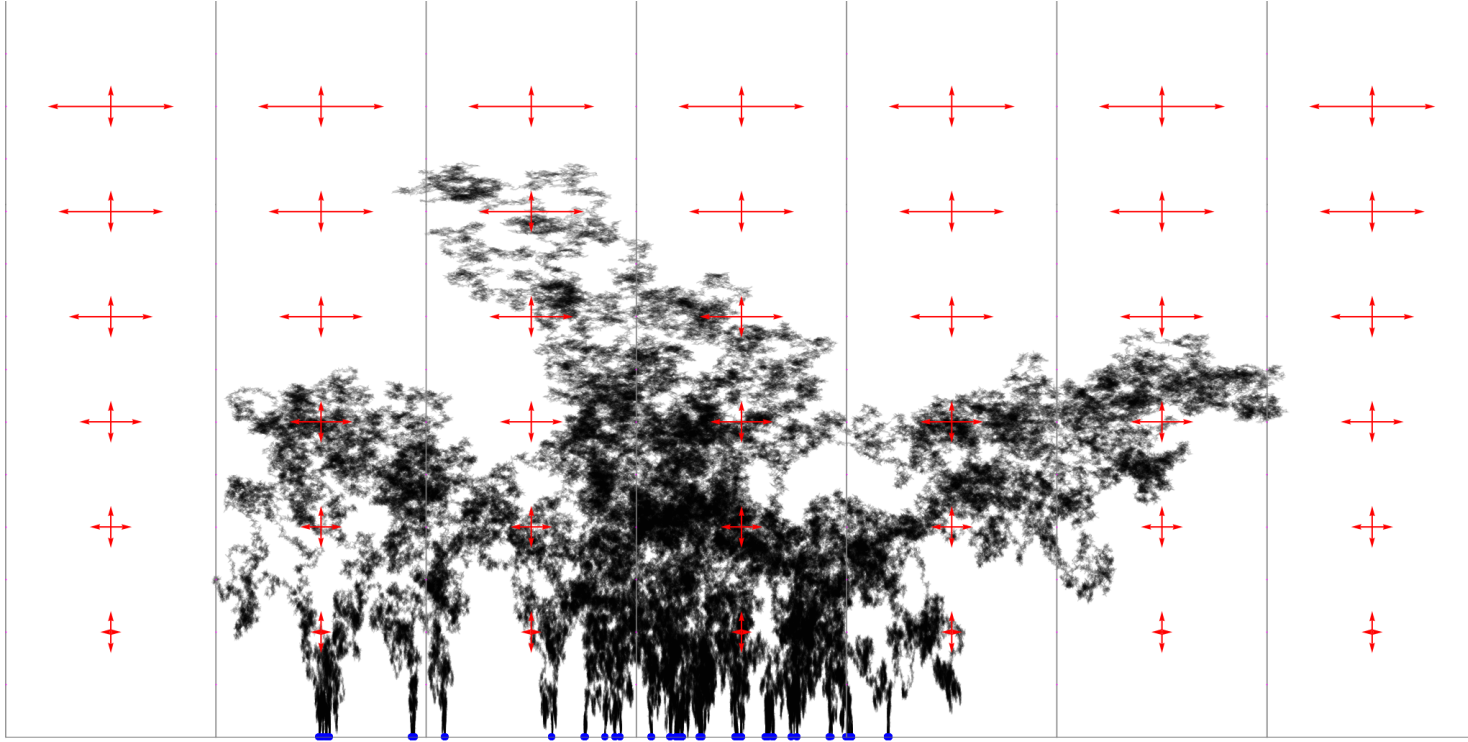




Boundary traces of diffusions

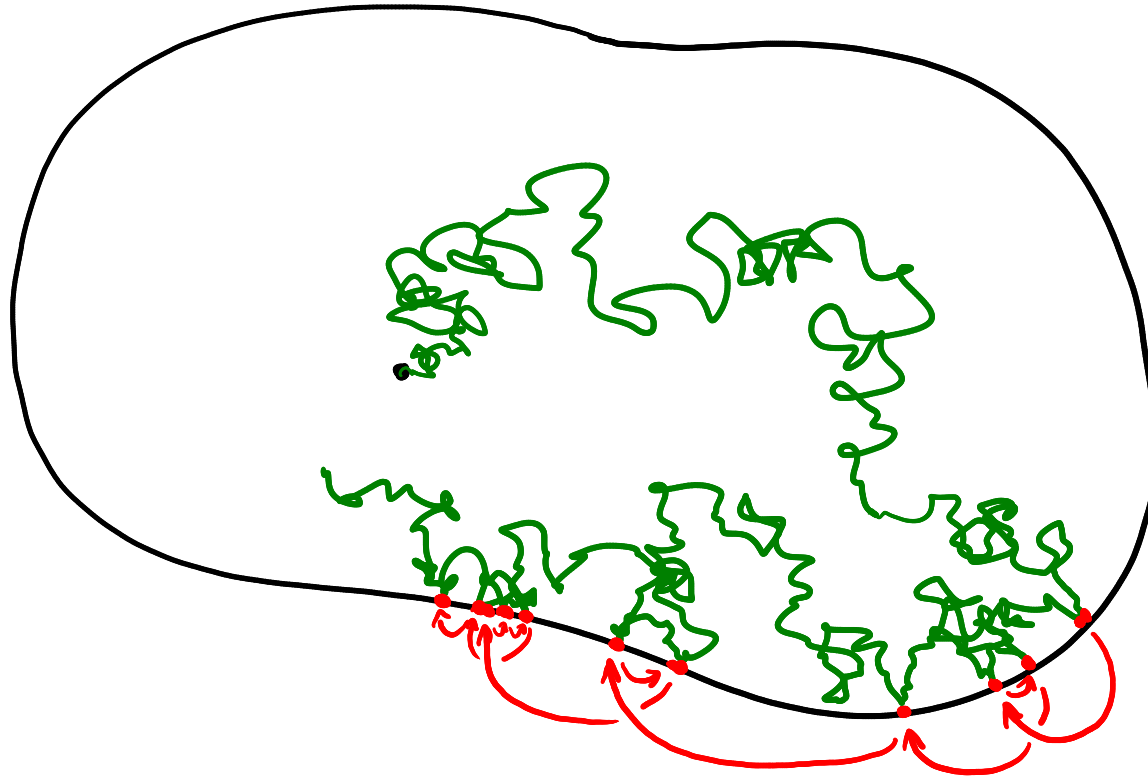
Mateusz Kwaśnicki

(Wrocław University of Science and Technology)

International Seminar on SDEs and Related Topics, October 17, 2025

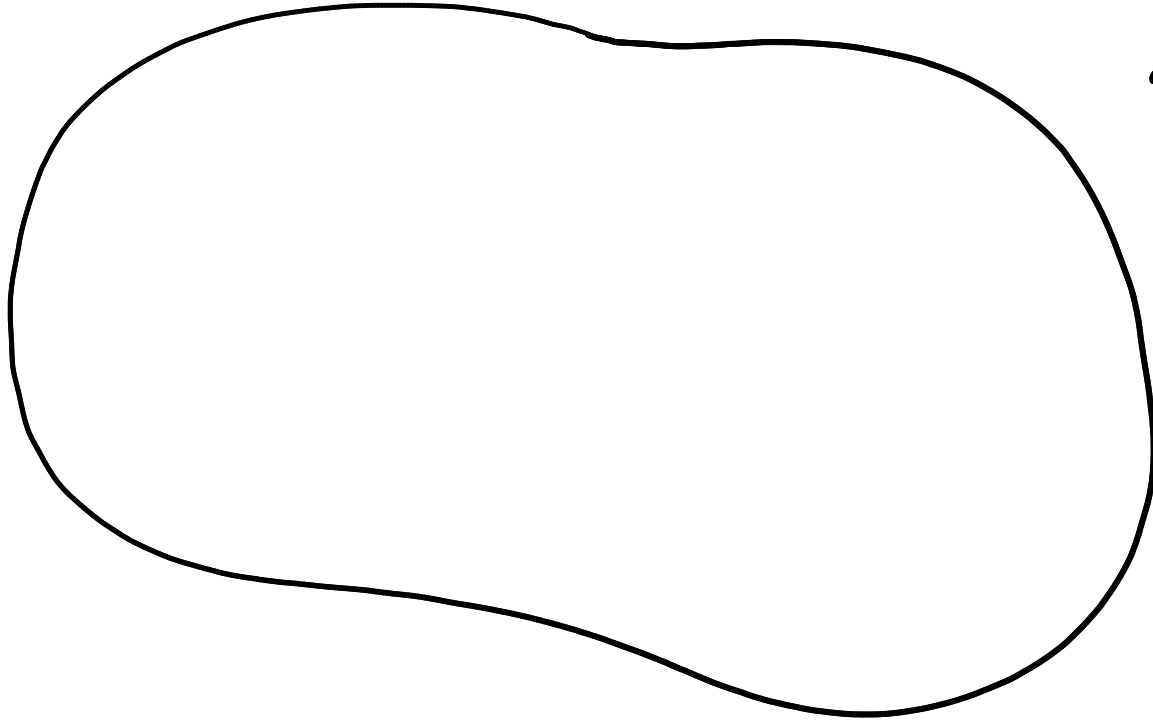
Outline

1. What is the "boundary trace"?
2. Dirichlet-to-Neumann maps
3. Toy (?) model: graphs
4. Molchanov-Ostrovski & Caffarelli-Silvestre
5. Half-plane: shift-invariant diffusions
6. Bell-shape



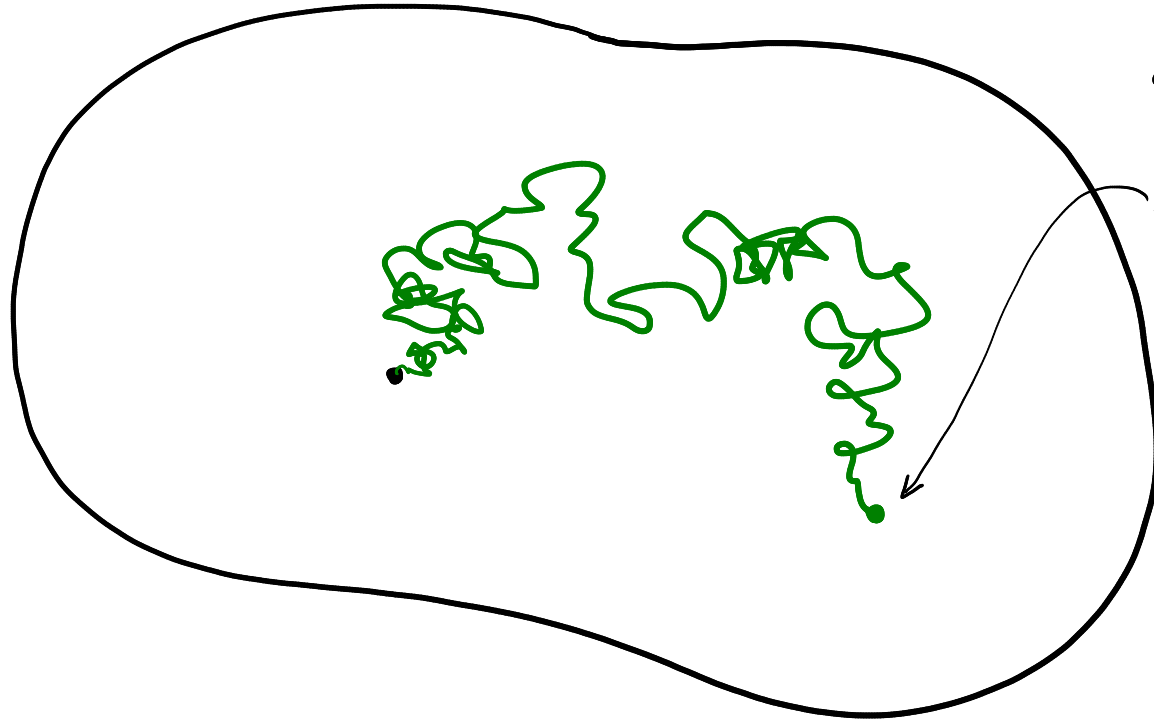
1. What is the "boundary trace"?

Boundary trace



D - a planar domain

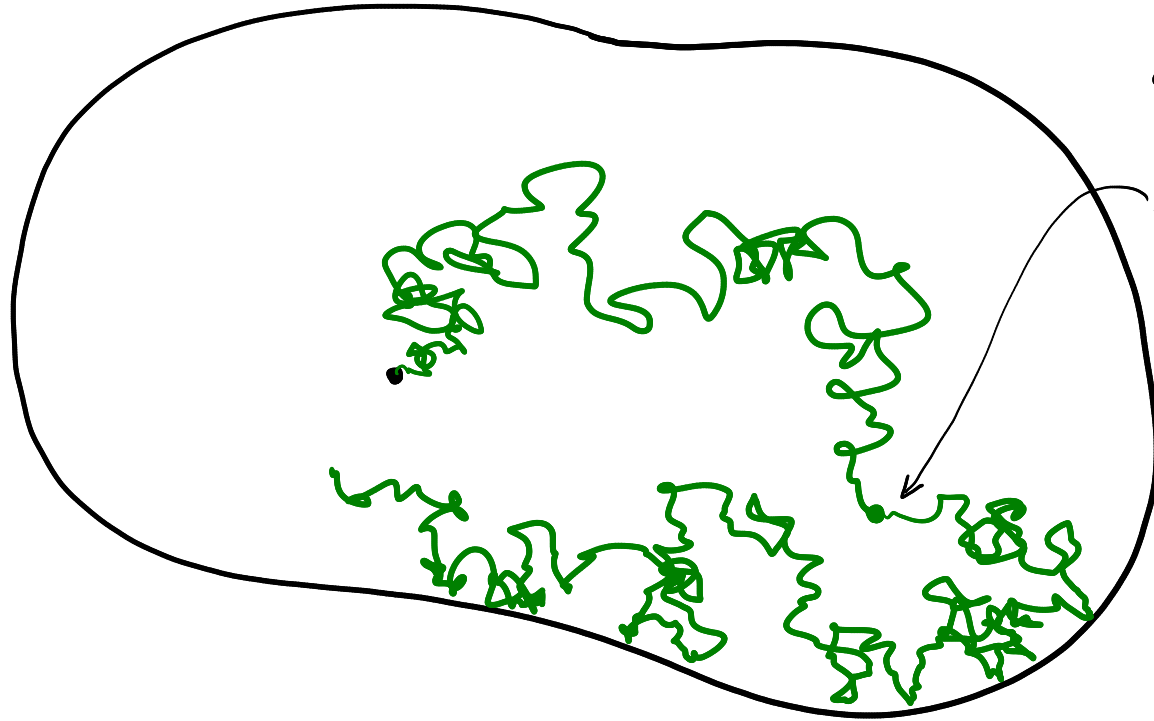
Boundary trace



D - a planar domain

X_t - a diffusion

Boundary trace



D - a planar domain

X_t - a diffusion
reflected at ∂D

Boundary trace



D - a planar domain

X_t - a diffusion
reflected at ∂D

Z - contact points at ∂D

Boundary trace



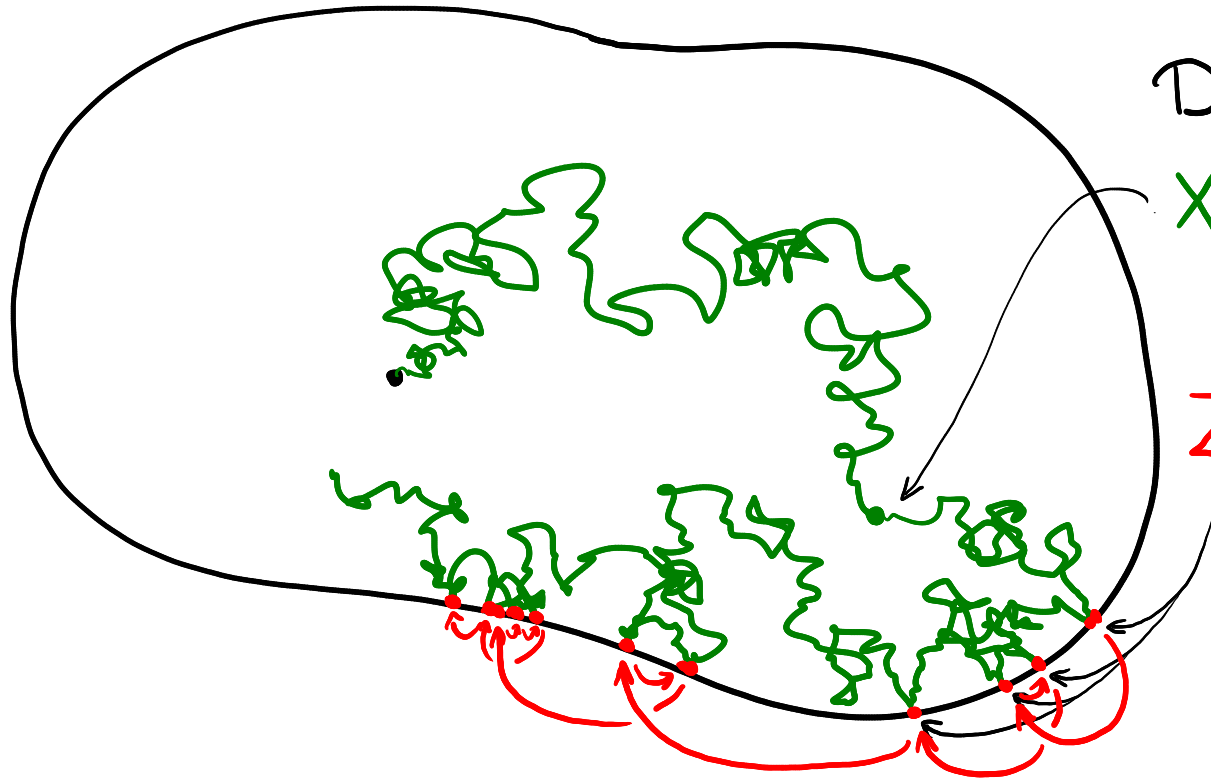
D - a planar domain

X_t - a diffusion
reflected at ∂D

Z - contact points at ∂D

Goal Describe Z in terms of X_t

Boundary trace process



D - a planar domain

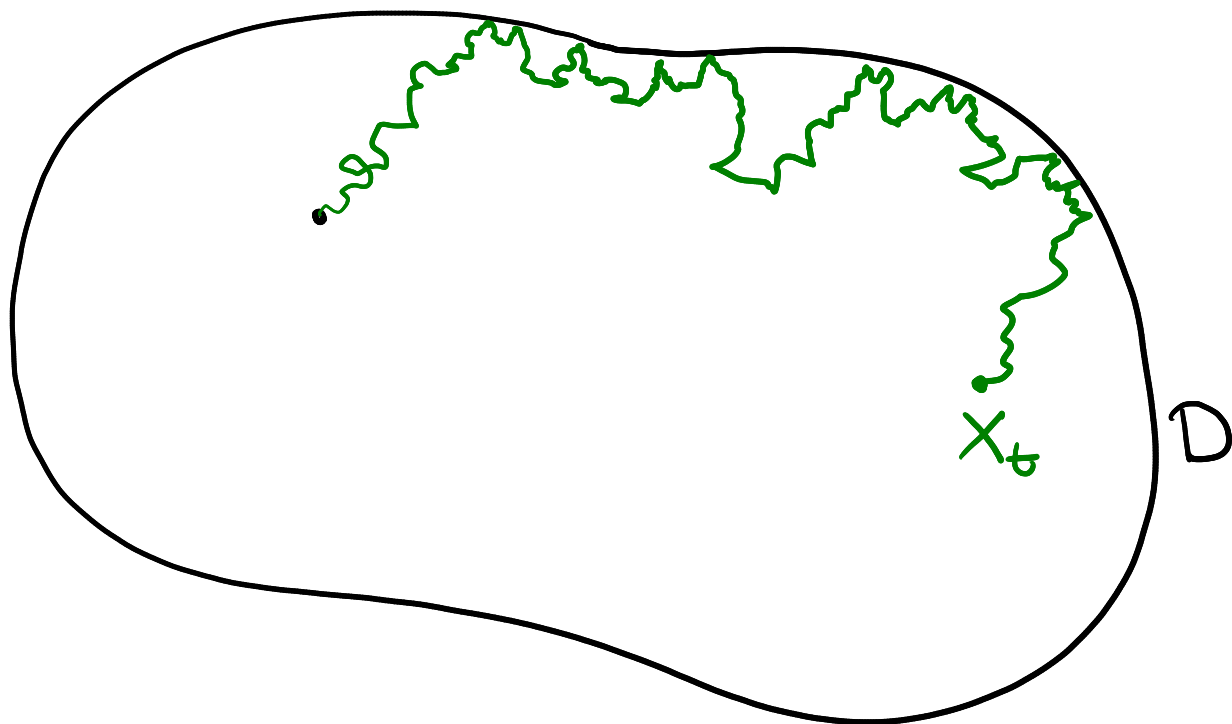
X_t - a diffusion
reflected at ∂D

Z_s - contact points at ∂D

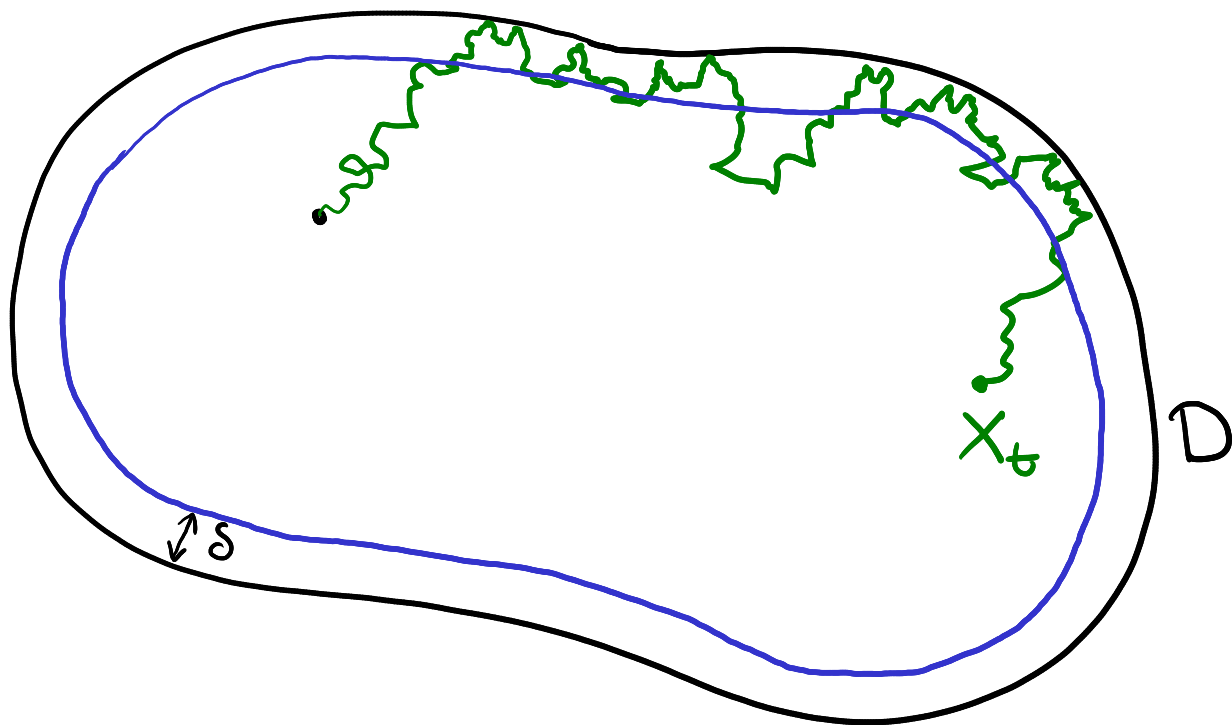
Z is a perfect set
turned into a process
using the local time

