# Regularisation of differential equations by multiplicative fractional noise

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# Overview

# Introduction

Regularization by noise: adding noise to certain deterministic systems makes them behave better.

Example: Let $f : \mathbb{R}^d \to \mathbb{R}^d$, and consider the following differential equation

$$dX_t = f(X_t)\,dt, \quad X_0 = x_0.$$

- Cauchy-Lipschitz/Picard–Lindelöf theorem: If $f \in \mathcal{C}^1_x$, then there exists a unique solution.
- For all $\alpha \in (0,1)$, there exists $f \in \mathcal{C}^\alpha_x$ such that it has infinitely many solutions.
  Indeed: for $f(x) = |x|^\alpha$, $x_0 = 0$, all of the below are solutions

$$X^c_t = \begin{cases} 0 & 0 \le t \le c \\ N_\alpha (t-c)^{1/(1-\alpha)} & t > c. \end{cases}$$

# Introduction

- For $\alpha = 0$ , it might even have a solution. For example, take $x_0 = 0$ and

$$f(x) = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0, \end{cases}$$

**Theorem (Zvonkin '74; Veretennikov '80)**

*Let $\sigma$ be strongly elliptic $+\ldots$ . For all $f \in C_x^0$ and all $x_0 \in \mathbb{R}^d$, the equation*

$$dX_t = f(X_t)\,dt + \sigma(X_t)\,dB_t, \qquad X_0 = x_0,$$

*admits a unique strong solution.*

# Introduction

Further works:

- [Krylov & Röckner; 05]: $f \in L^t_q L^x_p$, $d/p + 2/q < 1$, and recently crititcal cases [Röckner & Zhao; 21], 19 x [Krylov; 20-22],
- [Flandoli, Russo, Wolf; 03, 04], [Flandoli, Issoglio, Russo; '14]

$$X_t - Y_t = \int_0^t \big(f(X_s) - f(Y_s)\big)\, ds + \int_0^t \big(\sigma(X_s) - \sigma(Y_s)\big)\, dW_s$$

$$\partial_t u^i + L u^i = -f^i, \qquad u^i(T, x) = 0$$

$$\int_0^t f^i(X_s)\, ds = u^i(0, x) - u^i(t, X_t) + \int_0^t u^i_{x_j}(s, X_s)\, dB^j_s$$

Key points:

- Markovianity
- Itô calculus

# Introduction

- [Davie '07]: path-by-path uniqueness for $f \in \mathcal{C}^0_x$.

