$, $t \geq 0$, form the continuous in t solution process to (MVSDEa) such that its one-dimensional time marginals

$$\boldsymbol{\mathcal{L}_{X(t)} := (X(t))_*\mathbb{P},\ \ t \geq 0,}$$

i.e. the push forward or image measures of $\mathbb{P}$ under $X(t)$, satisfy (MVSDEb).

Obviously, the special case of the classical heat equation and classical Brownian motion is the case where $a_{ij}(t,x,\mu) = \delta_{ij}$ ($=$ Kronecker delta), $b_i(t,x,\mu) = 0$, i.e. (FPE) turns into

$$\frac{\partial}{\partial t}\mu_t = \Delta\mu_t,\ (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{HE'}$$

and (MVSDEa,b) into

$$dX(t) = dW(t),\ t > 0, \tag{BMa}$$
$$\mathcal{L}_{X(t)} = \mathcal{L}_{W(t)} = \mu_t,\ t \geq 0, \tag{BMb}$$

where, of course, each μ_t is absolutely continuous with respect to Lebesgue measure dx on $\mathbb{R}^d$ with density $u(t,x)$, so (HE') is really (HE).

Correspondence: McKean–Vlasov SDE $\longleftrightarrow$ nonlinear FPE

a) McKean–Vlasov SDE $\longrightarrow$ nonlinear FPE:
Consider (MVSDEa,b) and assume there exists a solution. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then by Itô's formula, since $\mu_t = (X(t))_*\mathbb{P}$, $t \geq 0$,

$$\int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) = \int_\Omega \varphi(X(t)(\omega))\,\mathbb{P}(\mathrm{d}\omega)$$

$$\underset{\text{Itô}}{=} \int_\Omega \varphi(X(0)(\omega))\,\mathbb{P}(\mathrm{d}\omega) + \int_\Omega \int_0^t L_{\mathcal{L}_{X(s)}}\varphi(X(s)(\omega))\,\mathrm{d}s\,\mathbb{P}(\mathrm{d}\omega)$$

$$= \int_{\mathbb{R}^d} \varphi(x)\mu_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^d} L_{\mu_s}\varphi(s,x)\mu_s(\mathrm{d}x)\mathrm{d}s$$

Hence $(\mu_t)_{t\geq 0}$ is a **distributional solution** of (FPE), more precisely a **probability solution**.

b) Nonlinear FPE $\longrightarrow$ McKean–Vlasov SDE:

Theorem 0 ([Barbu/R: SIAM 2018, AOP 2020])

Assume there exists a probability solution $[0,\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)$ *of* (FPE) *such that:*

(i) *For all* $T > 0$ *and* $1 \le i, j \le d$
- $a_{ij}, b_i \in L^1([0,T] \times U, \mu_t \mathrm{d}t)$ *for every ball* $U \subset \mathbb{R}^d$,
- $\displaystyle\int_0^T \int_{\mathbb{R}^d} \frac{|a_{ij}(t,x,\mu_t)| + |\langle x, b_i(t,x,\mu_t)\rangle|}{1+|x|^2}\, \mu_t(\mathrm{d}x)\, \mathrm{d}t < \infty$

(ii) $[0,\infty) \ni t \mapsto \mu_t$ *is weakly continuous.*

Then there exists a d-dimensional $(\mathcal{F}_t)$*-Brownian motion* $W(t)$, $t \ge 0$, *on a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ *and a continuous* $(\mathcal{F}_t)$*-progressively measurable map* $X : [0,\infty) \times \Omega \to \mathbb{R}^d$ *satisfying (MVSDEa,b).*

Remark

b, σ assumed to be **only measurable** in measure variable !

Literature on MVSDEs: Huge! See **e.g.** [Carmona/Delarue: Vol. I + II, Springer 2018] and the reference therein. Mostly, b, σ assumed to be **weakly continuous** in the measure variable μ. And: [Barbu/R: Springer LN 2024].

Literature on FPEs: Huge! See **e.g.** [Bogachev/Krylov/R/Shaposhnikov: AMS Monograph 2015], [Barbu/R: Springer LN 2024] and the references therein..