Goal Describe Z_s in terms of X_t

Local time

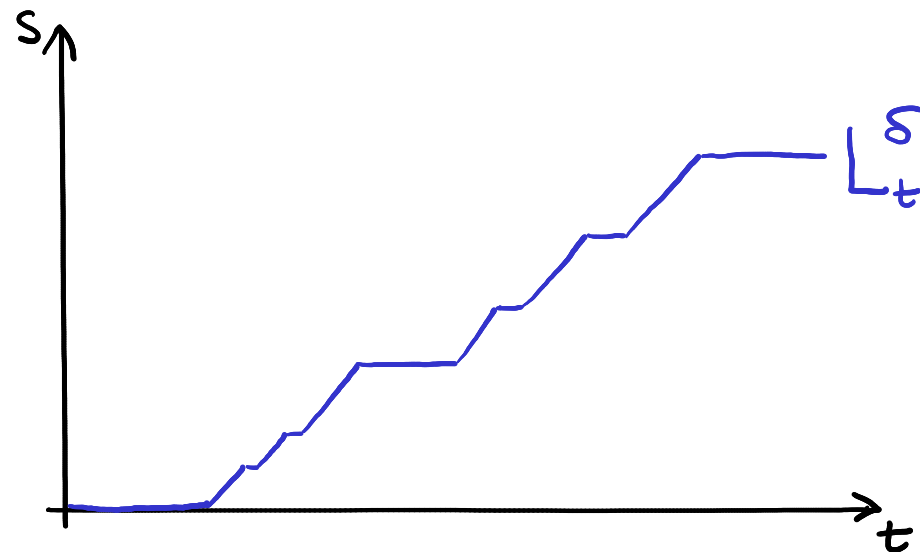


Local time

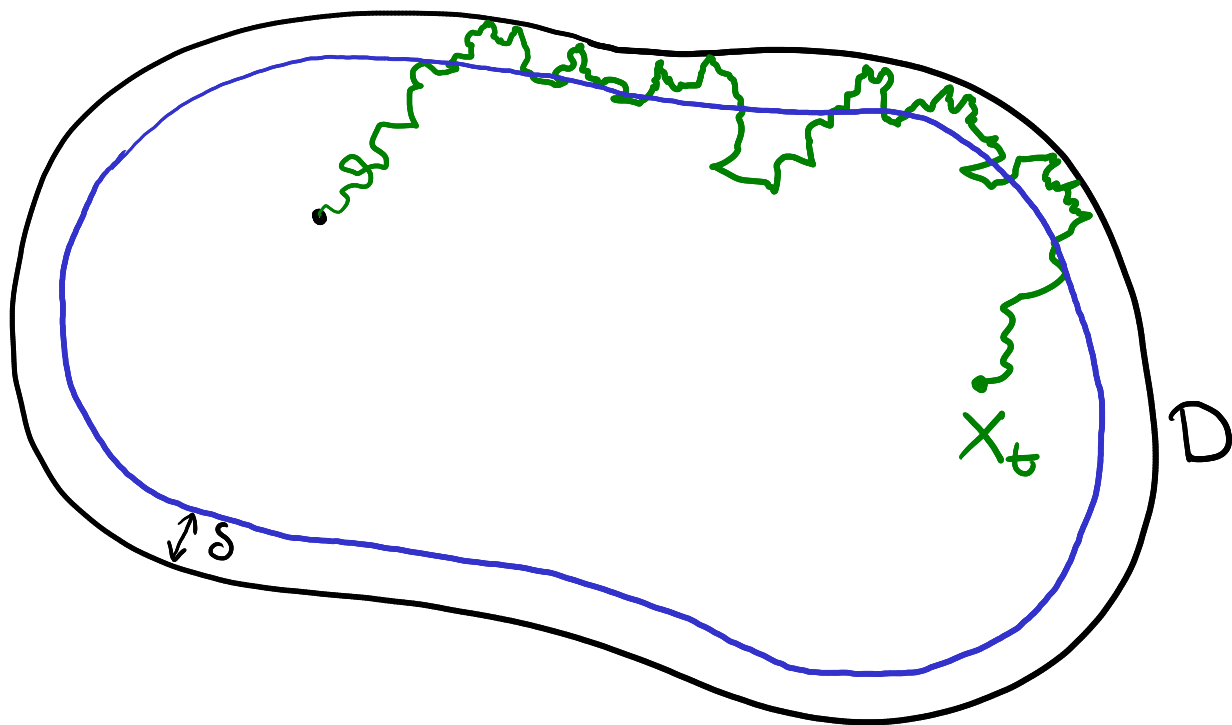


$$D^\delta = \{x \in D : \text{dist}(x, \partial D) < \delta\}$$

$$L_T^\delta = \int_0^T \mathbb{1}_{D^\delta}(X_t) dt$$



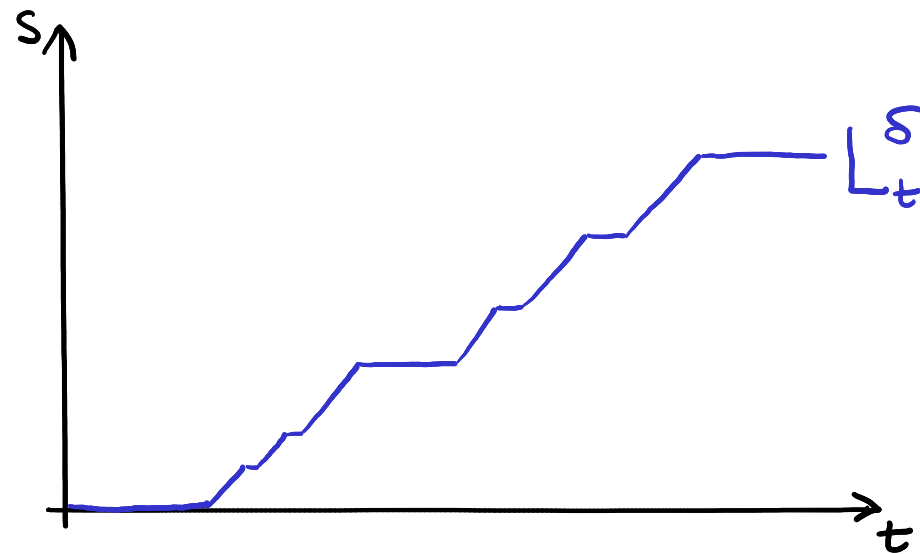
Local time



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$$L_T = \lim_{\delta \rightarrow 0^+} L_T^\delta$$



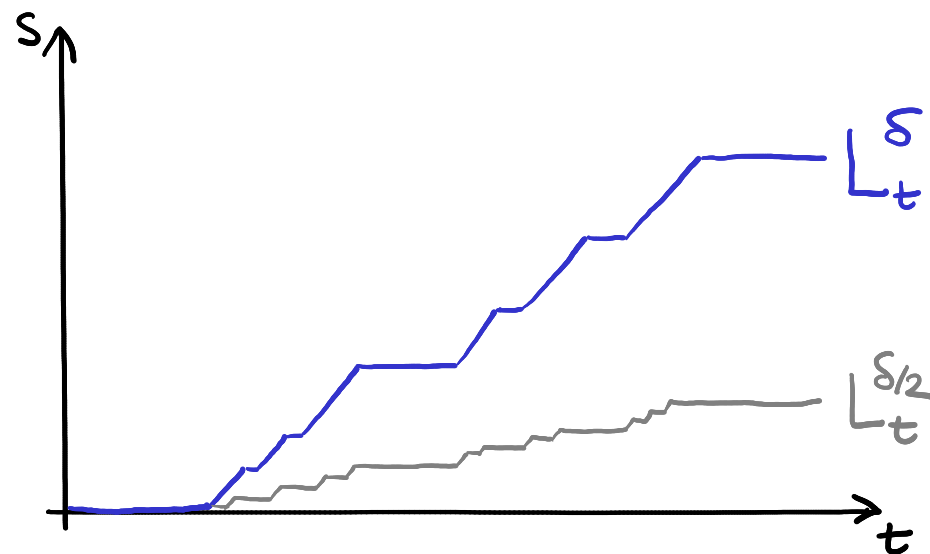
Local time



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$$L_T = \lim_{\delta \rightarrow 0^+} L_T^\delta = 0 \quad \ddot{\smile}$$



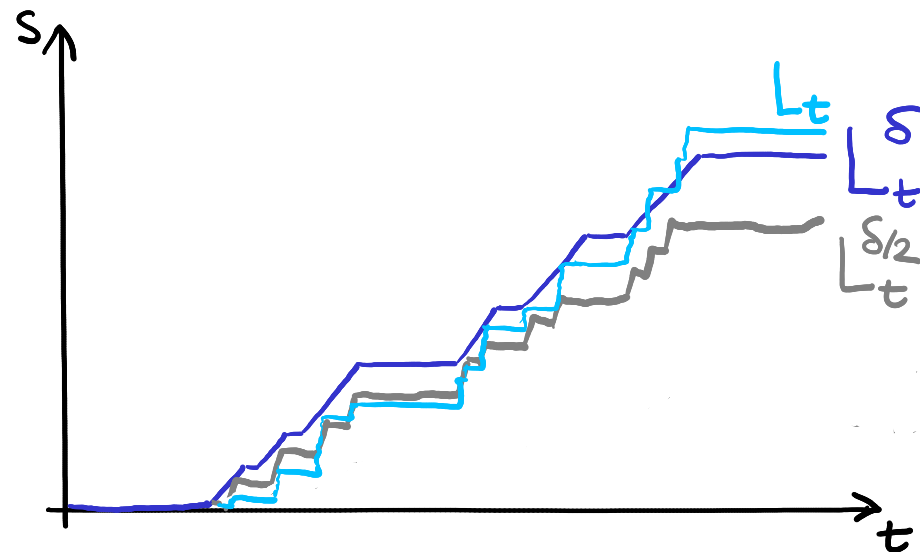
Local time



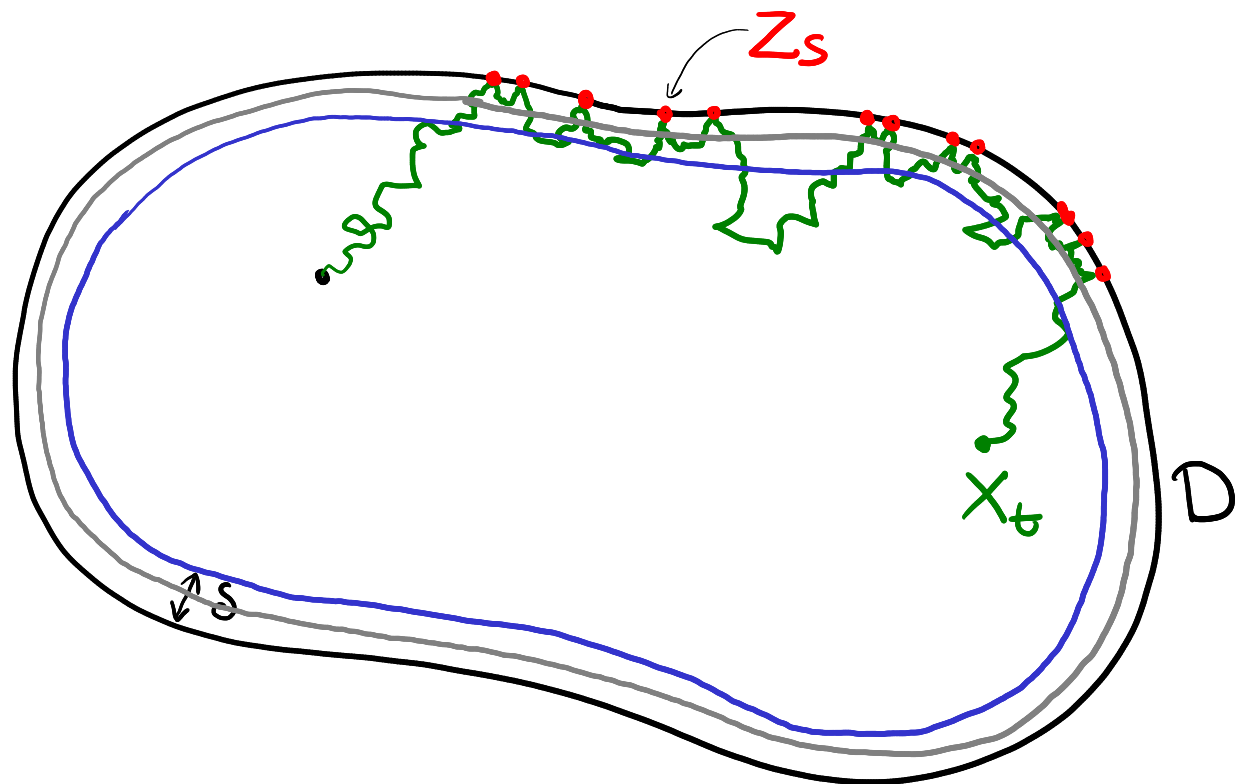
$$D^\delta = \{x \in D : \text{dist}(x, \partial D) < \delta\}$$

$$L_T^\delta = \frac{1}{\delta} \int_0^T \mathbb{1}_{D^\delta}(X_t) dt$$

$$L_T = \lim_{\delta \rightarrow 0^+} L_T^\delta \quad \text{nontrivial!}$$



Local time

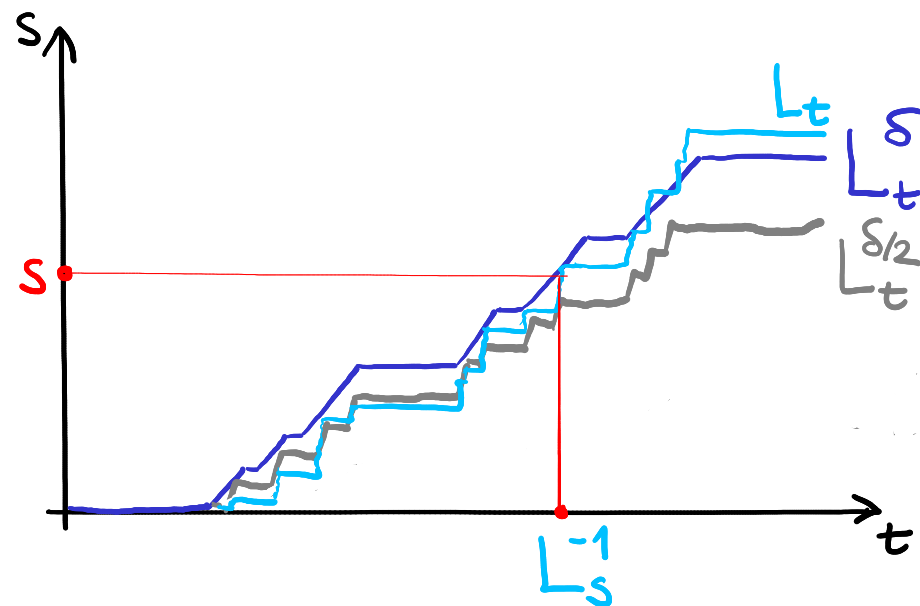


$$Z_s = X_{L_s^{-1}}$$

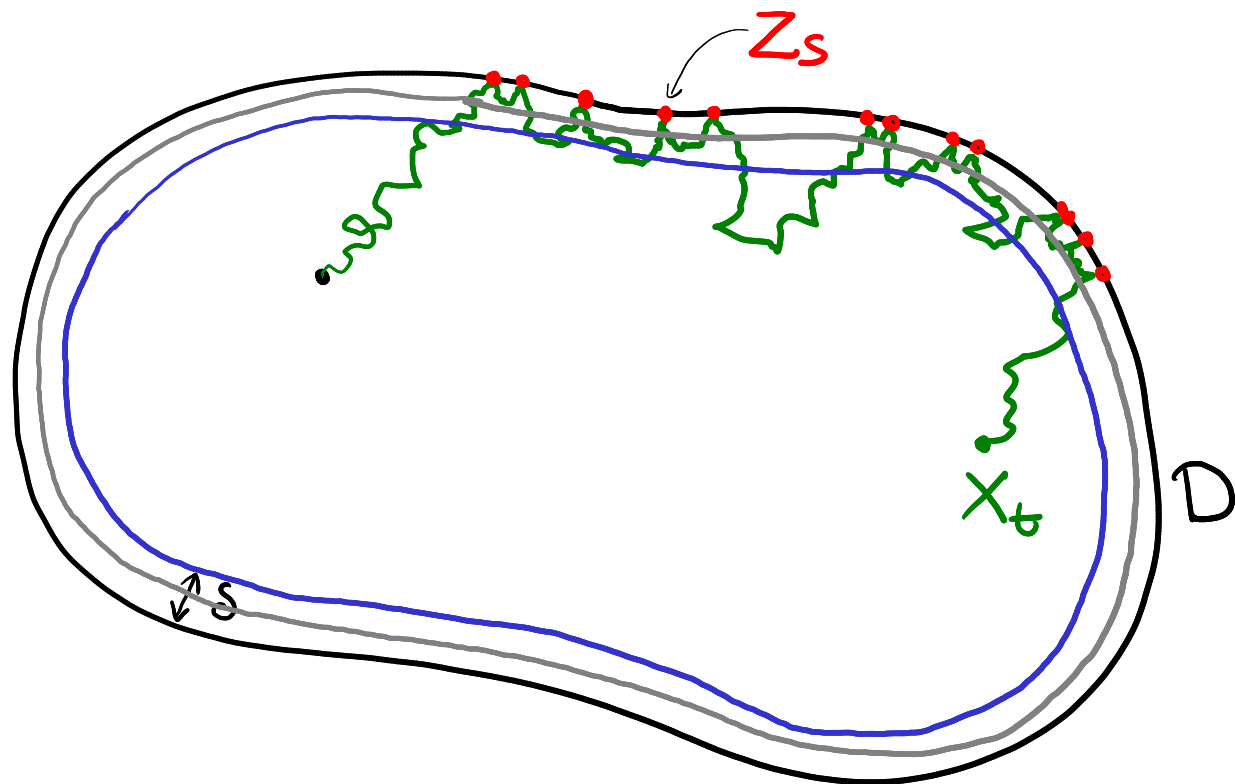
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Local time

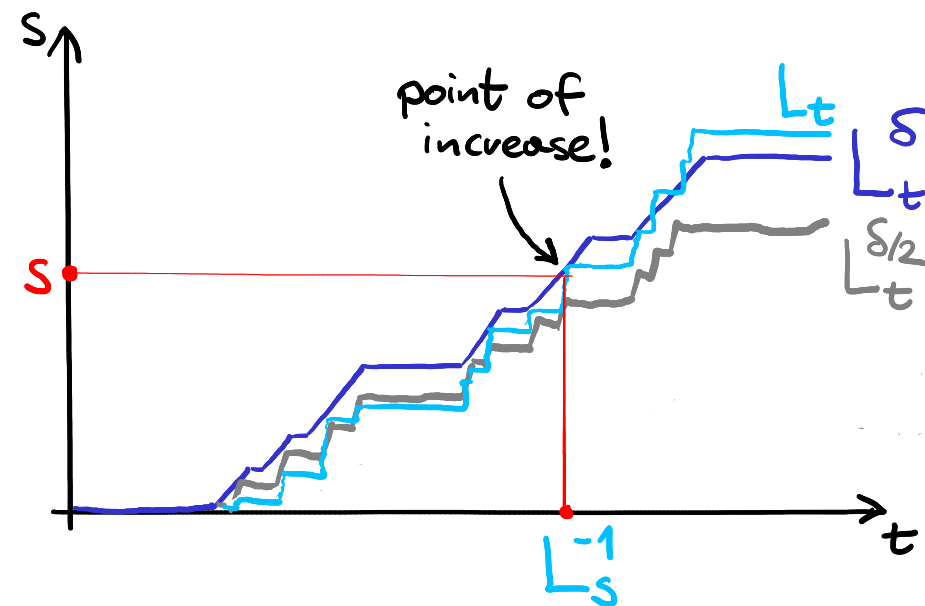


$$Z_s = X_{L_s^{-1}} \in \partial D, \text{ because } L_t \text{ increases at } t = L_s^{-1}$$

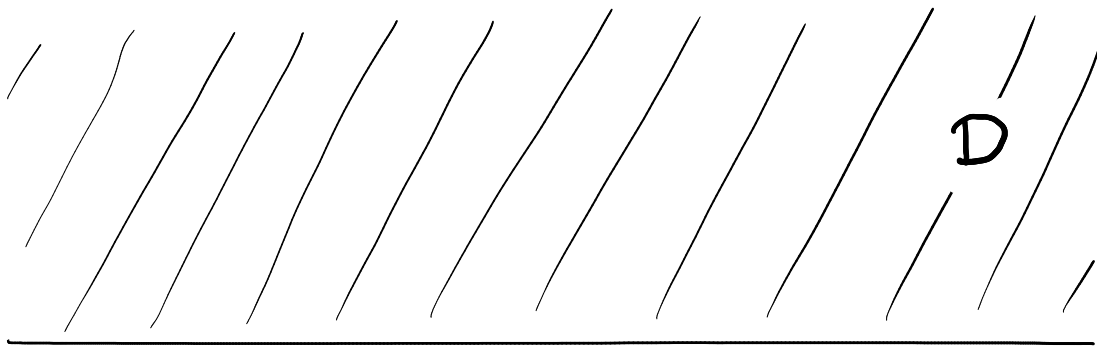
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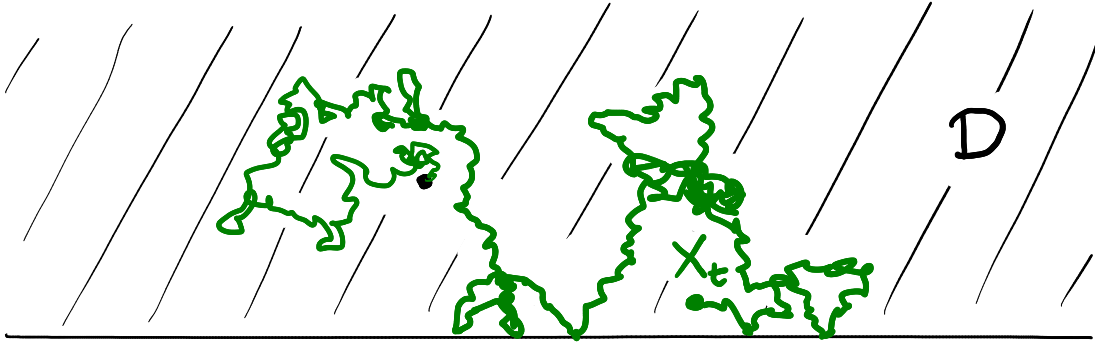


Example: Cauchy process [Spitzer, 1958]



D - half-plane
(or half-space)

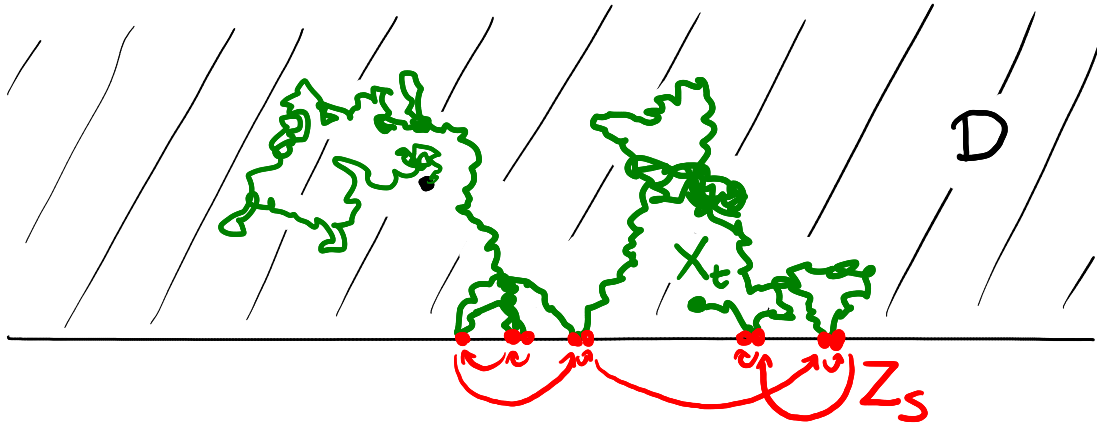
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X_t - Brownian motion