$$X_t = x_0 + \int_0^t f(X_s)\, ds + B_t$$

$$x \mapsto \int_0^t f(B_s + x)\, ds$$

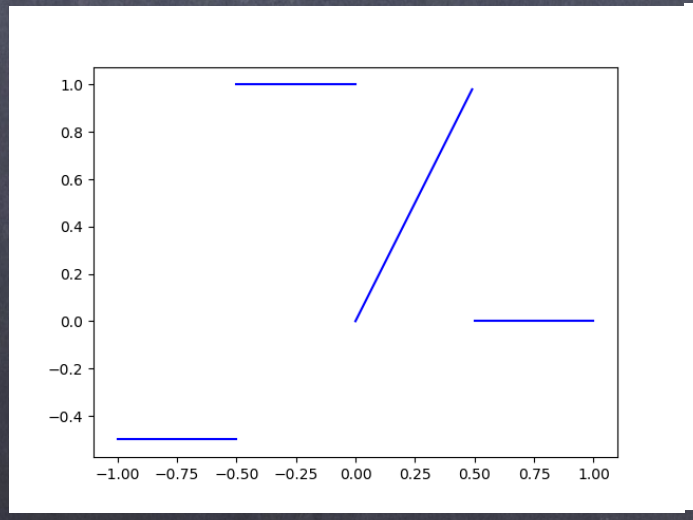


Figure: $x \mapsto f(x)$

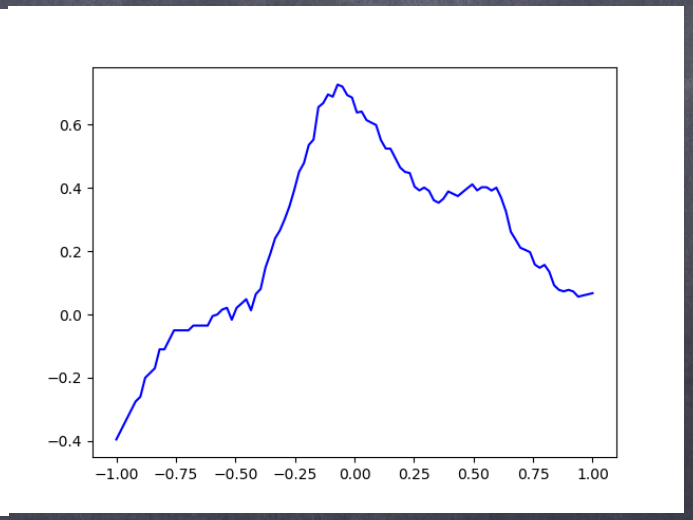


Figure: $x \mapsto \int_0^1 f(B_s + x)\, ds$

- [Catellier & Gubinelli '16]: $f \in \mathcal{C}^\alpha_x$, $\alpha > 1 - 1/2H$

$$X_t = x_0 + \int_0^t f(X_s)\, ds + B^H_s$$

- [Harang, Perkowski; 20], $f \in \mathcal{S}'$
- [Galeatti Gubinelli; 21], "noiseless" regularization by noise
- [Galeatti & Gerencsér; 22], $f \in L^q_t \mathcal{C}^\alpha_x$, $q \in (1, 2]$, $\alpha > 1 - 1/q^* H$
- [D. & Gerencsér; 22], multiplicative noise $\alpha \in (1 - 1/(2H)) \vee 0$, $H > 1/3$
- [Catellier, Duboscq; 22], multiplicative noise $\alpha \in (3/2 - 1/(2H)) \vee 0$, $H \in (1/4, 1/2)$

# Equations with fractional noise

We are interested in the equation

$$dX_t = f(X_t)\,dt + \sigma(X_t)\,dB_t^H, \qquad X_0 = x_{\cdot 0}$$

$B^H$ is a fractional Brownian motion of Hurst parameter $H \in (0,1)$:

$$\mathbb{E}[B_t^H \otimes B_s^H] = \frac{1}{2}(|t|^{2H} - |s|^{2H} - |t-s|^{2H})\mathbb{I}$$

$$B_t^H := \int_{-\infty}^{t} |t-s|^{H-1/2} + |s_-|^{H-1/2}\,dW_s$$

$$\mathcal{F}_t = \sigma(W_s, s \le t), \qquad B_t^H - \mathbb{E}^s B_t^H \perp\!\!\!\perp \mathcal{F}_s$$

- $\mathbb{P}(\Omega_H) = 1$, where $\Omega_H = \{B^H \in \mathcal{C}^{H-}\}$
- Young differential equation for $H \in (1/2, 1)$
- Rough DE for $H \in (1/3, 1/2)$

# The Young case $H \in (1/2, 1)$

## Assumption

*$f \in \mathcal{C}_x^\alpha$ for some $\alpha > 1 - 1/(2H)$, $\sigma \in \mathcal{C}_x^2$, and $\sigma\sigma^\top \succeq \lambda I$ for some $\lambda > 0$.*

## Definition

Given $\omega \in \Omega_H$, and $x_0 \in \mathbb{R}^d$, we say that a function $Y : [0,1] \to \mathbb{R}^d$ is a *solution* if $Y \in \mathcal{C}_t^\beta$ for some $\beta > 1 - H$ and it satisfies the equation.

## Definition

a stochastic process $(X_t)_{t \in [s_0,1]}$ is a *strong solution* if it is adapted and for almost all $\omega \in \Omega_H$, the function $X(\omega) : [0,1] \to \mathbb{R}^d$ is a solution.

## Theorem

*There exists a strong solution. Moreover, there exists an event $\widehat{\Omega} \subset \Omega_H$ of full probability such that for any $\omega \in \widehat{\Omega}$, $x_0 \in \mathbb{R}^d$, any two solutions coincide.*

# The Young case $H \in (1/2, 1)$

$$X_t - Y_t = \int_0^t f(X_s) - f(Y_s)\, ds + \int_0^t \sigma(X_s) - \sigma(Y_s)\, dB_s^H$$