3. Key Step 1: Identifying the parabolic p–Laplace equation as a nonlinear FPE

Recall: Coefficients in FPE only need to be measurable in μ. So, if for the solutions μ_t, $t \geq 0$, we have $\mu_t(dx) = u(t,x)dx$, $t > 0$, we can allow dependencies as

$$\begin{aligned} a_{ij}(t,x,\mu_t) &= \tilde{a}_{ij}(t,x,\Gamma_1(u)(t,x)), \\ b_i(t,x,\mu_t) &= \tilde{b}_i(t,x,\Gamma_2(u)(t,x)), \end{aligned} \qquad (*)$$

where $\tilde{b}_i, \tilde{a}_{ij} : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ are measurable and each Γ_i is a functional on the space of distributional solutions whose values are again measurable functions of t and x. Noting that

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}(\nabla(|\nabla u|^{p-2}u) - \nabla(|\nabla u|^{p-2})u),$$

we can rewrite (p–LE) as

$$\frac{\partial}{\partial t} u(t,x) = \Delta(|\nabla u(t,x)|^{p-2}u(t,x)) - \mathrm{div}(\nabla(|\nabla u(t,x)|^{p-2})u(t,x)),\ (t,x) \in (0,\infty)\times\mathbb{R}^d. \qquad (p\text{-LE}')$$

Hence we see that (p–LE), respectively (p-LE'), is of type (FPE) with a_{ij}, b_i as in $(*)$, where

$$\begin{aligned} \tilde{a}_{ij}(t,x,\Gamma_1(u)(t,x)) &= \quad \delta_{ij}|\nabla u(t,x)|^{p-2}, \\ \tilde{b}(t,x,\Gamma_2(u)(t,x)) &= \quad \nabla(|\nabla u(t,x)|^{p-2}). \end{aligned} \qquad (**)$$

4. Key Step 2: Solving the corresponding McKean–Vlasov SDE

The recipe to find the corresponding McKean–Vlasov SDE was already explained in Section 2, namely consider the associated Kolmogorov operator

$$L_u\varphi(t,x) = |\nabla u(t,x)|^{p-2}\Delta\varphi(x) + \nabla(|\nabla u(t,x)|^{p-2})\cdot\nabla\varphi(x),\ (t,x)\in(0,\infty)\times\mathbb{R}^d,$$

$\varphi \in C_0^\infty(\mathbb{R}^d)$, and then the corresponding McKean–Vlasov SDE is given by

$$dX(t) = \nabla(|\nabla u(t,X(t))|^{p-2})dt + |\nabla u(t,X(t))|^{\frac{p-2}{2}}dW(t),\ t>0, \qquad \text{(MVSDEa')}$$

$$\mathcal{L}_{X(t)} = u(t,x)dx,\ t>0. \qquad \text{(MVSDEb')}$$

Theorem I (arXiv: 2409.18744v1)

Let u be a probability solution to (p-LE') with initial condition $\nu \in \mathcal{P}(\mathbb{R}^d)$ in the sense of Definition 1 such that

$$|\nabla u|^{p-2} \in L^1_{loc}((0,\infty); W^{1,1}_{loc}(\mathbb{R}^d)).$$

and

$$\int_0^T \int_{\mathbb{R}^d} \left(|\nabla u|^{p-2} + |\nabla(|\nabla u|^{p-2})|\right) u \, dxdt < \infty, \quad \forall T > 0.$$

Then there exists a (probabilistically weak) solution $X = (X(t))_{t \geq 0}$ to the McKean–Vlasov SDE (MVSDEa',b') such that $(X(0))_* \mathbb{P} = \nu$.

Proof. Apply [Trevisan: EJP 2016] in the same way as in the proof of Theorem 0 in [Barbu/R: AOP 2020].