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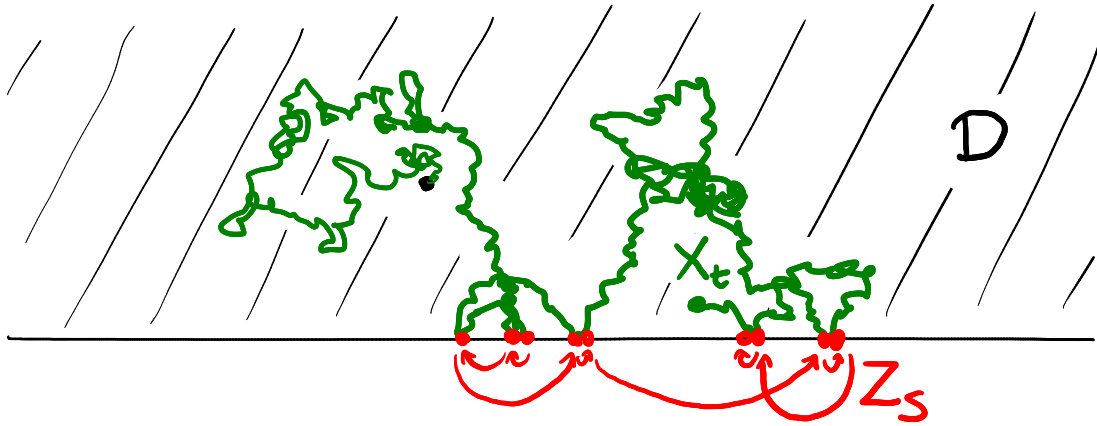


D - half-plane
(or half-space)

X_t - Brownian motion

Z_s - Cauchy process
(symmetric 1-stable Lévy)

Example: Cauchy process [Spitzer, 1958]



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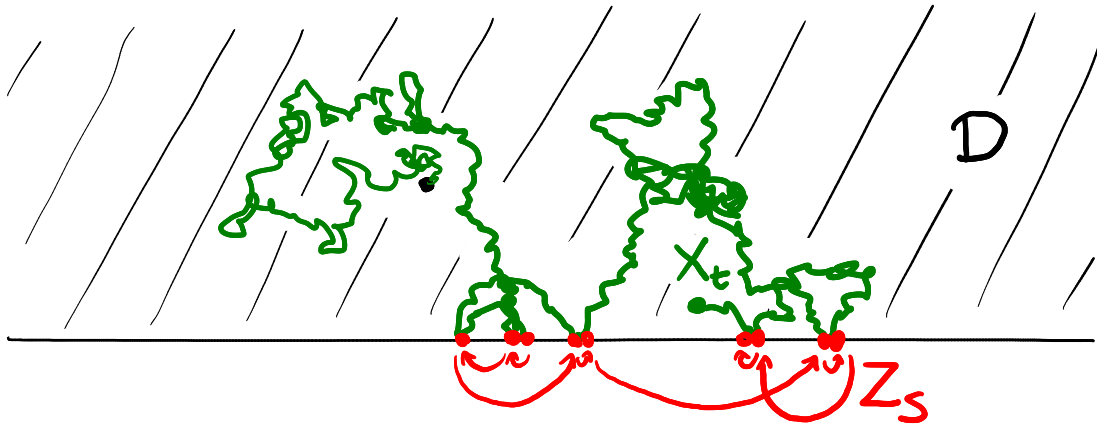
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- Markov
- translation invariant
- self-similar with $H=1$

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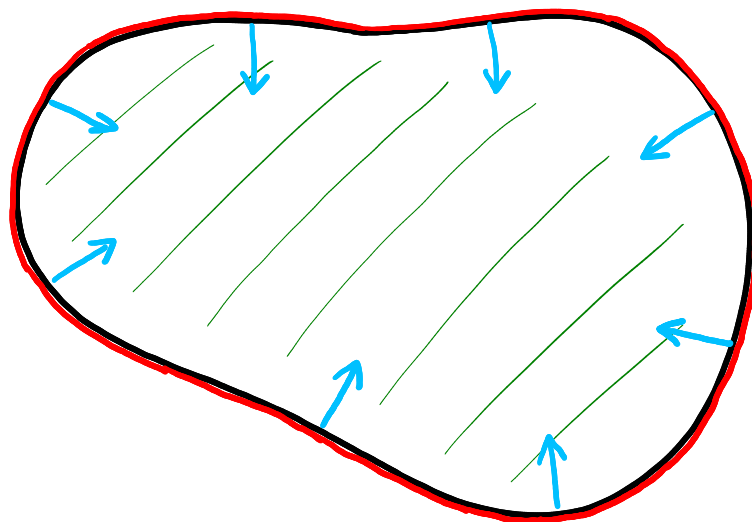
X_t - Brownian motion

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Proof: Z_s is:

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$$\left\{ \begin{array}{l} L_{\lambda^2 t} \stackrel{d}{=} \lambda L_t \\ L_{\lambda s}^{-1} \stackrel{d}{=} \lambda^2 L_s^{-1} \\ Z_{\lambda s} = X_{L_{\lambda s}^{-1}} \stackrel{d}{=} X_{\lambda^2 L_s^{-1}} \stackrel{d}{=} \lambda X_{L_s^{-1}} = \lambda Z_s \end{array} \right.$$



2. Dirichlet-to-Neumann maps

Half-Laplacian

Generator of the Cauchy process :

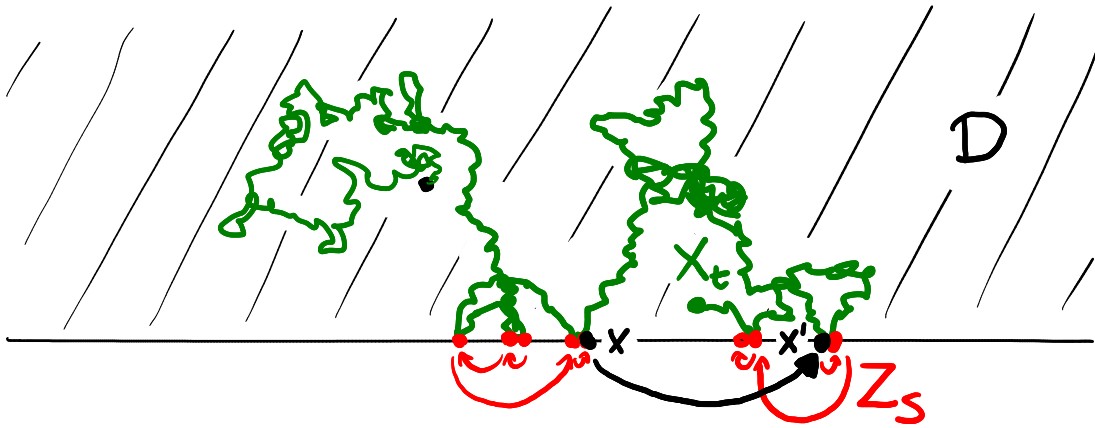
$$-(-\Delta)^{\frac{1}{2}} f(x) = \text{p.v.} \int_{-\infty}^{\infty} (f(x') - f(x)) \frac{1}{\pi} \frac{1}{|x' - x|^2} dx'$$

Half-Laplacian

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$$-(-\Delta)^{\frac{1}{2}} f(x) = \text{p.v.} \int_{-\infty}^{\infty} (f(x') - f(x)) \underbrace{\frac{1}{\pi} \frac{1}{|x' - x|^2}}_{\text{intensity of jumps from } x \text{ to } x'} dx'$$

intensity of jumps
from x to x'

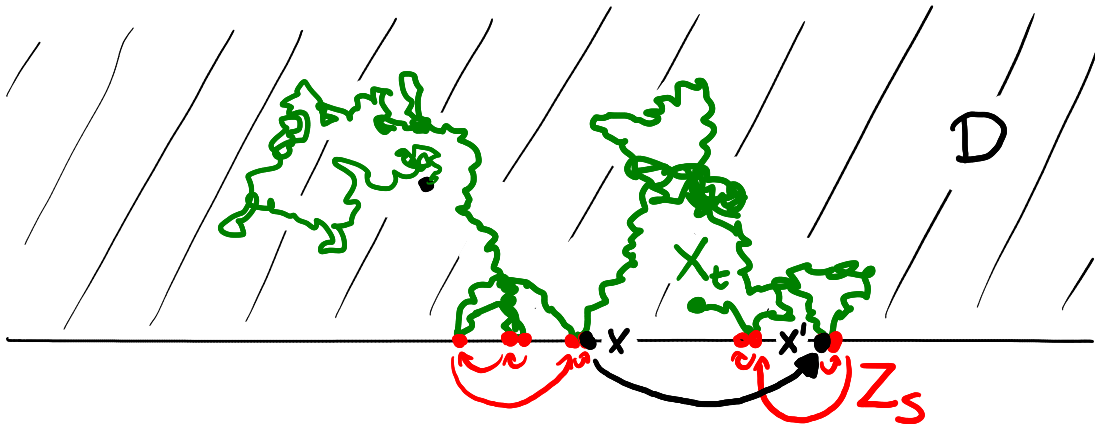


Half-Laplacian

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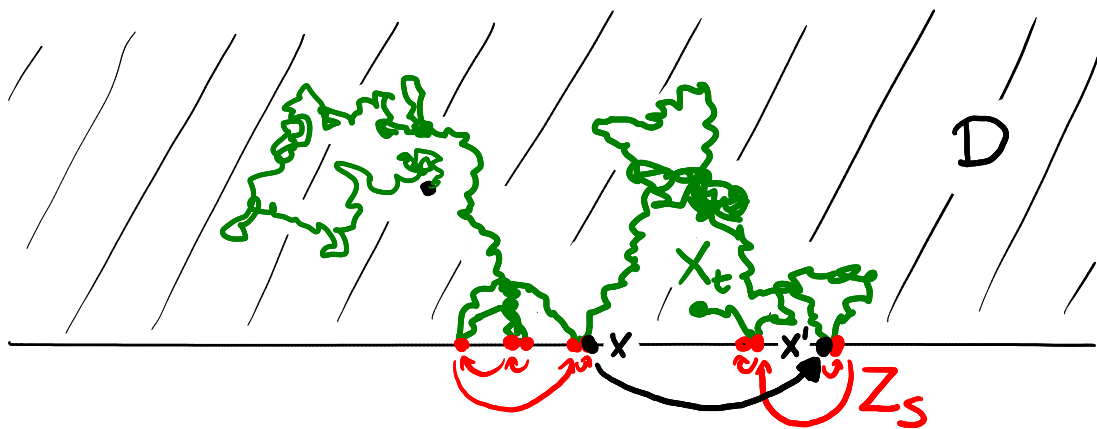
Fact $-(-\Delta)^{\frac{1}{2}}$ is the
Dirichlet-to-Neumann
map in D

Half-Laplacian

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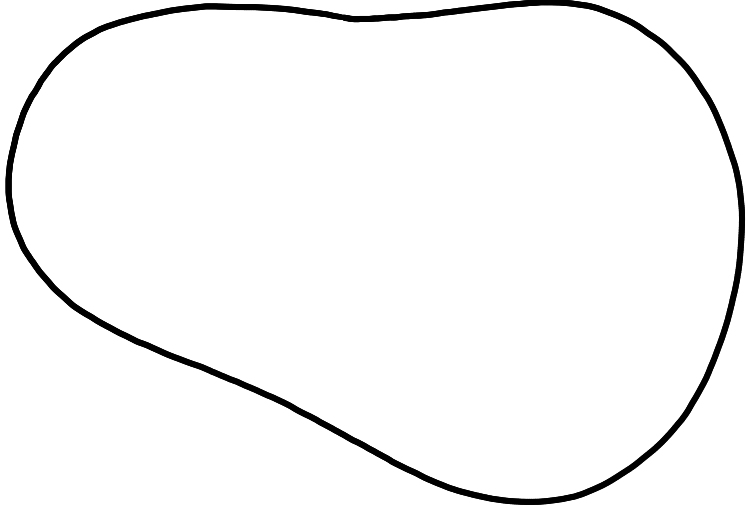
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Fact $-(-\Delta)^{\frac{1}{2}}$ is the
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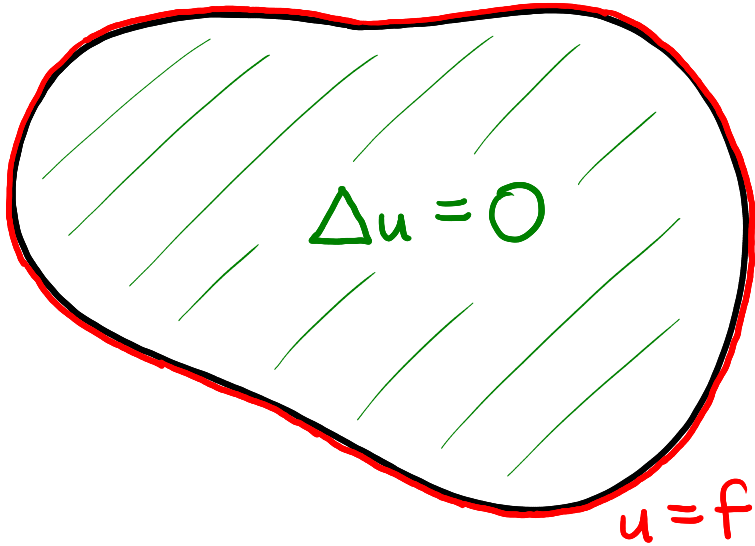
...and $\frac{1}{2}\Delta$ is the generator
of the Brownian motion

Dirichlet-to-Neumann map



D - a planar domain

Dirichlet-to-Neumann map

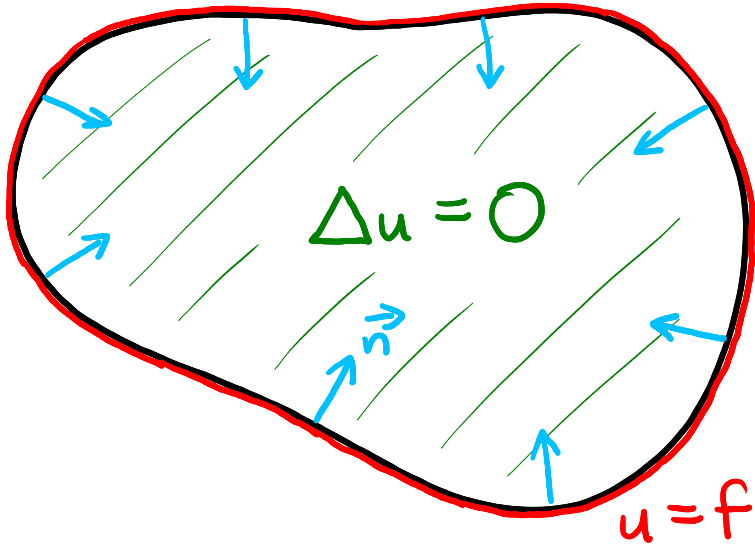


D - a planar domain

Dirichlet problem in D :

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

Dirichlet-to-Neumann map



D - a planar domain

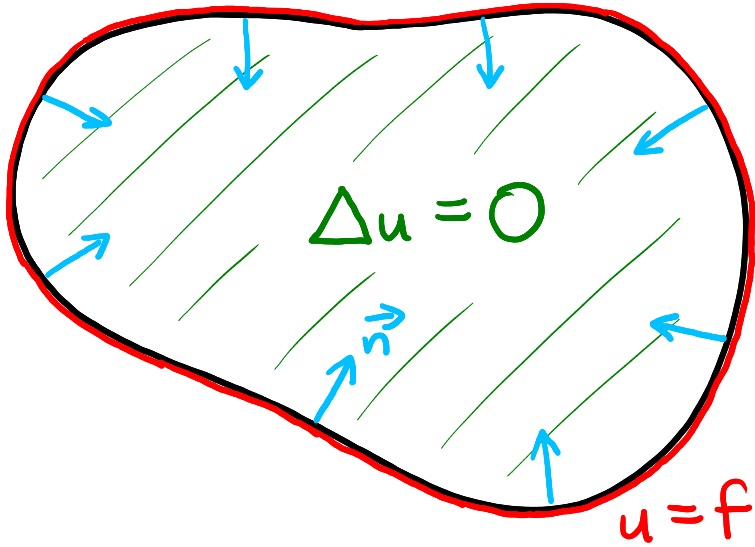
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Corresponding Neumann condition:

$$g = \frac{\partial u}{\partial \vec{n}} \quad \text{on } \partial D$$

Dirichlet-to-Neumann map



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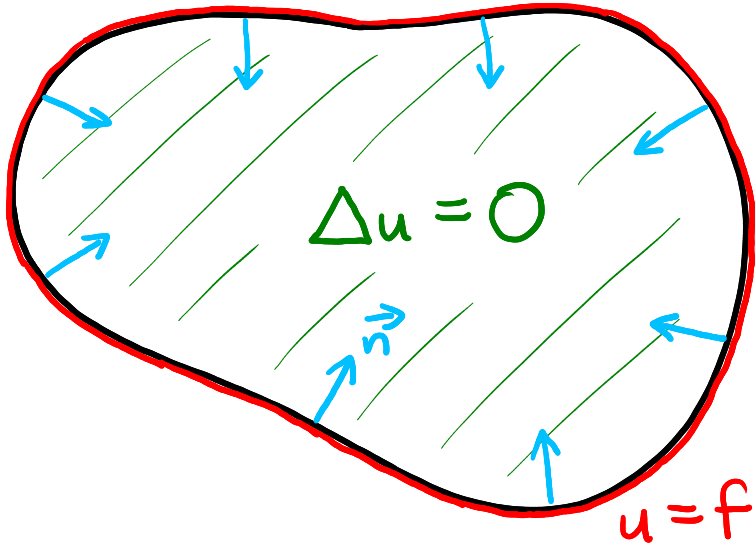
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Dirichlet-to-Neumann map



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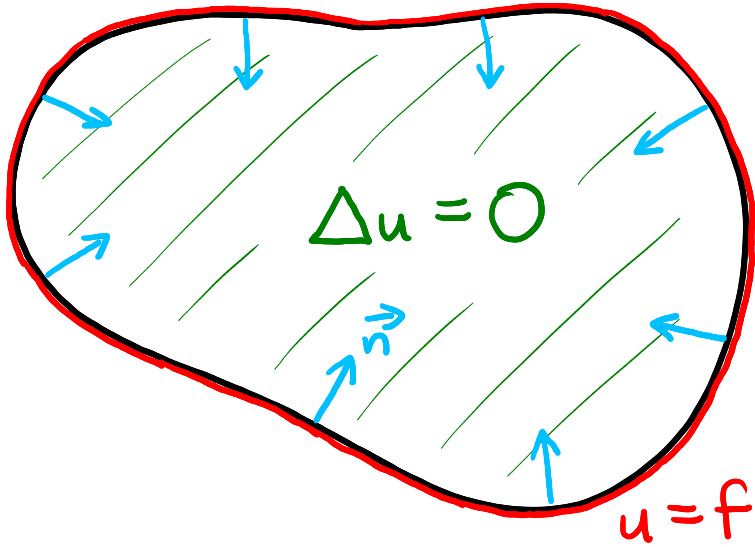
$$g = \frac{\partial u}{\partial \vec{n}} \quad \text{on } \partial D$$

Dirichlet-to-Neumann map: $f \mapsto g$

Notation: $Lu = \frac{1}{2}\Delta u$ (in D)

$Kf = g$ (on ∂D)

Dirichlet-to-Neumann map



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Question If L is the generator of X_t ,
is K the generator of Z_s ?

Boundary trace $\longleftrightarrow$ Dirichlet-to-Neumann map

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Short answer: YES

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Short answer: YES

- Brownian motion [Molchanov, 1964]
- diffusions [Sato, Ueno, 1965]
- more abstract [Motoo, 1965/66]
- extensions/variations [Hsu, 1986]
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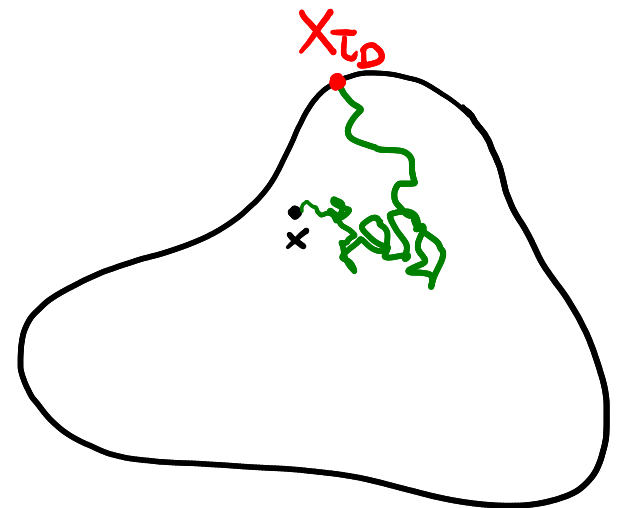
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Makes sense: [Kakutani, 1944]

$$u(x) = \mathbb{E}[f(X_{\tau_D}) \mid X_0 = x]$$

where τ_D - hitting time of ∂D



Boundary trace $\longleftrightarrow$ Dirichlet-to-Neumann map

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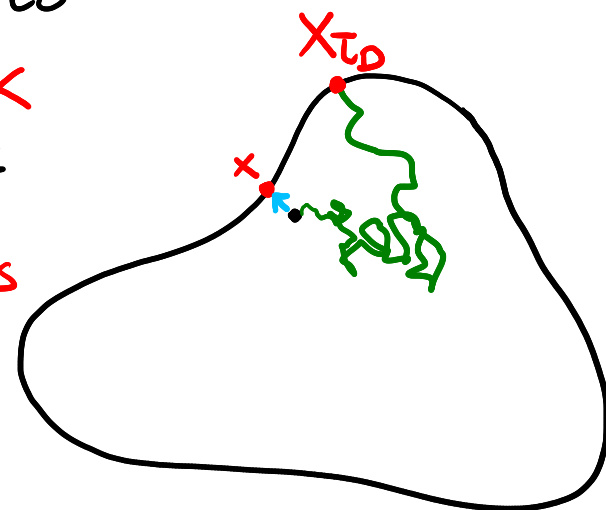
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$$u(x) = \mathbb{E}[f(X_{\tau_D}) \mid X_0 = x]$$

where τ_D - hitting time of ∂D

Distribution of X_{τ_D} "converges" as $x \rightarrow \partial D$ to

- kernel of K
- intensity of jumps of Z_s



Two questions

Question How irregular can X_t be?

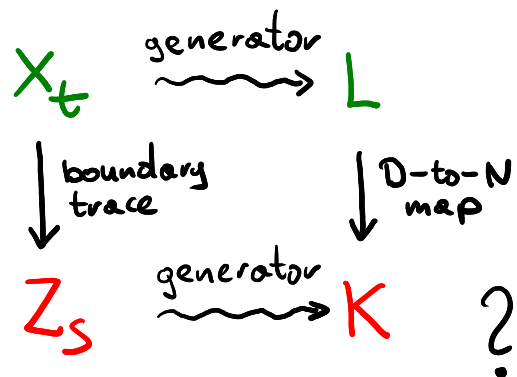
Question Is a given Z_s a boundary trace process?

Two questions

Question How irregular can X_t be?

In PDE some important K require L with very singular coefficients

Do we still have:



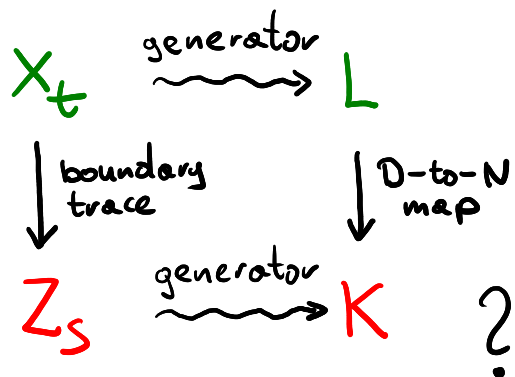
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Question Is a given Z_s a boundary trace process?

"Inverse problem":

given Z_s or K , determine X_t or L

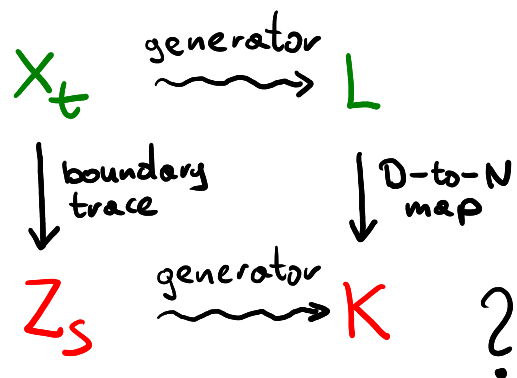
— a more general variant of the famous Calderón's inverse conductivity problem

Two questions

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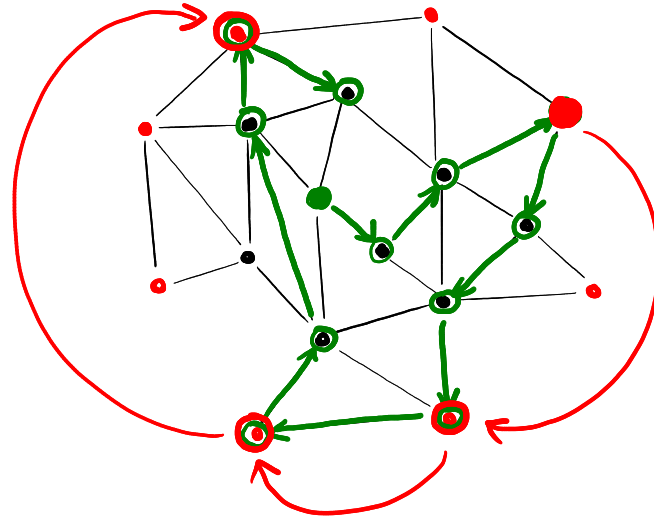
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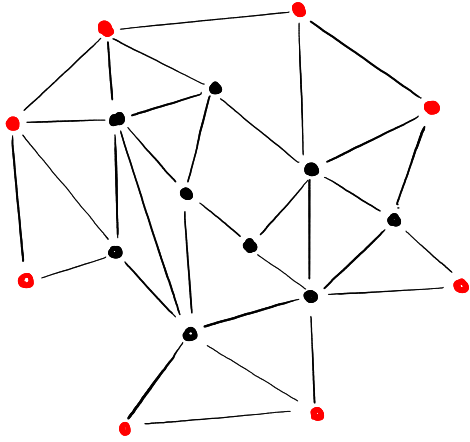
- a more general variant of the famous Calderón's inverse conductivity problem

Neither is fully resolved!



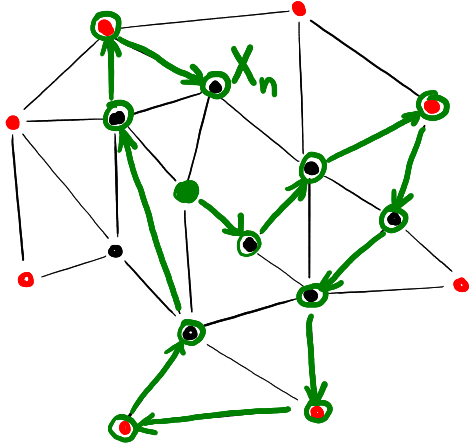
3. Toy (?) model: graphs

Boundary traces on graphs



D - a weighted planar graph with **boundary**

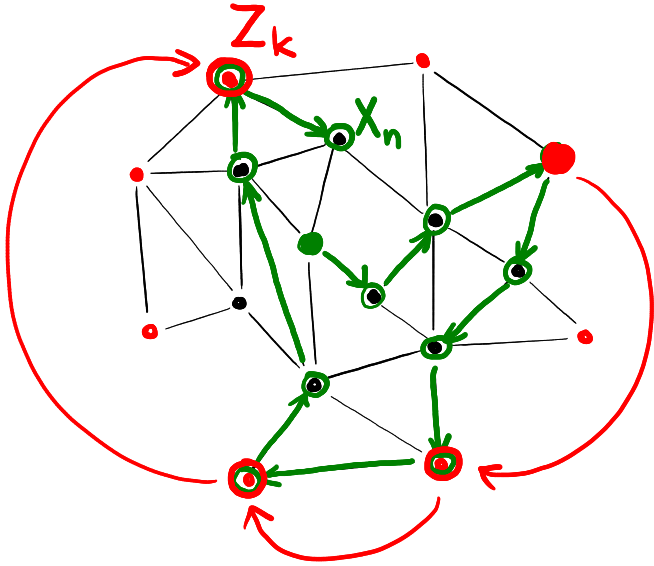
Boundary traces on graphs



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Boundary traces on graphs

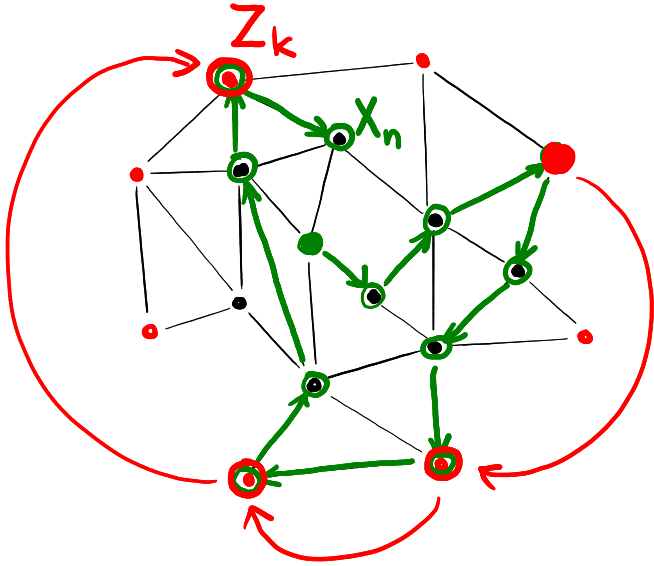


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Boundary traces on graphs



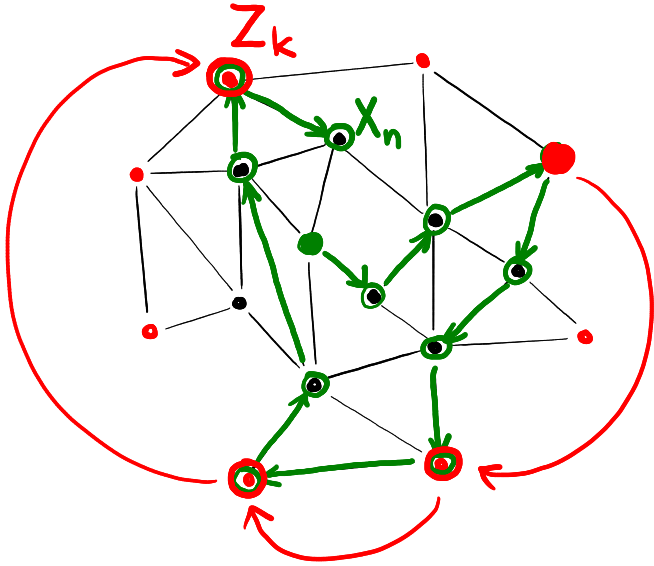
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Z_k - boundary trace

- Z_k is a Markov chain

Boundary traces on graphs



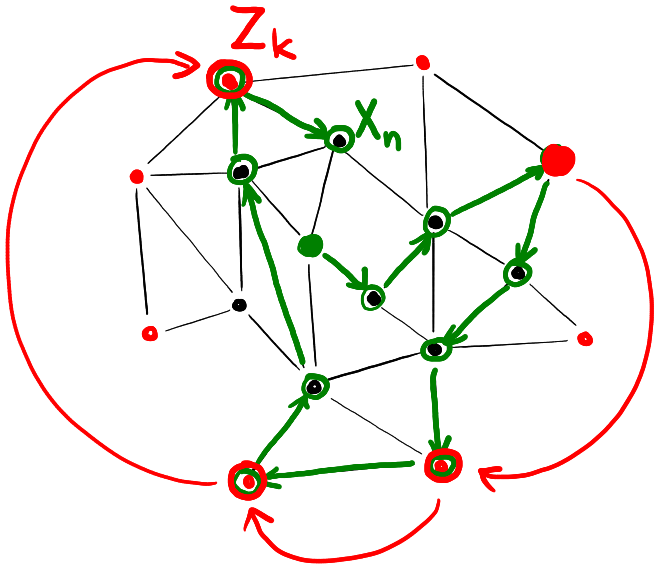
D - a weighted planar graph with **boundary**

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Z_k - boundary trace

- Z_k is a Markov chain
- X_n - symmetric $\Rightarrow Z_k$ - symmetric

Boundary traces on graphs



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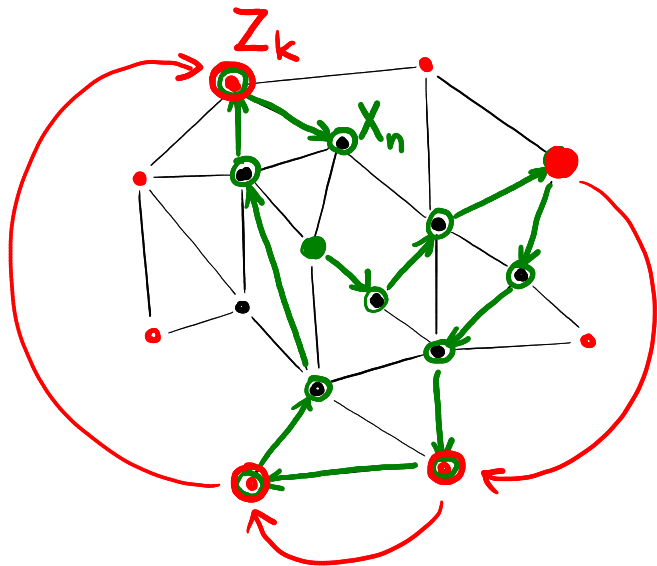
- Z_k is a Markov chain
- X_n - symmetric $\Rightarrow Z_k$ - symmetric
- generator matrix of Z_k (= transition rates) fully characterised in terms of "cyclic total positivity" (at least in continuous time)

[Colin de Verdière, 1994]

[Curtis, Ingerman, Morrow, 1995]

[Postnikov, 2006⁺]

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Generator matrix coincides with the Dirichlet-to-Neumann map

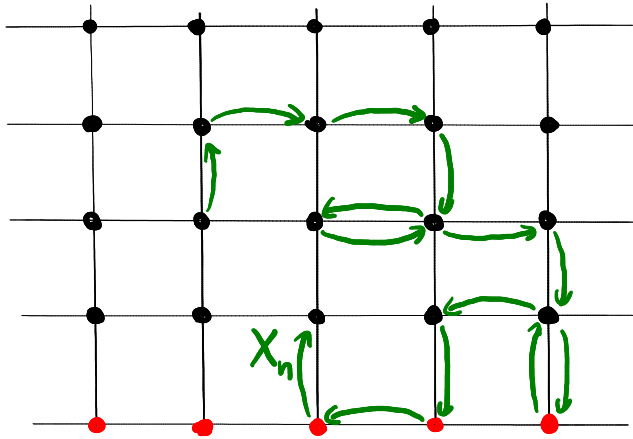
No regularity issues

[Colin de Verdière, 1994]

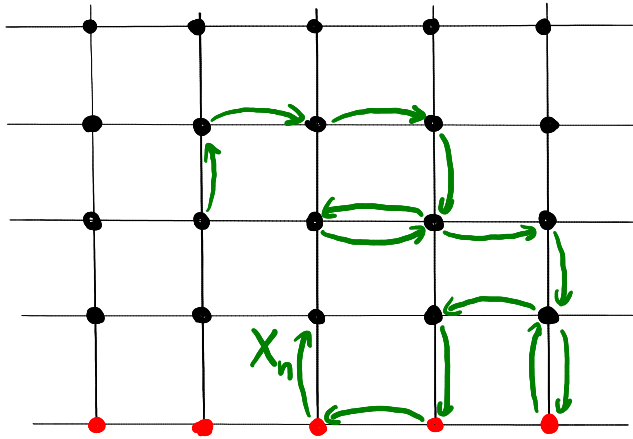
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Boundary traces on lattices



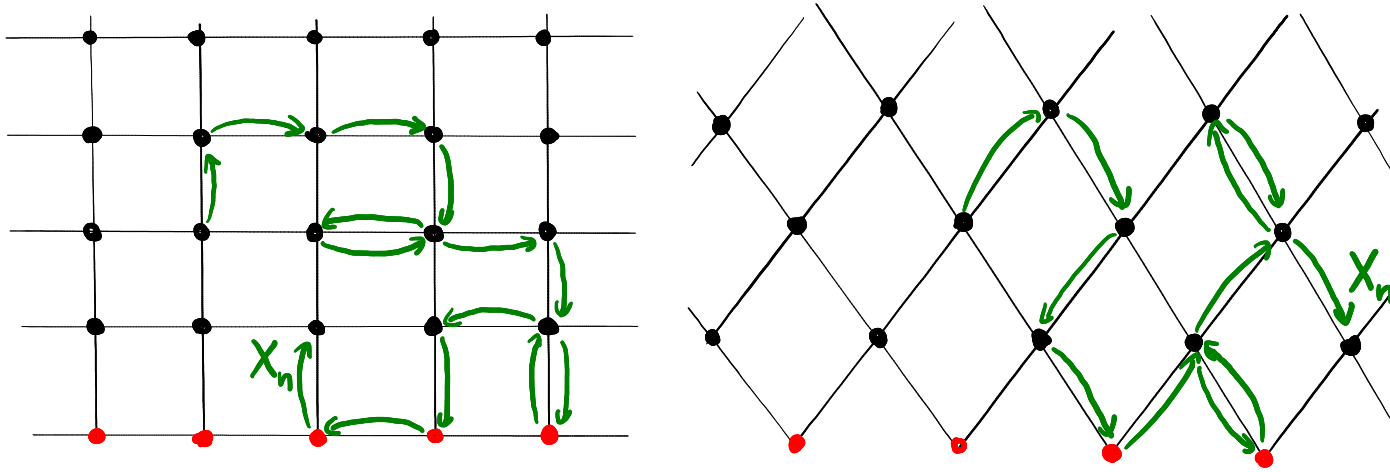
Boundary traces on lattices



Natural assumption: transition rates only depend on the "height"

Z_k has connections to complete monotonicity and the bell-shape [Wszota, 2025+]

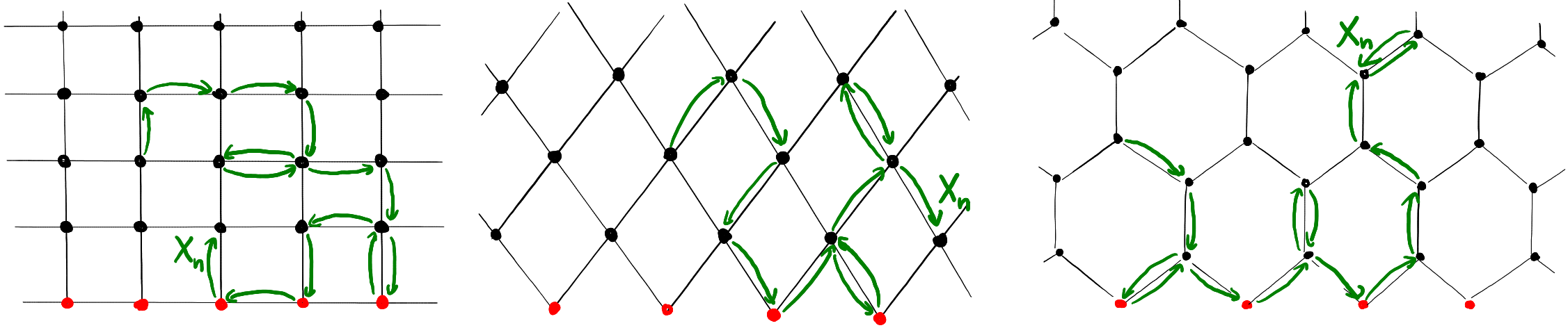
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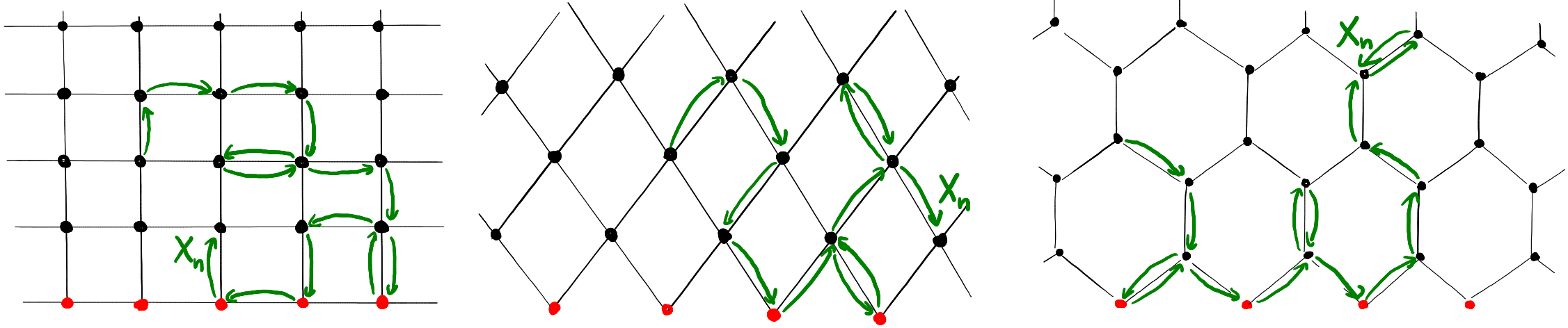
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Natural assumption: transition rates only depend on the "height"

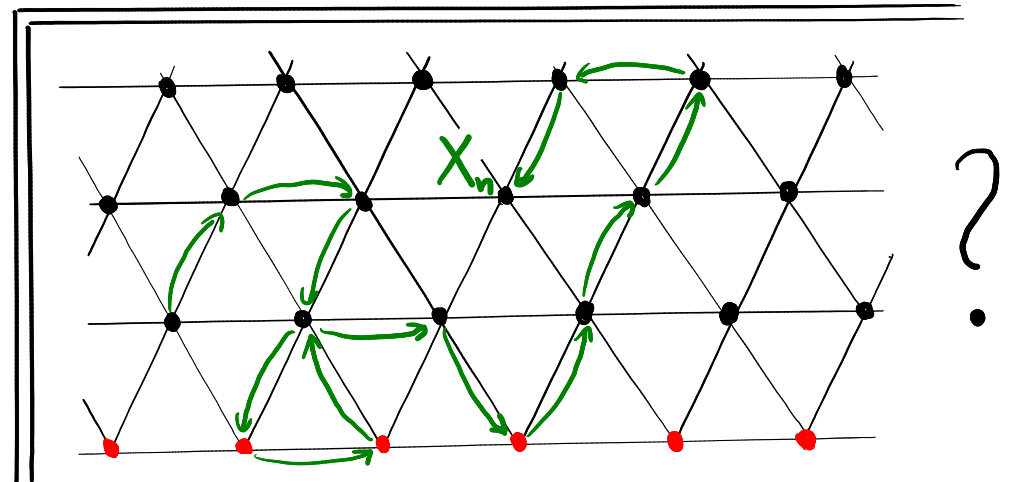
Z_k has connections to complete monotonicity
and the bell-shape [Wszoł, 2025⁺]

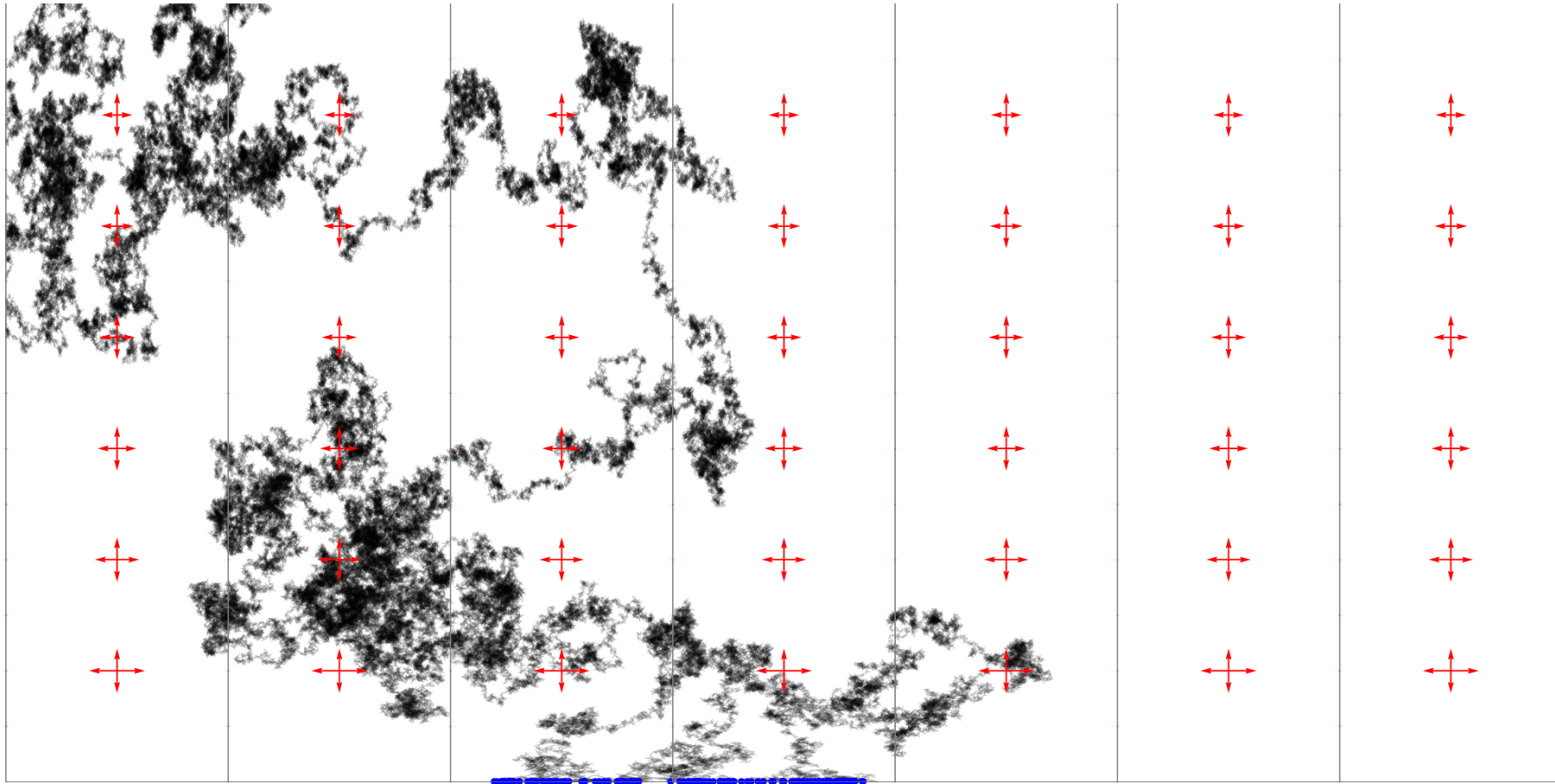
Boundary traces on lattices



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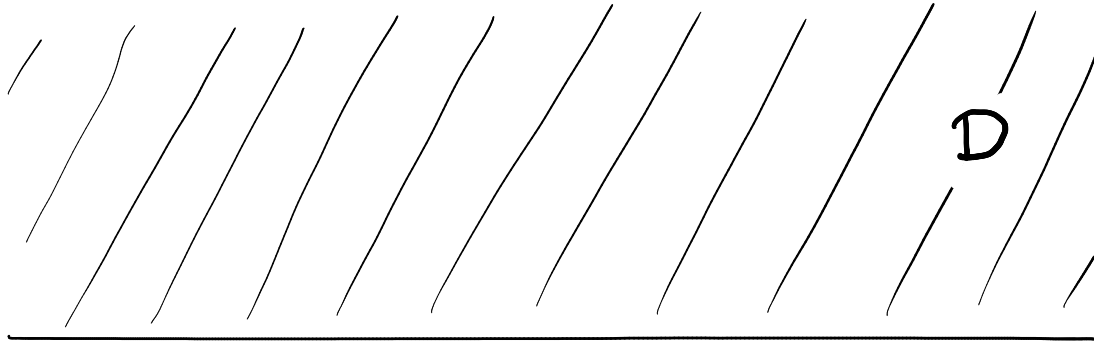