Remark

The already challenging so-called **Nemytskii-case**, *where $\Gamma_i(u)(t,x) = u(t,x)$ in $(*)$ has received more and more attention in the last years (see, for instance, [Barbu/R: Springer LN 2024] and the references therein). The coefficients in (p-LE'), however, even depend on u via its first- and second-order derivatives. To the best of our knowledge, the relation of such nonlinear FPEs to McKean–Vlasov SDEs has not been studied before.*

5. Key Step 3: The corresponding nonlinear Markov process: p-Brownian motion

To obtain the nonlinear Markov process from (MVSDEa',b'), we need to take into account initial conditions, which we choose to be Dirac measures δ_y, $y \in \mathbb{R}^d$. So, we impose in (p-LE') that $u(0,x)dx = \delta_y$ and in (MVSDEb') that $\mathcal{L}_{X(0)} = \delta_y$. In both cases $p = 2$ and $p > 2$, for such initial conditions the distributional solution to (HE) and (p-LE') are explicitly known, namely in case $p = 2$, for $y \in \mathbb{R}^d$ it is given by the classical **Gaussian heat kernel**

$$u^y(t,x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{1}{4t}|x-y|^2\right), \; (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

and in case $p > 2$ by the famous **Barenblatt solution** (see [Kamin/Vázquez 1988])

$$w^y(t,x) := t^{-k}\left(C_1 - qt^{-\frac{kp}{d(p-1)}}|x-y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \; (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

where $k := \left(p-2+\frac{p}{d}\right)^{-1}$, $q := \frac{p-2}{p}\left(\frac{k}{d}\right)^{\frac{1}{p-1}}$, $C_1 \in (0,\infty)$ such that $\int_{\mathbb{R}^d} w^y(t,x)dx = 1$ for all $t > 0$, and $f_+ := \max(f,0)$. Then, we consider the **path laws** of the corresponding solutions $X^y(t)$, $t \geq 0$, of (BMa,b), respectively, (MVSDEa',b') with w^y replacing u, namely

$$\boldsymbol{P_y := (X^y)_*\mathbb{P},\; y \in \mathbb{R}^d,} \qquad \textbf{(MP)}$$

i.e. the push forward or image measure of $\mathbb{P}$ under the map $X^y : \Omega \to C([0,\infty),\mathbb{R}^d)$ (= all continuous paths in $\mathbb{R}^d$).

Classical case $p = 2$:

P_y, $y \in \mathbb{R}^d$, form a Markov process in the sense of e.g. [Dynkin 1965], also called **Brownian motion**, which is uniquely determined by $u^y(t)$, $y \in \mathbb{R}^d$, $t > 0$.

Our result in [arxiv:2409.18744v1]: For $d \geq 2$, $\boldsymbol{p > 2\left(1 + \frac{1}{d}\right)}$:
$P_y, y \in \mathbb{R}^d$, form a nonlinear Markov process in the sense of McKean: [PNAS 1966], which is uniquely determined by $w^y(t, \cdot), y \in \mathbb{R}^d, t > 0$, and (MVSDEa',b'), hence may be called ***p*–Brownian motion**.

To state the latter more precisely in Theorem II below, we define for $0 \leq r \leq t$:

- $C([r, \infty), \mathbb{R}^d) :=$ space of all continuous paths in $\mathbb{R}^d$ starting at r, equipped with its Borel σ-algebra $\mathcal{B}(C([r, \infty), \mathbb{R}^d))$.
- $\pi_t^r : C([r, \infty), \mathbb{R}^d) \to \mathbb{R}^d$, $\pi_t^r(w) := w(t)$; $\mathcal{F}_r := \sigma(\{\pi_s^0 : 0 \leq s \leq r\})$.
- $P_{r, w^y(r, \cdot)} := (X^{r, w^y(r, \cdot)})_* \, \mathbb{P} =$ **path law of** $X^{r, w^y(r, \cdot)}$ on $C([r, \infty), \mathbb{R}^d)$, where $X^{r, w^y(r, \cdot)}$ for $r > 0$ is the unique solution of the McKean–Vlasov SDE

$$dX(t) = \nabla(|\nabla w^y(t, X(t))|^{p-2})dt + |\nabla w^y(t, X(t))|^{\frac{p-2}{2}} dW(t), \quad t \geq r,$$
$$\mathcal{L}_{X(t)} = w^y(t, x)dx, \; t \geq r.$$

Theorem II (arXiv: 2409.18744v1)