4. Molchanov-Ostrovski & Caffarelli-Silvestre

Example: symmetric α -stable Lévy process

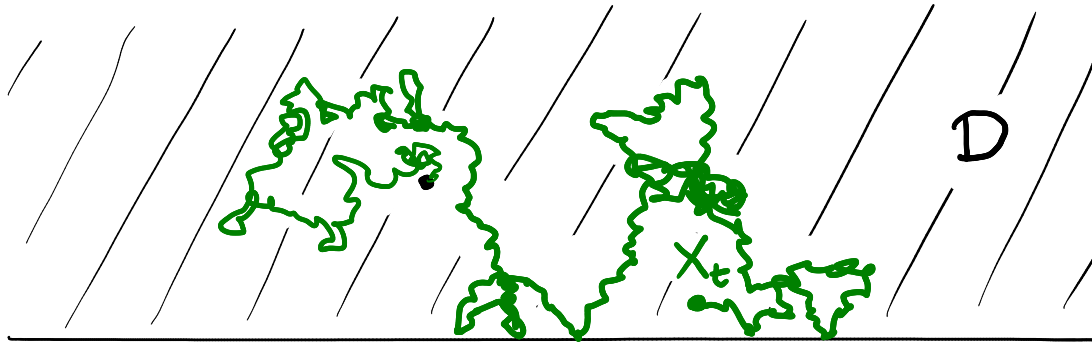
[Molchanov, Ostrovski, 1969]



D - half-plane
(or half-space)

Example: symmetric α -stable Lévy process

[Molchanov, Ostrovski, 1969]

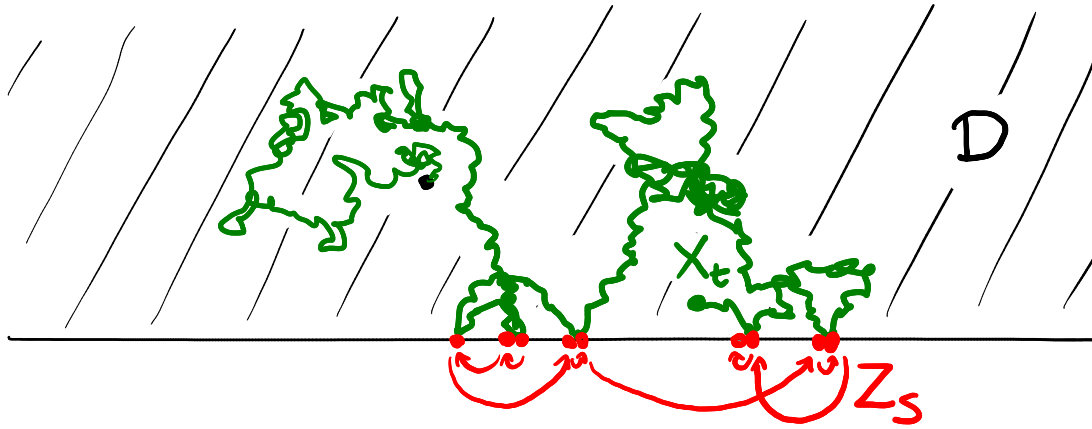


D - half-plane
(or half-space)

X_t - Brownian motion

Example: symmetric α -stable Lévy process

[Molchanov, Ostrovski, 1969]



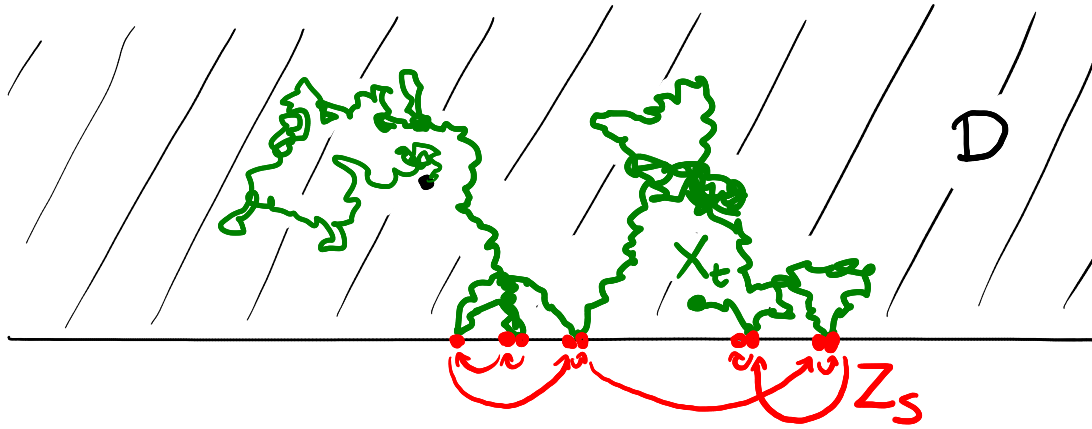
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(symmetric 1-stable Lévy)

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(or half-space)

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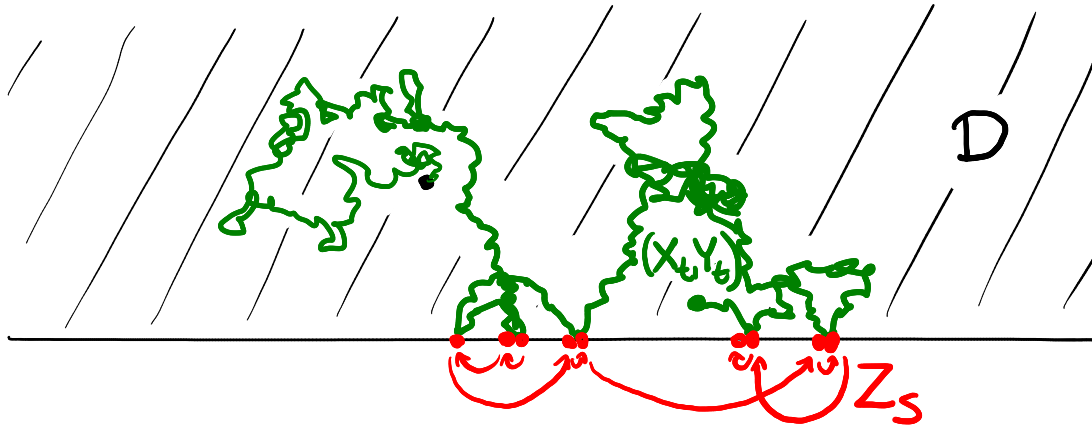
$$L_{\lambda^2 t} \stackrel{d}{=} \lambda L_t$$

$$L_{\lambda s}^{-1} \stackrel{d}{=} \lambda^2 L_s^{-1}$$

$$Z_{\lambda s} = X_{L_{\lambda s}^{-1}} \stackrel{d}{=} X_{\lambda^2 L_s^{-1}} \stackrel{d}{=} \lambda X_{L_s^{-1}} = \lambda Z_s$$

Example: symmetric α -stable Lévy process

[Molchanov, Ostrovski, 1969]



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X_t - Brownian motion in $\mathbb{R}$

Y_t - Brownian motion in $[0, \infty)$

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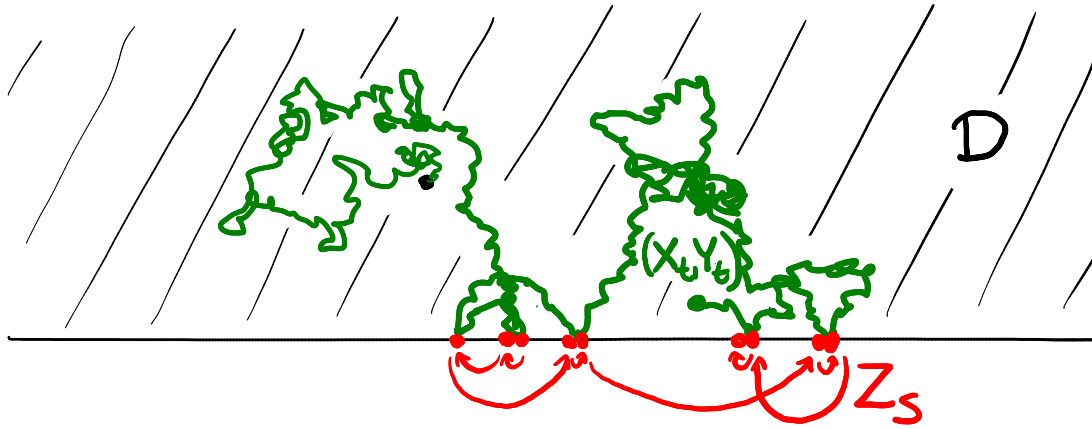
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Example: symmetric α -stable Lévy process

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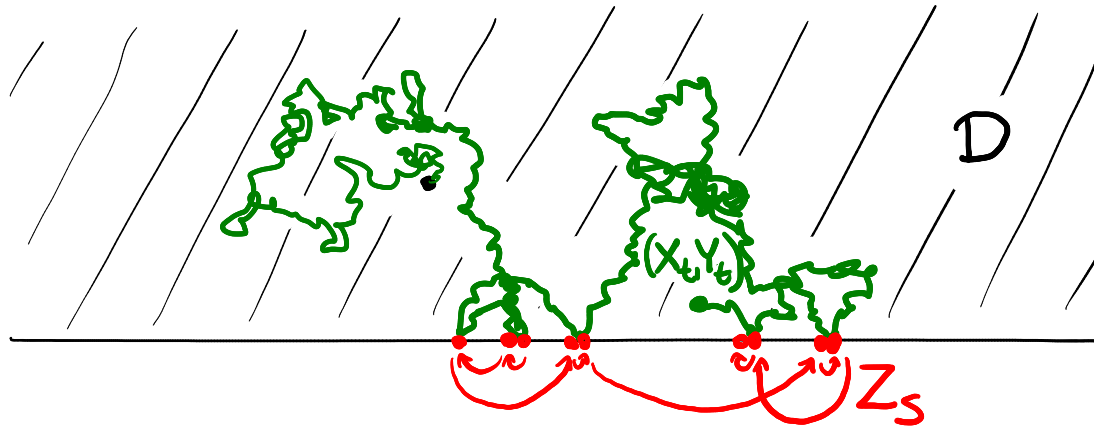
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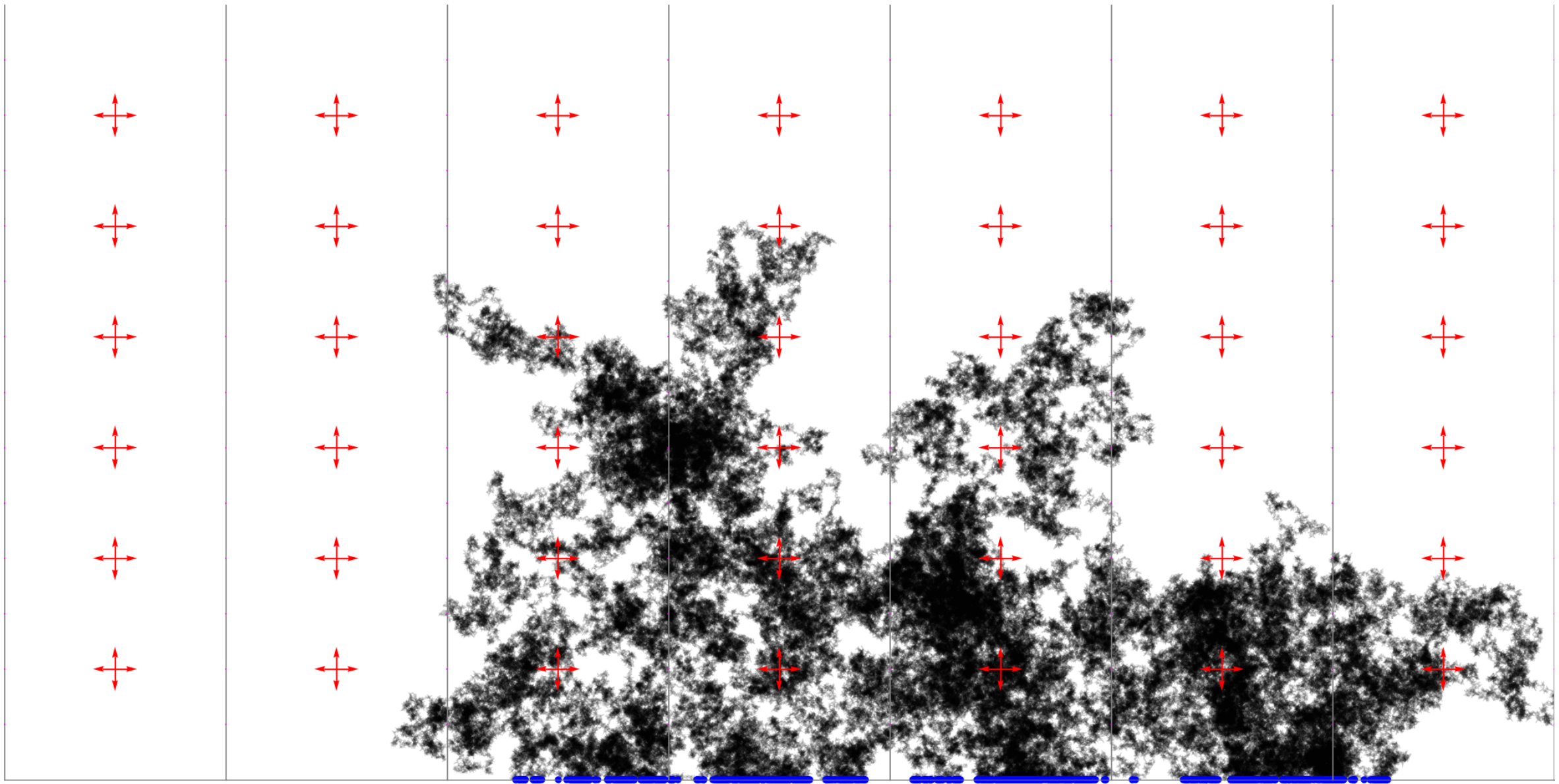
Y_t - Bessel process in $[0, \infty)$

Z_s - ~~Cauchy process~~
symmetric α -stable Lévy

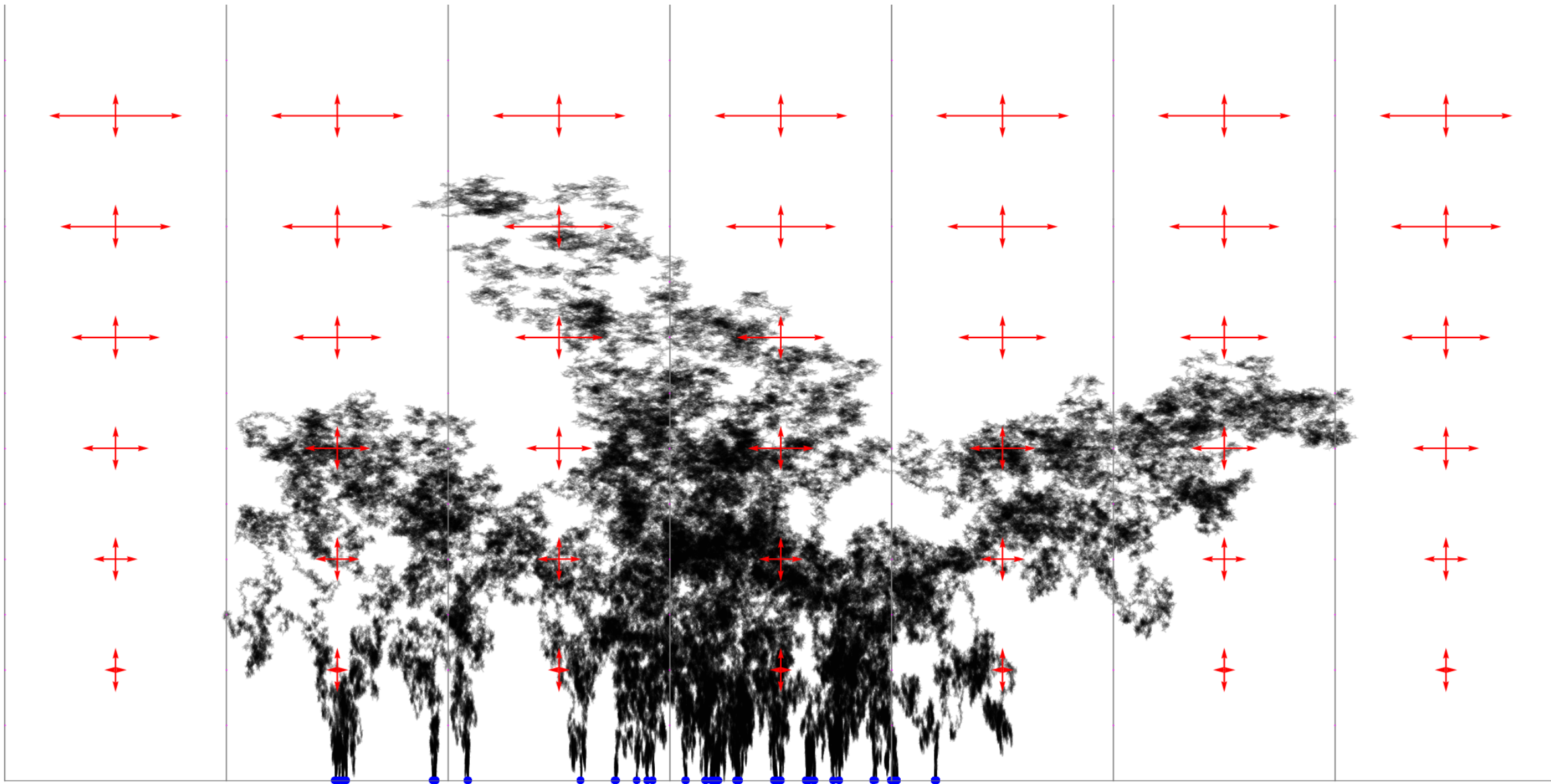
$$L_{\lambda t}^{\frac{2}{\alpha}} \stackrel{d}{=} \lambda L_t$$

$$L_{\lambda s}^{-1} \stackrel{d}{=} \lambda^{\frac{2}{\alpha}} L_s^{-1}$$

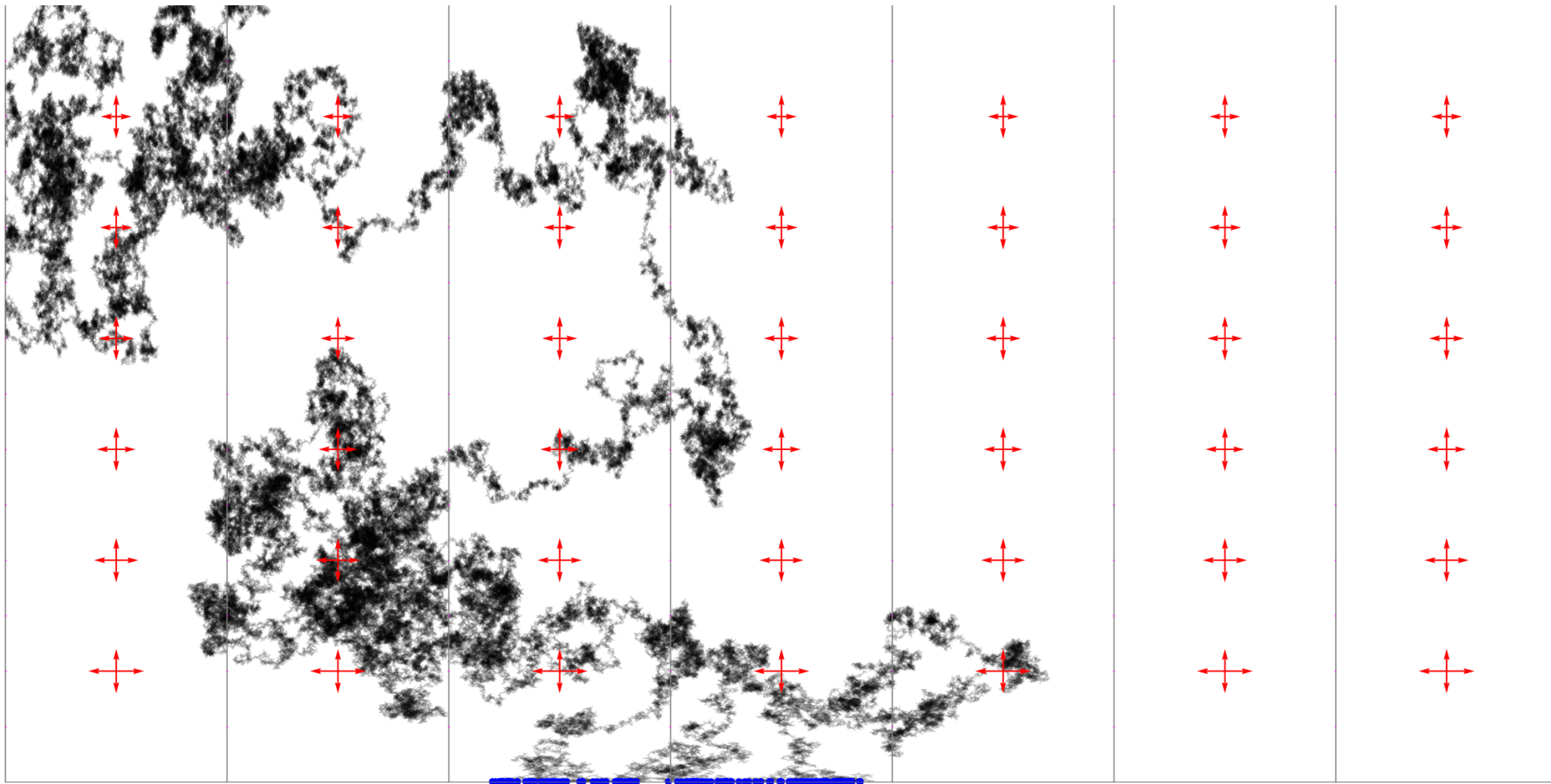
$$Z_{\lambda s} = X_{L_{\lambda s}^{-1}} \stackrel{d}{=} X_{\lambda^{\frac{2}{\alpha}} L_s^{-1}} \stackrel{d}{=} \lambda^{\frac{1}{\alpha}} X_{L_s^{-1}} = \lambda^{\frac{1}{\alpha}} Z_s$$



$\alpha=1$: Cauchy process

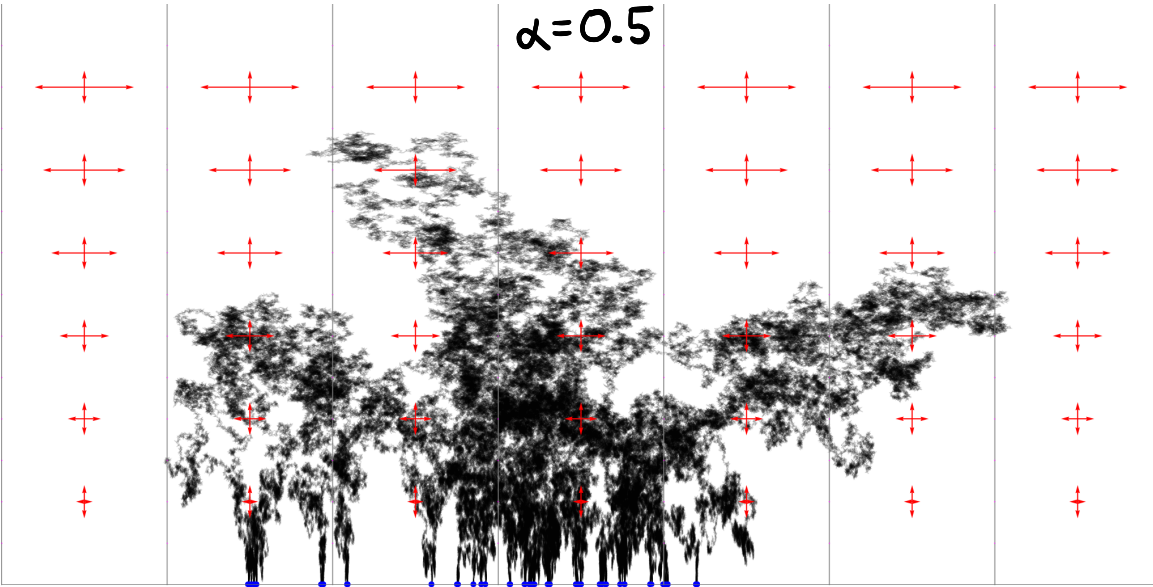


$$\alpha = 0.5$$

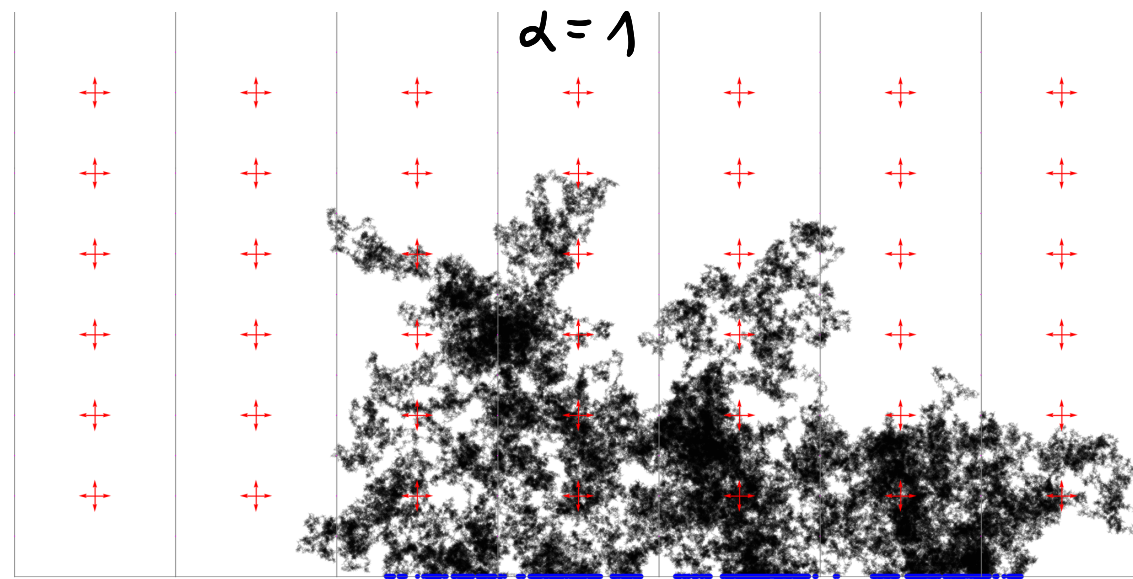


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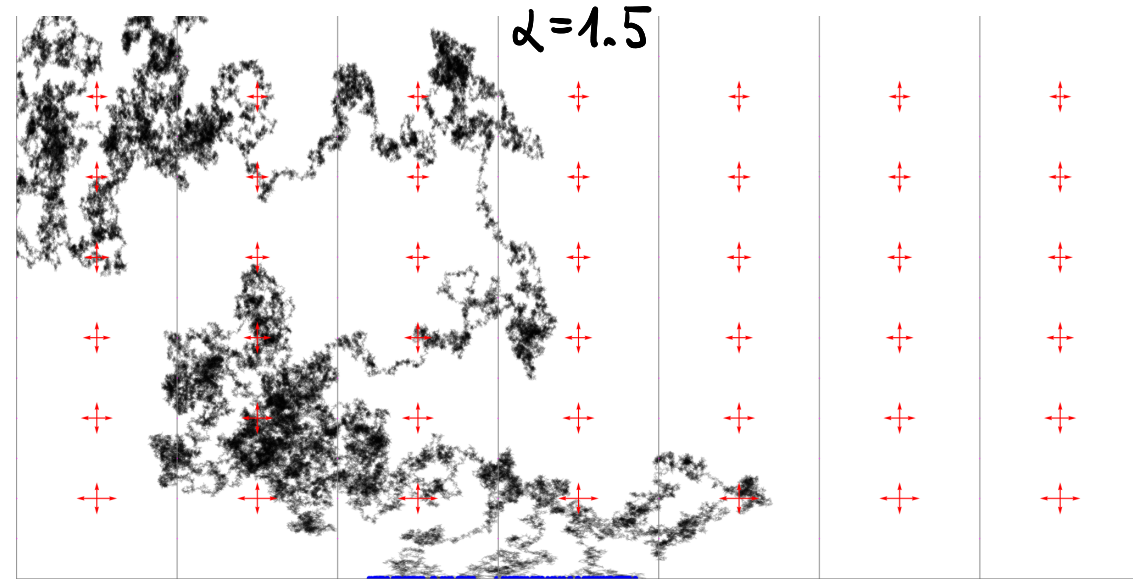
$\alpha=0.5$



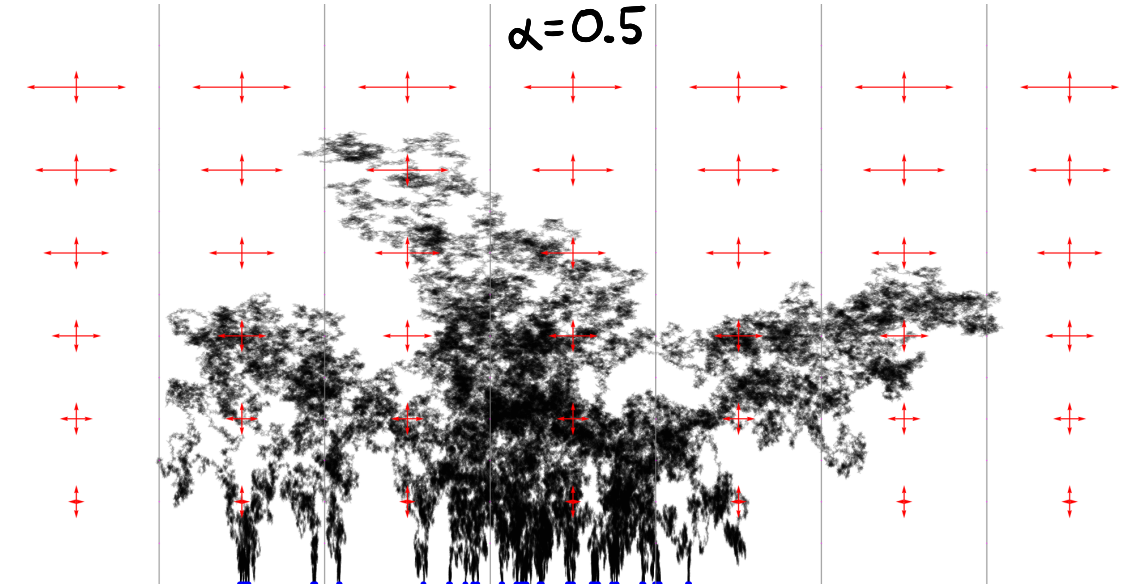
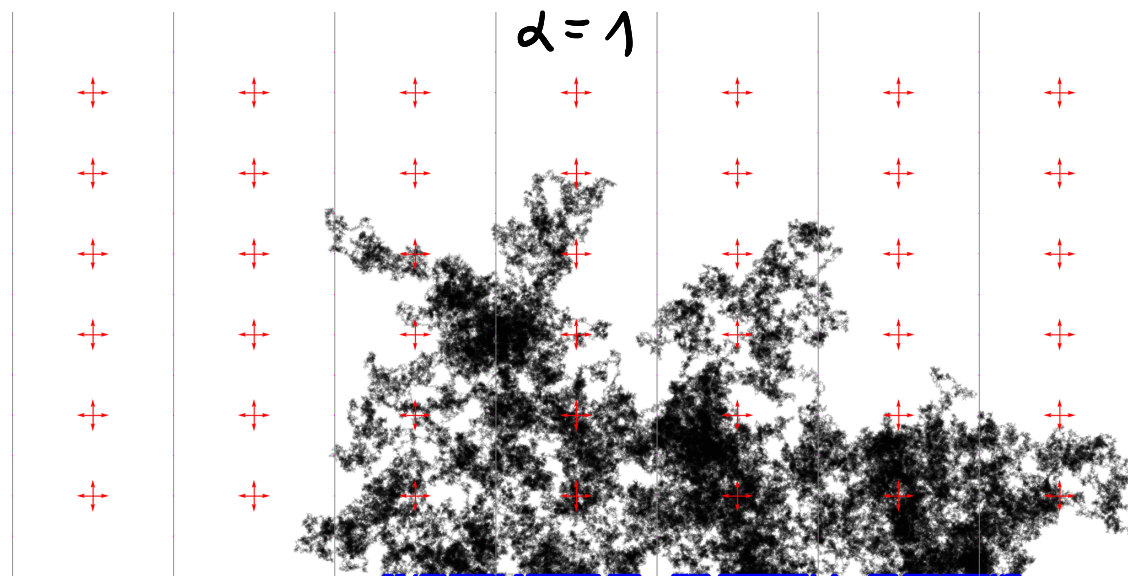
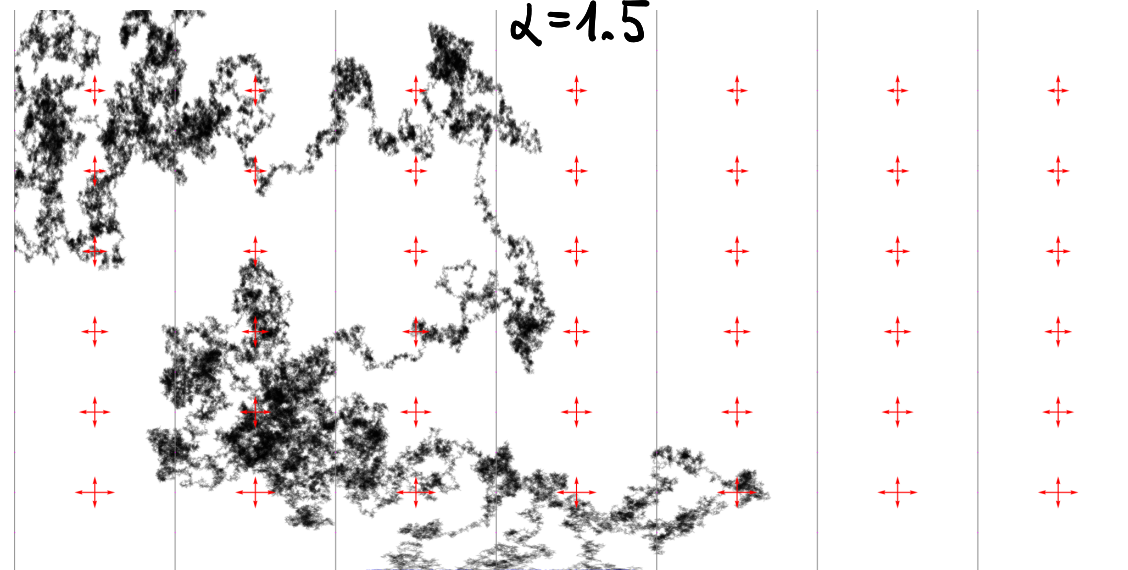
$\alpha=1$



$\alpha=1.5$



Independence of X_t and Y_t
plays a key role

$\alpha = 0.5$  $\alpha = 1$  $\alpha = 1.5$ 

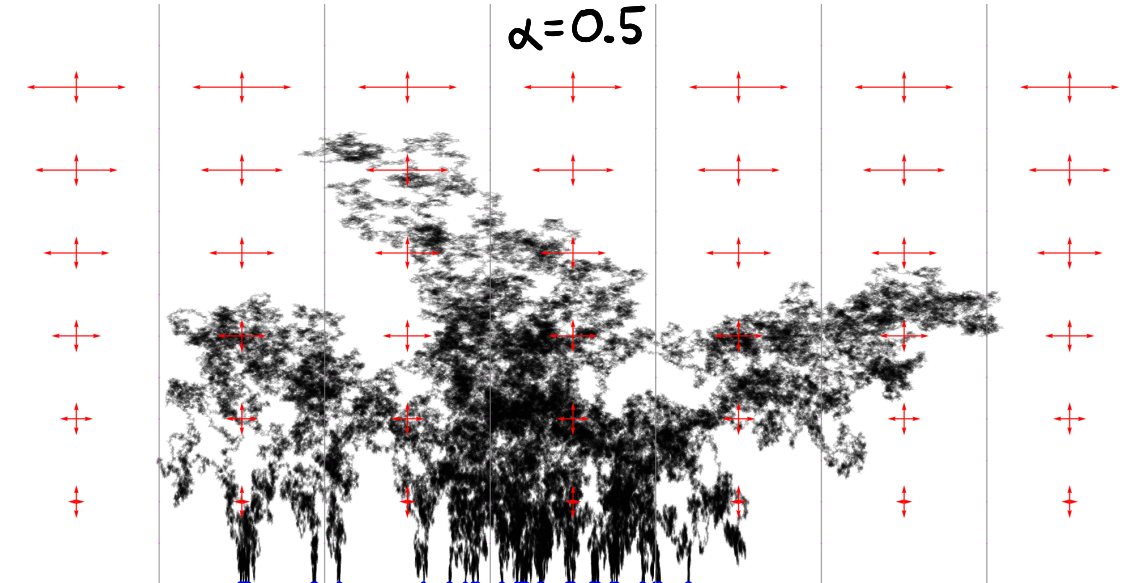
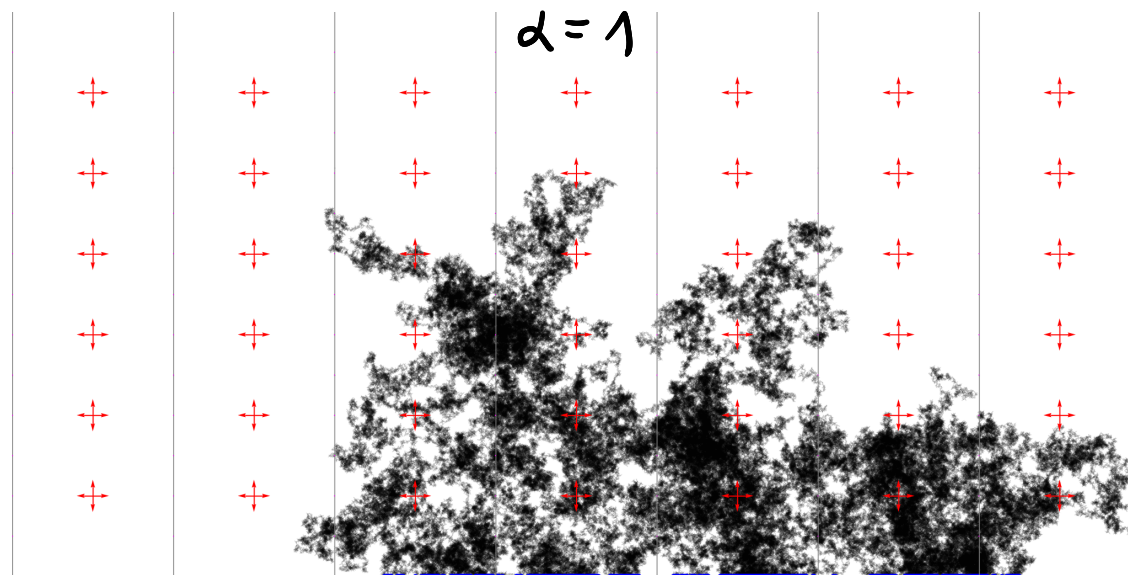
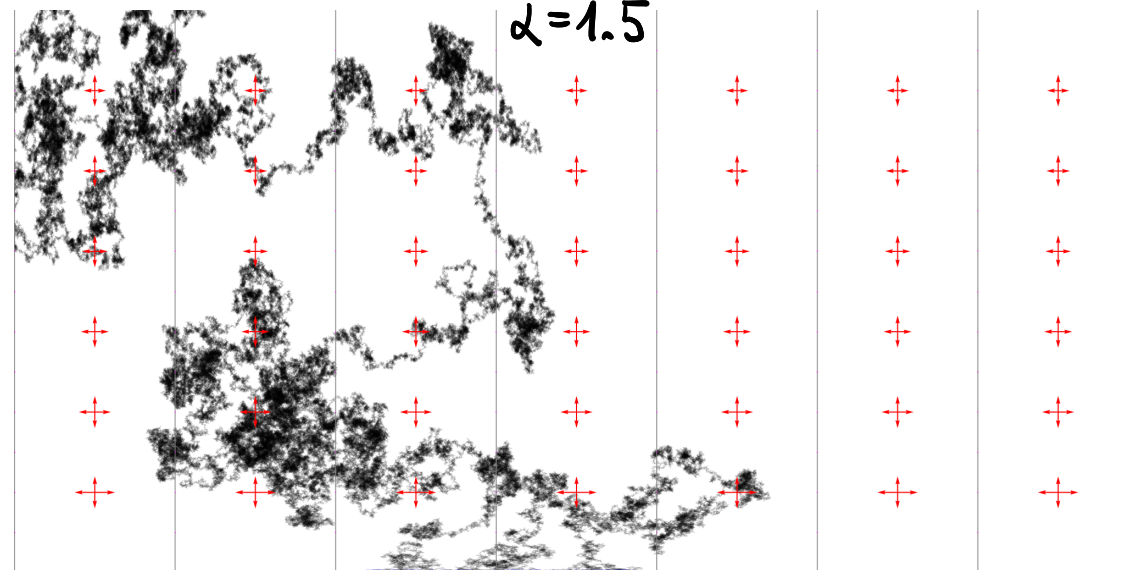
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$$Z_s = X_{L_s^{-1}}$$

X_t — Brownian motion

L_s^{-1} — $\frac{\alpha}{2}$ -stable subordinator

↗ independent

$\alpha = 0.5$  $\alpha = 1$  $\alpha = 1.5$ 

Independence of X_t and Y_t
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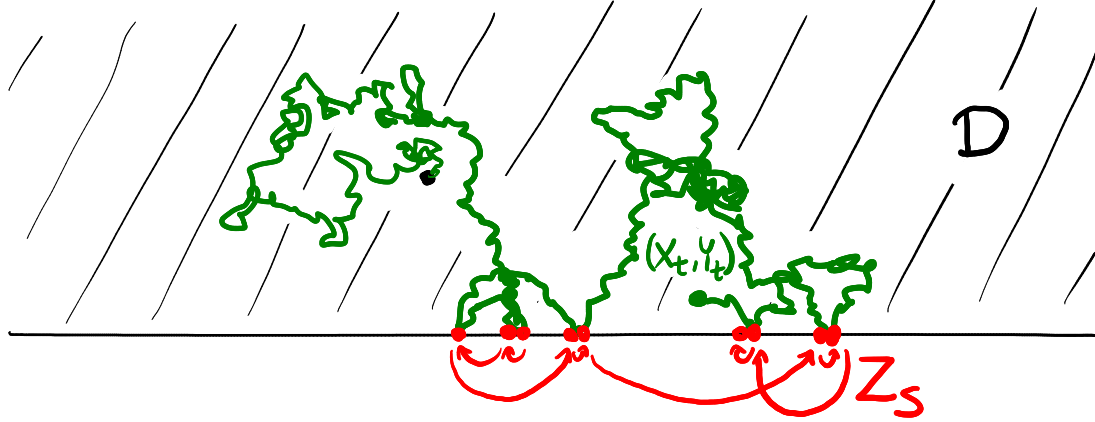
$$Z_s = X_{L_s^{-1}} \quad \leftarrow \begin{array}{l} \text{subordinate} \\ \text{Brownian} \\ \text{motion} \end{array}$$

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Subordinate Brownian motions



D - half-plane
(or half-space)

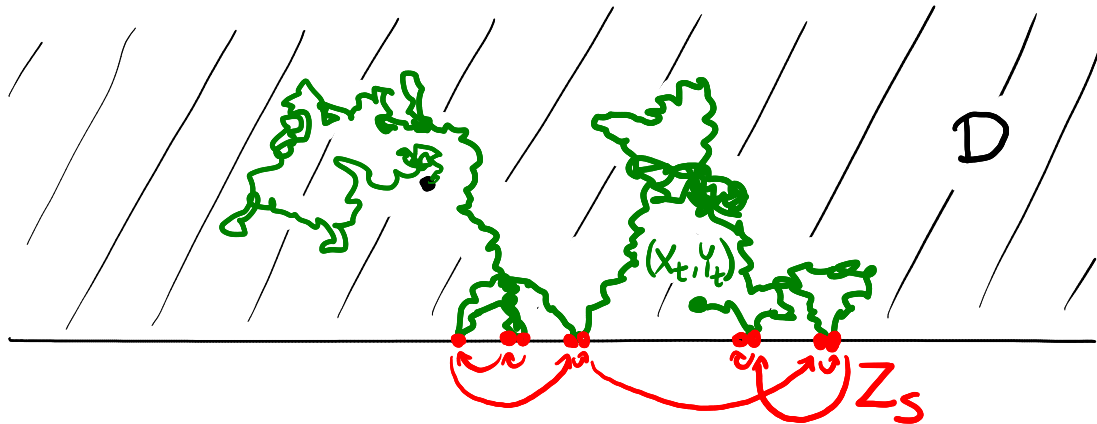
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Subordinate Brownian motions



L_s^{-1} is a subordinator
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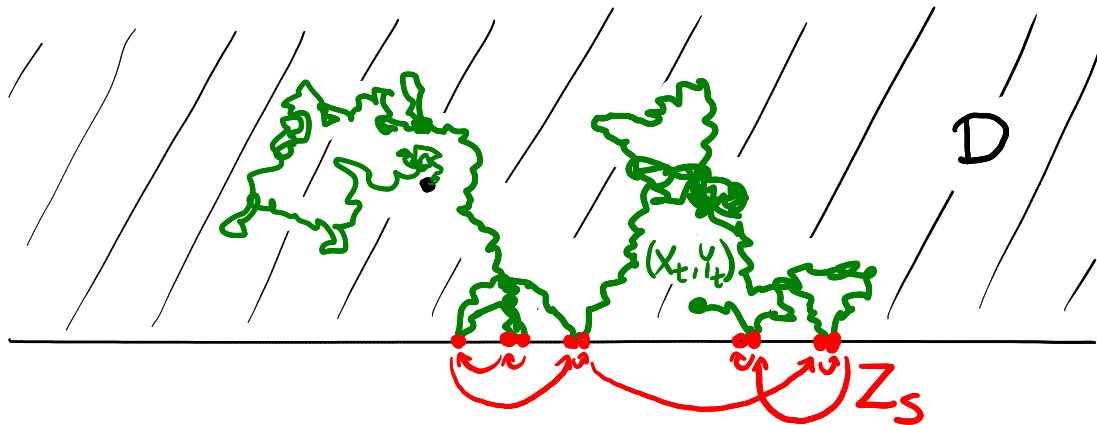
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Subordinate Brownian motions