Let $d \geq 2$, $p > 2(1 + \frac{1}{d})$. Then $P_y, y \in \mathbb{R}^d$, from (**MP**) *above satisfy the* **nonlinear Markov property** *of [McKean: PNAS 1966], i.e. for all $y \in \mathbb{R}^d, 0 < r \leq t$, $A \in \mathcal{B}(\mathbb{R}^d)$*

$$P_y\left[\pi_t^0 \in A | \mathcal{F}_r\right](w) = p_{y,(r,\pi_r^0(w))}[\pi_t^r \in A] \text{ for } P_y - a.e. \ w \in C([0,\infty), \mathbb{R}^d),$$

where $p_{y,(r,z)}$, $z \in \mathbb{R}^d$, is a regular conditional probability kernel from $\mathbb{R}^d$ to $\mathcal{B}(C([r,\infty),\mathbb{R}^d))$ of

$$P_{r,w^y(r,\cdot)}\left[\ \cdot\ \middle| \pi_r^r = z\right], \quad z \in \mathbb{R}^d,$$

(i.e., in particular, $p_{y,(r,z)}$ is a probability measure on $C([r,\infty),\mathbb{R}^d)$ and $p_{y,(r,z)}[\pi_r^r = z] = 1$ for all $z \in \mathbb{R}^d$).

Definition 2

Let $d \geq 2$, $p > 2(1 + \frac{1}{d})$. We call the family $(P_y)_{y \in \mathbb{R}^d}$ of probability measures on $C([0,\infty),\mathbb{R}^d)$ from (**MP**)*, which form a nonlinear Markov process in the sense of McKean, by analogy to the case $p = 2$, a* ***p*-Brownian motion**.

The main ingredient of the proof of Theorem II, which is quite involved, is a restricted linearized uniqueness result presented in the next section.

6. Restricted linearized uniqueness

For $\delta \in (0,\infty)$ set

$$w_\delta(t,x) := w^0(t+\delta,x), \quad \varrho_\delta(t,x) := |\nabla w_\delta(t)|^{p-2}(x),\ (t,x) \in [0,\infty) \times \mathbb{R}^d,$$

and consider the linearized version of equation (FPE)

$$\frac{d}{dt}\, u = \Delta\big(\varrho_\delta u\big) - \operatorname{div}\big(\nabla \varrho_\delta u\big),\ t > 0, \tag{ℓFPE}$$

which is a linear Fokker–Planck equation obtained by a priori fixing the coefficients ϱ_δ and $\nabla\varrho_\delta$ in place of $|\nabla u|^{p-2}$ and $\nabla(|\nabla u|^{p-2})$ in the nonlinear equation (FPE). Distributional and probability solutions to (ℓFPE) are defined analogously to Definition 1.

Remark 1

(i) *w_δ is a probability solution to (ℓFPE) with initial condition $w_0(\delta,x)dx$ and $w_\delta \in \big(L^1 \cap L^\infty\big)\big((0,T) \times \mathbb{R}^d\big)$ for all $T > 0$.*

(ii) *Since $w_\delta(t,\cdot)$ has support in a ball in $\mathbb{R}^d$ and (ℓFPE) is considered on all of $\mathbb{R}^d$, (ℓFPE) is a highly degenerate linear PDE. This makes the proof of Theorem III below complicated.*

Theorem III (“restricted linearized uniqueness after time $\delta \in (0, \infty)$”)

Let $d \geq 2$, $p > 2(1 + \frac{1}{d})$, $\delta, T \in (0, \infty)$. Let $[0, T] \ni t \mapsto \nu_t \in \mathcal{P}(\mathbb{R}^d)$ be a probability solution of (ℓFPE) *such that*

(i) $\nu_0(dx) = w_\delta(0, x)dx$, $\nu_t(dx) = v(t, x)dx$ *for a.e.* $t \in (0, T)$,

(ii) $\exists C \in (0, \infty)$ *such that* $v \leq C\, w_\delta$ *dtdx–a.e. on* $(0, T) \times \mathbb{R}^d$.

Then $v = w_\delta$ *dtdx-a.e. on* $(0, T) \times \mathbb{R}^d$.

Proof.

By PDE–methods. (Hard!) The special form of w_δ, ϱ_δ and $\nabla \varrho_\delta$ is heavily used. □