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The Lévy measure (intensity of jumps) of L_s^{-1} has a
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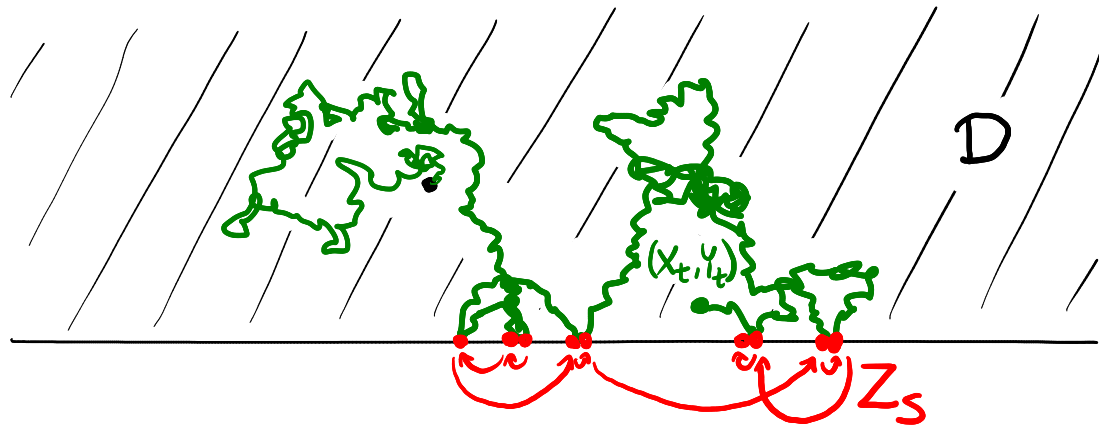
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Subordinate Brownian motions



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$Z_s = X_{L_s^{-1}}$ - subordinate Brownian motion

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Converse is true if we allow gap diffusions [Krein, 1951-53]
[Kotani, Watanabe, 1981]

Example: symmetric α -stable Lévy process (revisited)

The generator of Z_s :

$$-(-\Delta)^{\frac{\alpha}{2}} f(x) = \text{p.v.} \int_{-\infty}^{\infty} (f(x') - f(x)) c_{\alpha} \frac{1}{|x' - x|^{1+\alpha}} dx$$

is the Dirichlet-to-Neumann map:

$$\begin{cases} \Delta_x u + \frac{\partial^2 u}{\partial y^2} + \frac{1-\alpha}{y} \frac{\partial u}{\partial y} = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

$$\downarrow$$
$$Kf = c_{\alpha} \lim_{y \rightarrow 0^+} \frac{u(\cdot, y) - u(\cdot, 0)}{y^{\alpha}}$$

[Caffarelli, Silvestre, 2007]

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[Caffarelli, Silvestre, 2007]

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Thm [K, Mucha, 2018]

- $K = -\psi(-\Delta)$ for a complete Bernstein function ψ

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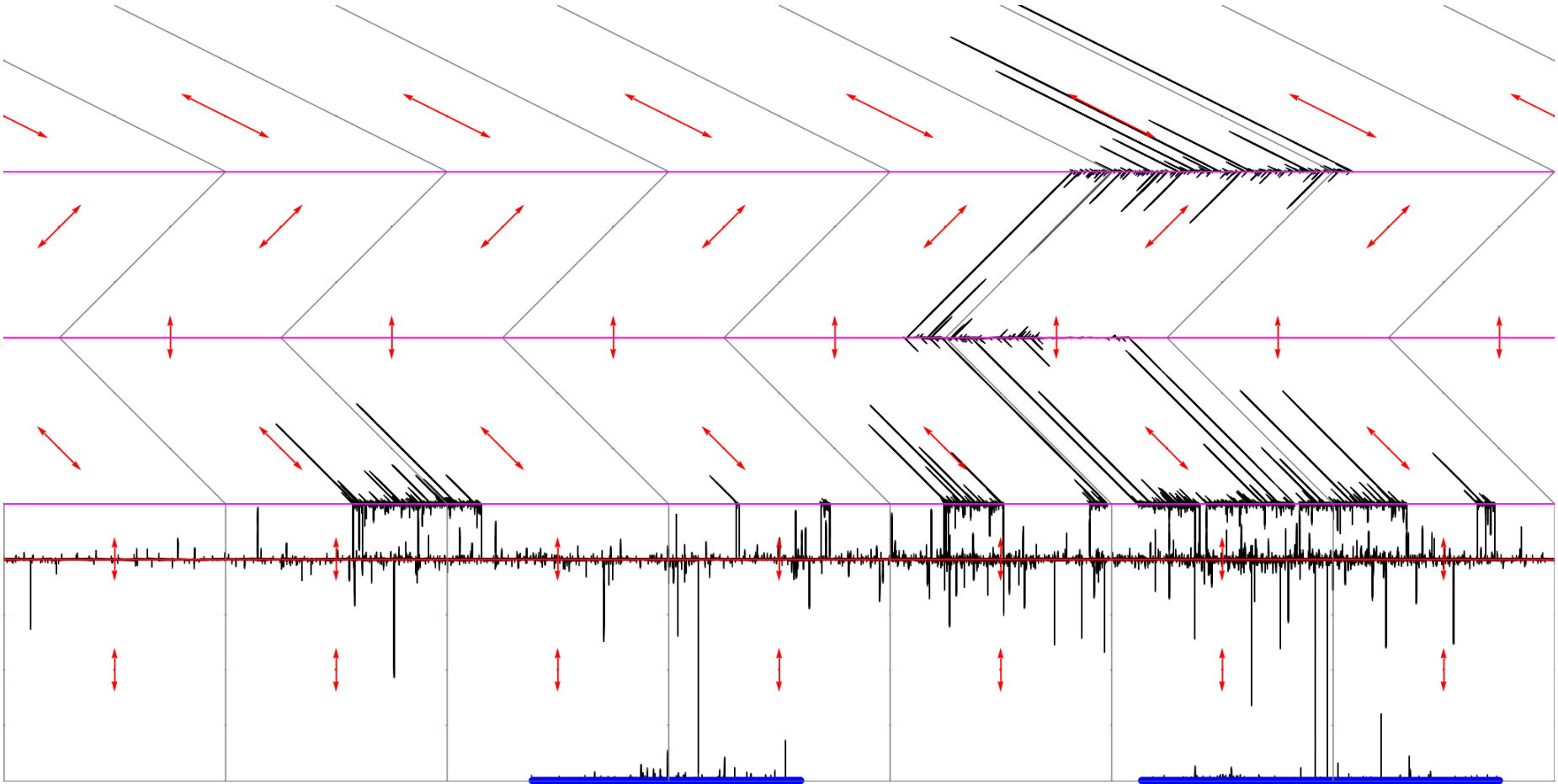
[Caffarelli, Silvestre, 2007]

$$Lu = 0$$

Thm [K, Mucha, 2018]

- $K = -\psi(-\Delta)$ for a complete Bernstein function ψ
- Converse is true if we allow singular L

Question What other K have similar representation?



5. Half-plane: shift-invariant diffusions

Motivating questions

- Are **asymmetric α -stable Lévy processes** boundary traces?
- More generally: which **Lévy processes** are boundary traces?
- Even more generally: which **Markov processes** are boundary traces?

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Rem Answers depend on how singular diffusions are allowed,
but very little seems to be known about **existence**

Uniqueness is better understood [Astala, Lousas, Päivärinta, 2005]

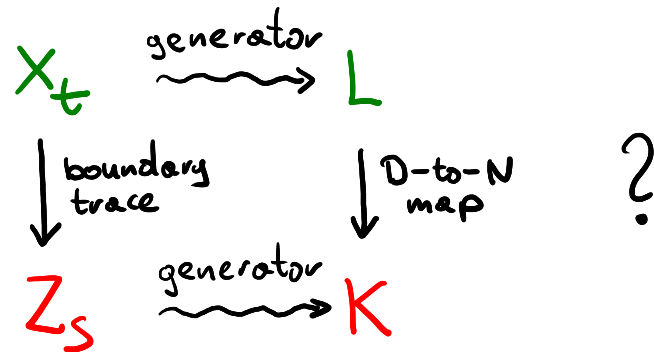
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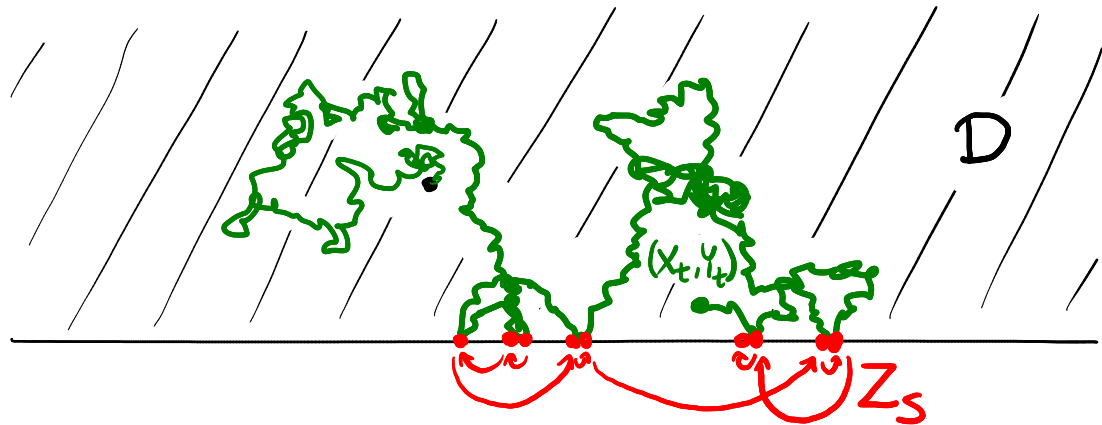
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- Do we still have:



Half-plane



D - half-plane
(or half-space)

X_t - Brownian motion in $\mathbb{R}$ $\overset{\text{independent}}{\leftarrow}$

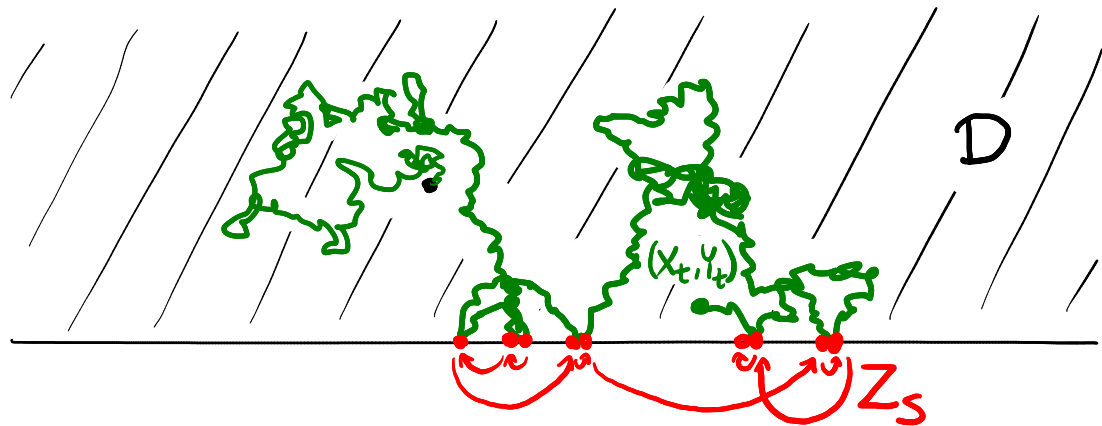
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L_s^{-1} - inverse local time of Y_t at 0

$Z_s = X_{L_s^{-1}}$ - subordinate
Brownian motion

Independence of X_t and Y_t
plays a key role

Half-plane



Every x -symmetric diffusion is equivalent to that

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(or half-space)

X_t - Brownian motion in $\mathbb{R}$ $\overset{\text{independent}}{\longleftrightarrow}$

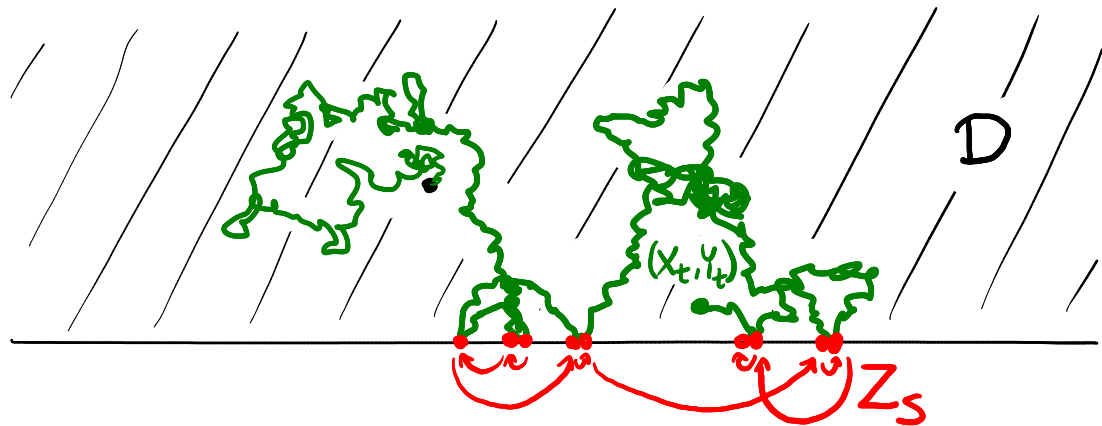
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Half-plane



Every x-symmetric diffusion
is equivalent to that

Without symmetry we no longer
have X_t and Y_t independent!

D - half-plane
(or half-space)

X_t - Brownian motion in $\mathbb{R}$ ^{independent}

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Shift-invariant diffusions

Natural assumption: coefficients only depend on the "height",
that is, the law of (X_t, Y_t) is invariant under horizontal shifts

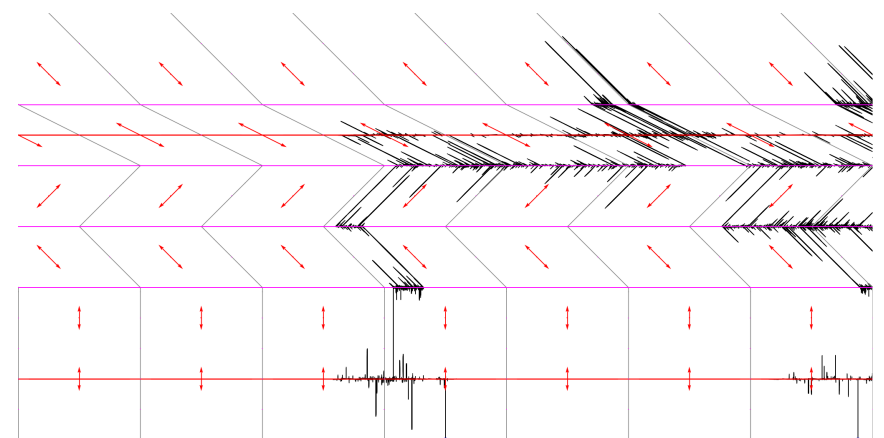
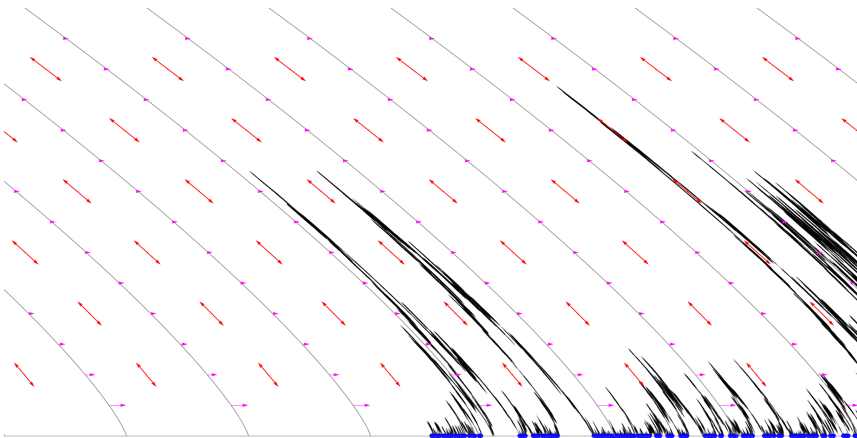
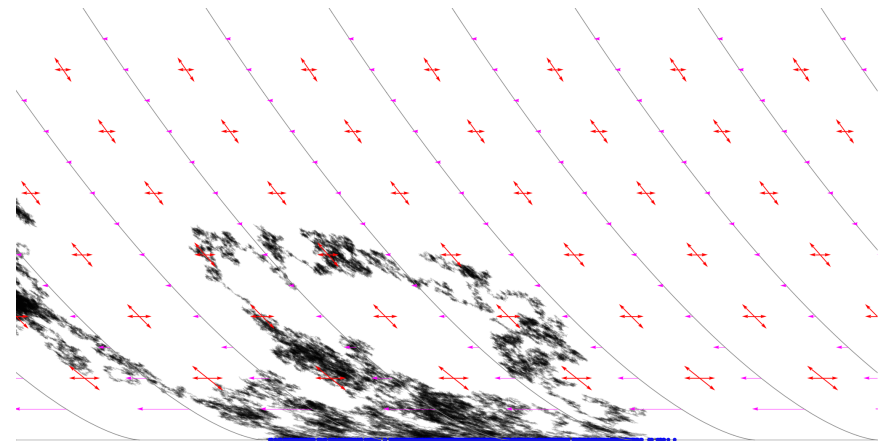
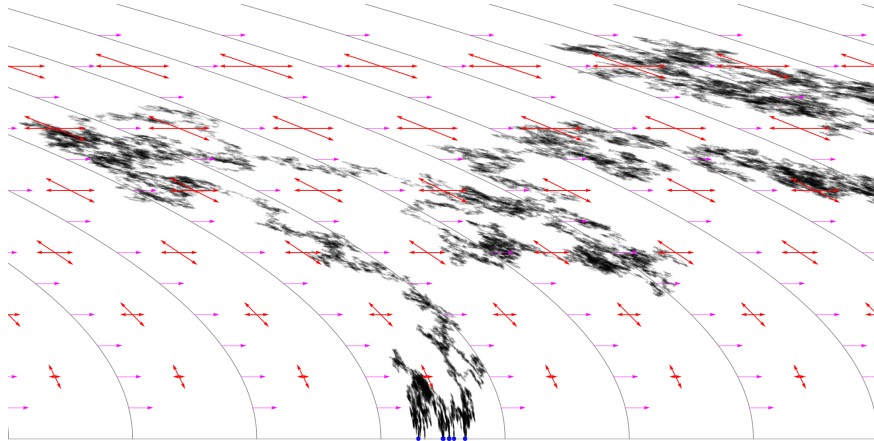
Think:

$$dX_t = \sqrt{a(Y_t)} dW_1(t) + b(Y_t) dW_2(t) + c(Y_t) dt$$

$$dY_t = \sqrt{d(Y_t)} dW_2(t) + e(Y_t) dt$$

Shift-invariant diffusions

Natural assumption: coefficients only depend on the "height"



Key ideas

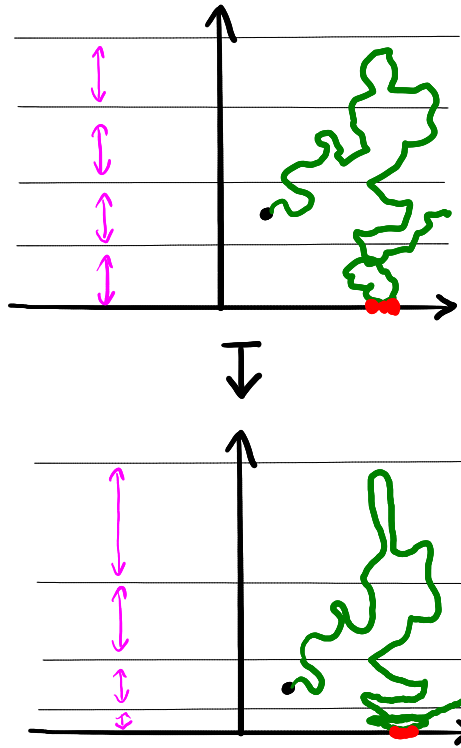
① Transform (X_t, Y_t) without affecting Z_s :

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Key ideas

① Transform (X_t, Y_t) without affecting Z_s :

- scale Y_t

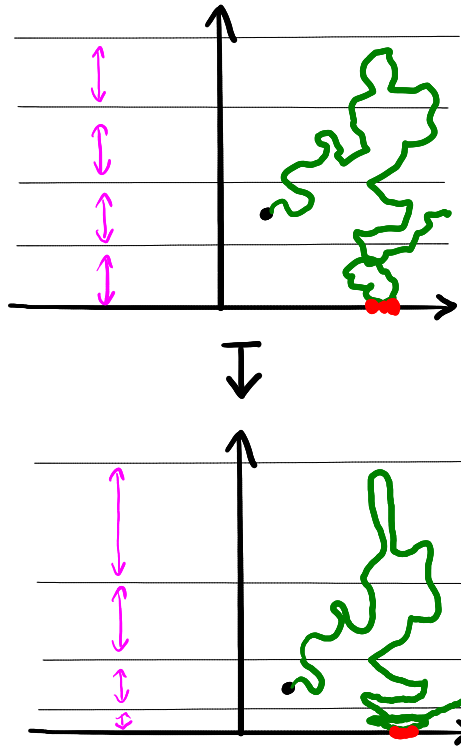


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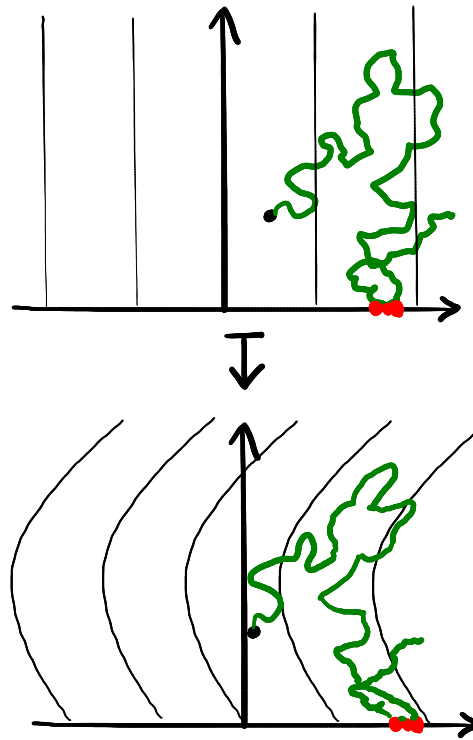
Key ideas

① Transform (X_t, Y_t) without affecting Z_s :

- scale Y_t



- adjust profile

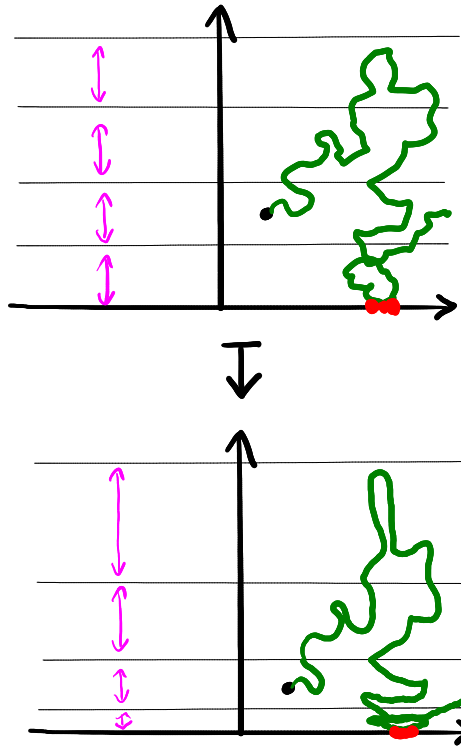


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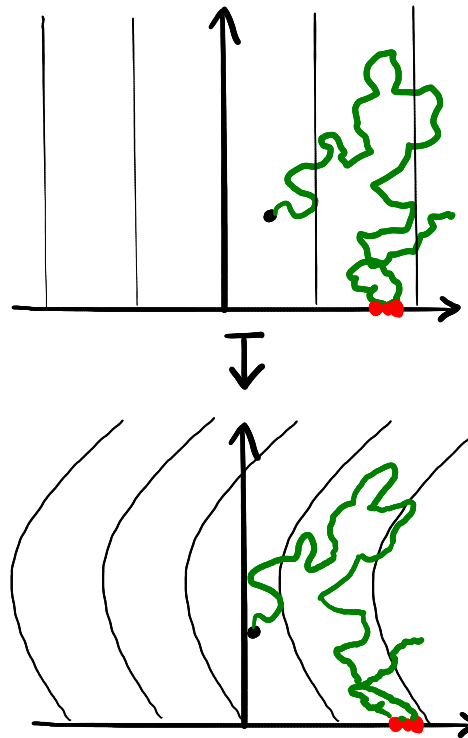
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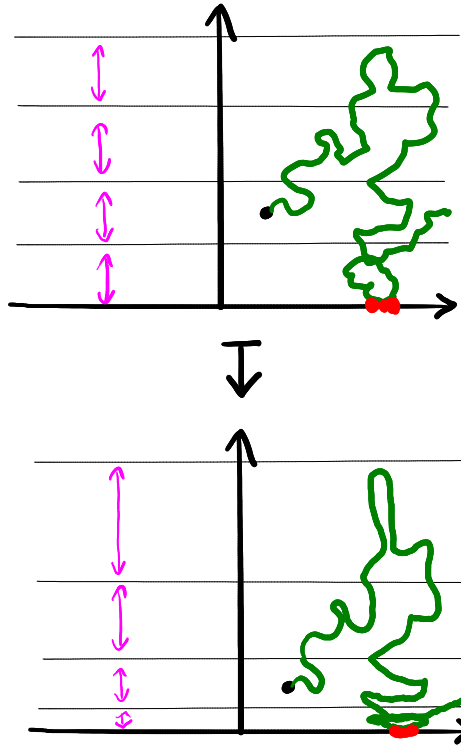
- change time

$$dX_t = \sqrt{a(Y_t)} dW_1(t) + b(Y_t) dW_2(t) + \cancel{c(Y_t) dt} \quad dY_t = \cancel{\sqrt{d(Y_t)}} dW_2(t) + \cancel{e(Y_t) dt}$$

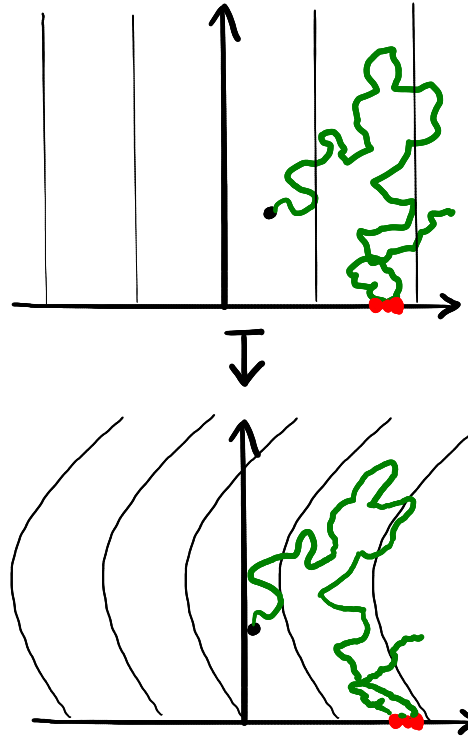
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- scale Y_t



- adjust profile



- change time

... so that:

- Y_t is the reflected Brownian motion
- X_t is a martingale

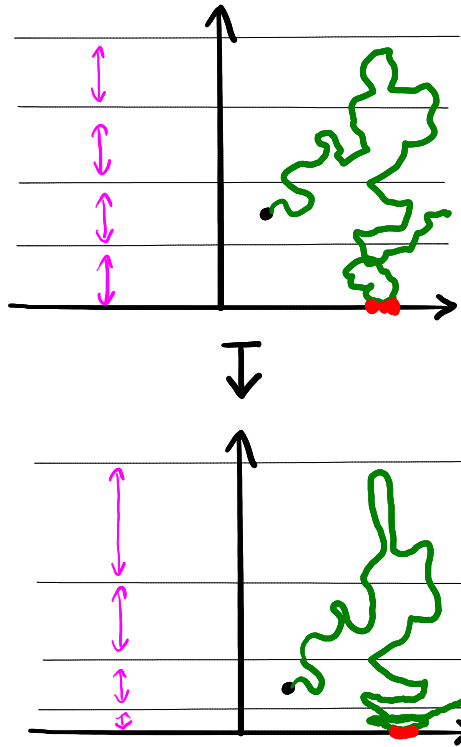
$$dX_t = \sqrt{a(Y_t)} dW_1(t) + b(Y_t) dW_2(t)$$

$$dY_t = dW_2(t)$$

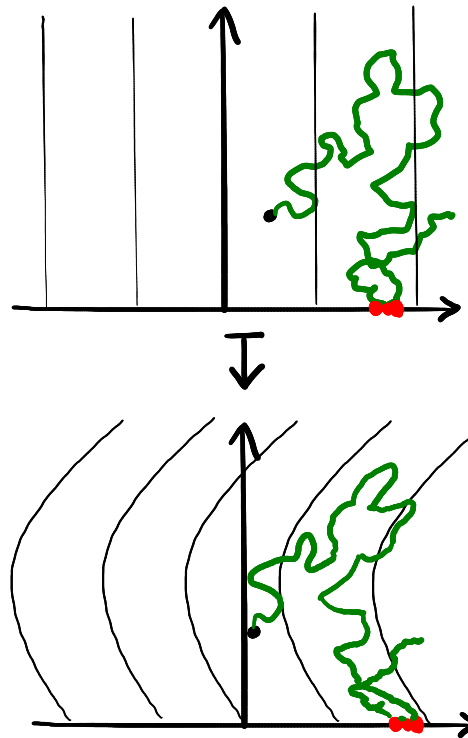
Key ideas

① Transform (X_t, Y_t) without affecting Z_s :

- scale Y_t



- adjust profile



- change time

... so that:

- Y_t is the reflected Brownian motion
- X_t is a martingale

Then (Y_t, X_t) is a Markov additive process

$$dX_t = \sqrt{a(Y_t)} dW_1(t) + b(Y_t) dW_2(t)$$

$$dY_t = dW_2(t)$$

Key ideas (continued)

② $X_t = W_{A_t} + B_t$ where

- W_t is a Brownian motion independent from Y_t
- A_t is increasing, B_t is a martingale
- A_t and B_t are additive functionals of Y_t

[Çinlar, 1972]

Key ideas (continued)

$$\textcircled{2} \quad X_t = W_{A_t} + B_t \quad \text{where} \quad dA_t \overset{\text{measure}}{\downarrow} = a(Y_t) dt, \quad dB_t \overset{\text{locally } L^2}{\downarrow} = b(Y_t) dY_t$$

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[Çinlar, 1972]

$$\textcircled{3} \quad \hat{X}_t = \mathbb{E}[e^{i\zeta X_t} | Y] = e^{-\frac{\zeta^2}{2} A_t + i\zeta B_t} \quad \text{solves SDE:}$$

$$d\hat{X}_t = i\zeta \hat{X}_t dB_t - \frac{\zeta^2}{2} \hat{X}_t dA_t - \frac{\zeta^2}{2} \hat{X}_t d\langle B \rangle_t$$

Key ideas (continued)

$$d\hat{X}_t = i\zeta \hat{X}_t d\mathcal{B}_t - \frac{\sigma^2}{2} \hat{X}_t d\mathcal{A}_t - \frac{\sigma^2}{2} \hat{X}_t d\langle \mathcal{B} \rangle_t$$

④ Analogous ODE:

$$-\frac{1}{2}\varphi''(y) = i\zeta \varphi'(y)b(y) - \frac{\sigma^2}{2}\varphi(y)a(y) - \frac{\sigma^2}{2}\varphi(y)(b(y))^2$$

[de Branges, 1960-62]

$$(\varphi(0) = 1, \varphi(\infty) = 0)$$

[Eckhardt, Kostenko, 2015]

Key ideas (continued)

$$d\hat{X}_t = i\zeta \hat{X}_t dB_t - \frac{\zeta^2}{2} \hat{X}_t dA_t - \frac{\zeta^2}{2} \hat{X}_t d\langle B \rangle_t$$

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Key ideas (continued)

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$$\textcircled{5} \quad d(\hat{X}_t \varphi(Y_t)) = \hat{X}_t \varphi'(Y_t) dL_t + [\text{martingale}] \quad / \quad \mathbb{E}\left[\int_0^{L_s^{-1}} \dots\right]$$

$$\mathbb{E}[\hat{X}_{L_s^{-1}} \varphi(Y_{L_s^{-1}})] - 1 = \int_0^s \mathbb{E}[\hat{X}_{L_s^{-1}} \varphi'(Y_{L_s^{-1}})] ds$$

Key ideas (continued)

$$\mathbb{E} \left[\hat{X}_{L_s^{-1}} \overbrace{\varphi(Y_{L_s^{-1}})}^{= \varphi(0)=1} \right] - 1 = \int_0^s \mathbb{E} \left[\hat{X}_{L_s^{-1}} \overbrace{\varphi'(Y_{L_s^{-1}})}^{= \varphi'(0)} \right] ds$$

Key ideas (continued)

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Key ideas (continued)

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 & \quad \parallel \\
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 & \quad \parallel \\
 & \mathbb{E} \left[e^{i\xi Z_s} \right] \\
 & \quad \parallel \\
 & \varphi'(0) \int_0^s \mathbb{E} \left[\hat{X}_{L_s^{-1}} \right] ds \\
 & \quad \parallel \\
 & \varphi'(0) \int_0^s \mathbb{E} \left[e^{i\xi Z_s} \right] ds
 \end{aligned}$$

• Characteristic exponent: $\mathbb{E} \left[e^{i\xi Z_s} \right] = e^{s \varphi'(0)}$

Key ideas (continued)

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• Characteristic exponent: $\mathbb{E}[e^{i\xi Z_s}] = e^{s\varphi'(0)}$

• $\xi \mapsto -\frac{\varphi'(0)}{\xi}$ is a Nevanlinna-Pick function [de Branges, 1970-72]
[Eckhardt, Kostenko, 2015]

Key ideas (continued)

$$\begin{aligned} \mathbb{E}[\hat{X}_{L_s^{-1}} \overbrace{\varphi(Y_{L_s^{-1}})}^{=\varphi(0)=1}] - 1 &= \int_0^s \mathbb{E}[\hat{X}_{L_s^{-1}} \overbrace{\varphi'(Y_{L_s^{-1}})}^{=\varphi'(0)}] ds \\ &\stackrel{||}{=} \mathbb{E}[\hat{X}_{L_s^{-1}}] \stackrel{||}{=} \varphi'(0) \int_0^s \mathbb{E}[\hat{X}_{L_s^{-1}}] ds \\ &\stackrel{||}{=} \mathbb{E}[e^{i\xi Z_s}] \stackrel{||}{=} \varphi'(0) \int_0^s \mathbb{E}[e^{i\xi Z_s}] ds \end{aligned}$$

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- $\xi \mapsto -\frac{\varphi'(0)}{\xi}$ is a Nevanlinna-Pick function [de Branges, 1970-72]
[Eckhardt, Kostenko, 2015]
- The Lévy measure of Z_s has a density ν such that $\nu(z)$ and $\nu(-z)$ are completely monotone [Rogers, 1983]

Main result

Thm [K, 2023]

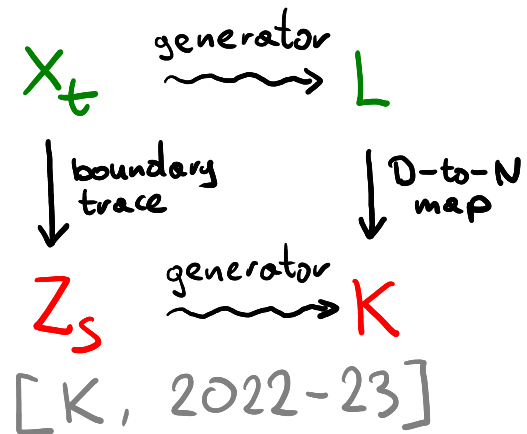
A Lévy process Z_s is the boundary trace of a shift-invariant diffusion if and only if the Lévy measure (intensity of jumps) of Z_s has a density function ν such that $\nu(z)$ and $\nu(-z)$ are completely monotone

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Bonus:

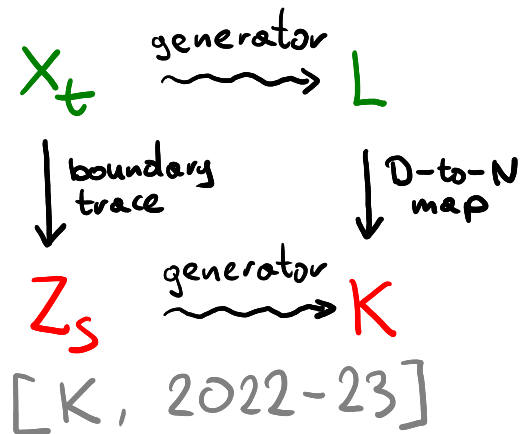


Main result

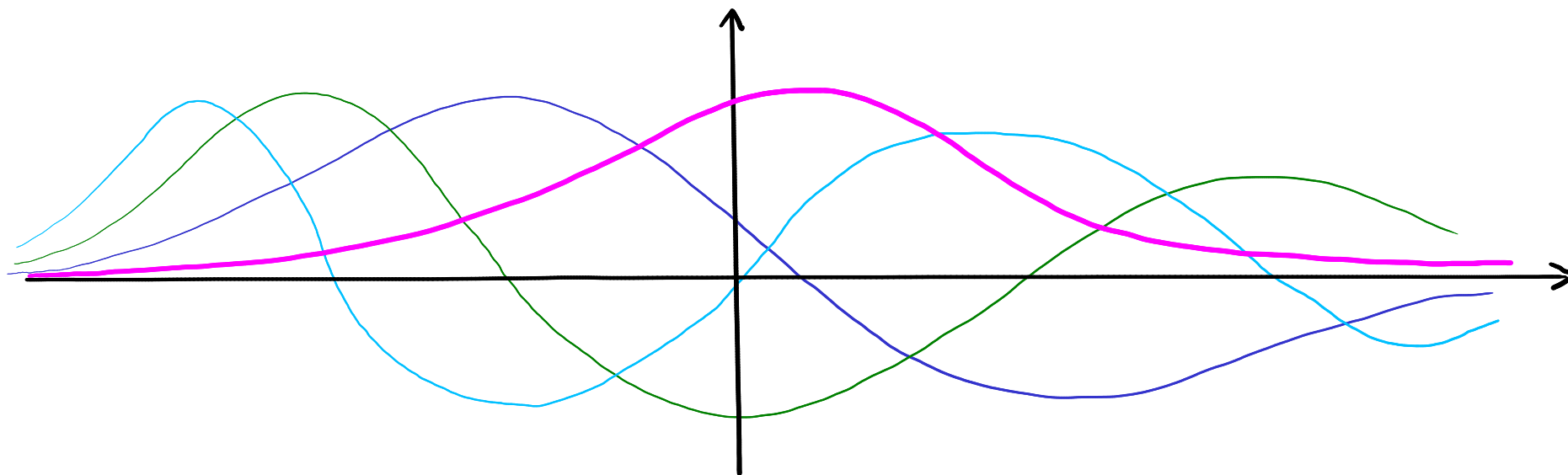
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A Lévy process Z_s is the boundary trace of a shift-invariant diffusion if and only if the Lévy measure (intensity of jumps) of Z_s has a density function ν such that $\nu(z)$ and $\nu(-z)$ are completely monotone

Bonus:



No need for
gap diffusions!

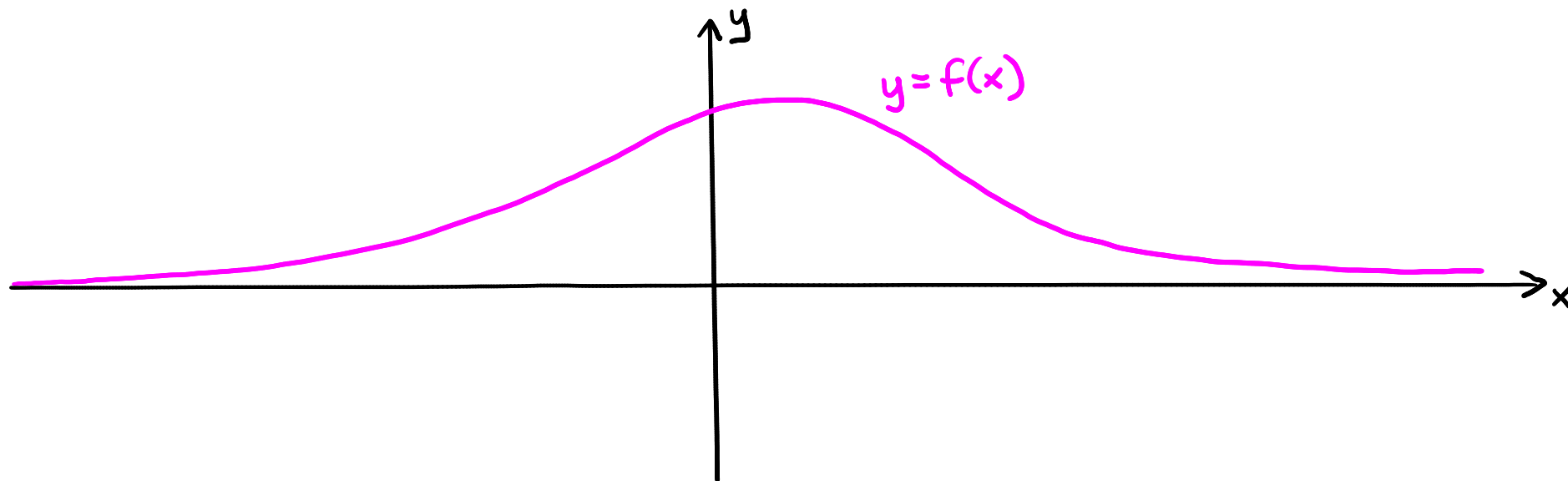


6. Bell-shape

Bell-shape

Def $f: \mathbb{R} \rightarrow [0, \infty)$ is bell-shaped if

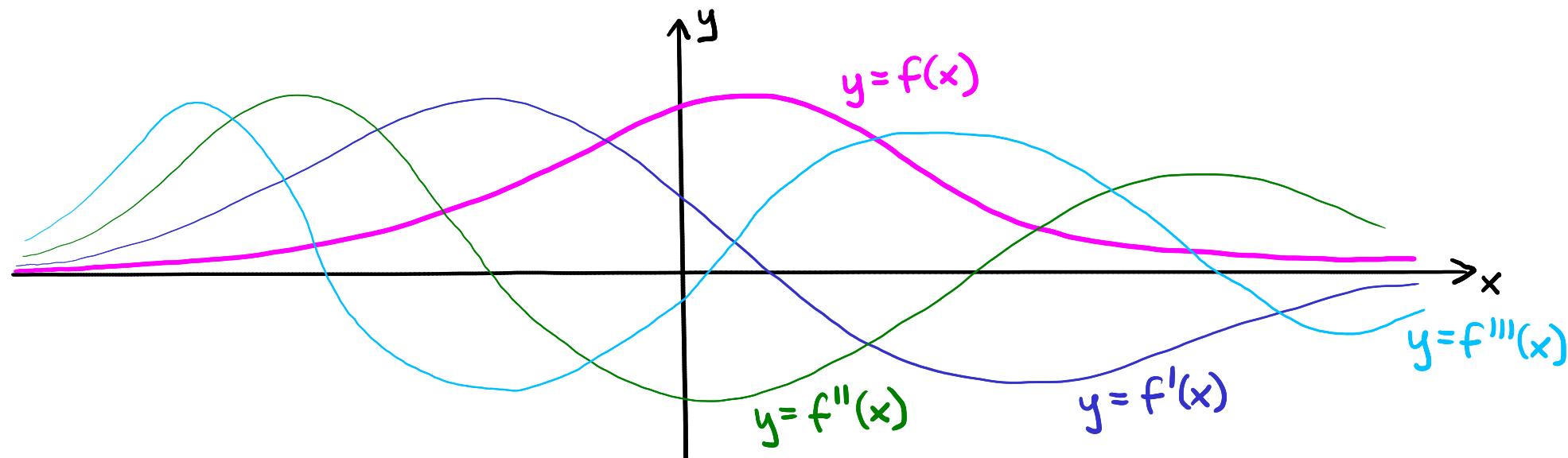
- $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$
- $f^{(n)}(x)$ changes sign n times for $n=0,1,2,\dots$



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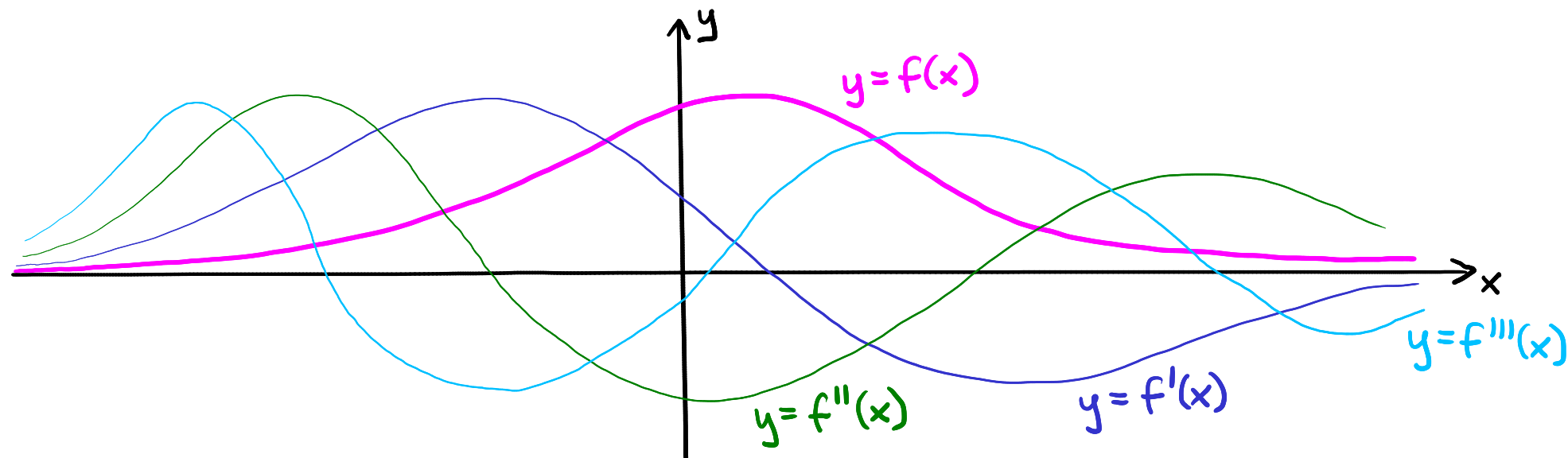
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Examples: e^{-x^2} , $\frac{1}{1+x^2}$, $x^{-\frac{3}{2}} e^{-\frac{1}{x}} \mathbb{1}_{(0,\infty)}(x)$, stable densities

Main result (continued)

Thm [K, 2025⁺]

The distribution X_{τ_D} of X_t at the first hitting time of ∂D has a bell-shaped density

Main result (continued)

Thm [K, 2025⁺]

The distribution X_{τ_D} of X_t at the first hitting time of ∂D has a bell-shaped density

The converse is not true, but very close