Strategy: We want to study the regularity of the map

$$Z \mapsto \int_0^t \big(f(X_r + Z_r) - f(X_r)\big)\, dr$$

Main tool: Stochastic Sewing Lemma and its modifications

# The Young case $H \in (1/2, 1)$

**Theorem (K. Lê ; 18)**

*Let $p \geq 2$, $0 \leq S < T$ and let $A_{s,t} \in L_p(\Omega)$ for $S \leq s \leq t \leq T$ with $A_{s,t} \in \mathcal{F}_t$. Suppose that for some $\varepsilon_1, \varepsilon_2 > 0$ and $C_1, C_2$*

$$\|A_{s,t}\|_{L_p(\Omega)} \leq C_1 |t-s|^{1/2+\varepsilon_1}$$
$$\|\mathbb{E}^s(A_{s,t} - A_{s,u} - A_{u,t})\|_{L_p(\Omega)} \leq C_2 |t-s|^{1+\varepsilon_2}$$

*Then $\exists!$ adapted process $(\mathcal{A}_t)_{t\in[S,T]}$ such that $\mathcal{A}_S = 0$,*

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L_p(\Omega)} \leq N_1 |t-s|^{1/2+\varepsilon_1}$$
$$\|\mathbb{E}^s(\mathcal{A}_t - \mathcal{A}_s - A_{s,t})\|_{L_p(\Omega)} \leq N_2 |t-s|^{1+\varepsilon_2}$$

*Moreover, $\mathcal{A}$ satisfies the bounds*

$$\|\mathcal{A}_s - \mathcal{A}_t\|_{L_p(\Omega)} \leq NC_1 |t-s|^{1/2+\varepsilon_1} + NC_2 |t-s|^{1+\varepsilon_2}.$$

## Lemma

*Let $X$ be a strong solution. There exists $N \geq 0$ such that for all adapted stochastic processes $Z$ the following bound holds*

$$\Big\| \int_s^t \big(f(\tilde{X}_r + Z_r) - f(\tilde{X}_r)\big)\, dr \Big\|_{L_p(\Omega)} \leq N \|f\|_{\mathcal{C}_x^\alpha} \|Z\|_{\mathscr{C}_p^0} |t-s|^{1/2+\varepsilon}$$
$$+ N \|f\|_{\mathcal{C}_x^\alpha} \Big( \big\| (1 + [B^H]_{\mathcal{C}^{H-}|\mathbb{F}}) Z \big\|_{\mathscr{C}_p^0} + [Z]_{\mathscr{C}_p^{1/2}} \Big) |t-s|^{1+\varepsilon}.$$

Skech: The increments of the process that we want to study are

$$\mathcal{A}_t - \mathcal{A}_s = \int_s^t f(B_r^H + z) - f(B_r^H)\, dr$$

we will study instead

$$A_{s,t} = \int_s^t \mathbb{E}^s \big(f(B_r^H + z) - f(B_r^H)\big)\, dr$$

# The Young case $H \in (1/2, 1)$

$$|A_{s,t}| \leq \int_s^t \left| \mathbb{E}^s \big( f(B_r^H + z) - f(B_r^H) \big) \right| \, dr \left( B_r^H = (B_r^H - \mathbb{E}^s B_r^H) + \mathbb{E}^s B_r^H \right)$$

$$= \int_s^t |\mathcal{P}_{(r-s)^{2H}} f(\mathbb{E}^s B_r + z) - \mathcal{P}_{(r-s)^{2H}} f(\mathbb{E}^s B_r)| \, dr$$

$$\lesssim |z| \int_s^t [\mathcal{P}_{(r-s)^{2H}} f]_{\mathcal{C}^1} \, dr \lesssim |z| \int_s^t (r-s)^{-H(1-\alpha)} \|f\|_{\mathcal{C}^\alpha} \, dr$$

$$\leq |z| \|f\|_{\mathcal{C}_x^\alpha} (t-s)^{1+H(\alpha-1)} = N|z| \|f\|_{\mathcal{C}_x^\alpha} (t-s)^{1/2+\varepsilon}$$

$$\|A_{s,t}\|_{L_p} \leq N|z|\|f\|_{\mathcal{C}^\alpha_x}(t-s)^{1+H(\alpha-1)}.$$

Also $\mathbb{E}^s(A_{s,t} - A_{s,u} - A_{u,t}) = 0$. Moreover, $A_{s,t}$ is "close" to $\mathcal{A}_s - \mathcal{A}_t$, so by SSL

$$\Big\| \int_s^t f(B^H_r + z) - f(B^H_r)\, dr \Big\|_{L_p} \leq N|z|\|f\|_{\mathcal{C}^\alpha_x}(t-s)^{1+H(\alpha-1)},$$

i.e.,

$$z \mapsto \int_0^\cdot f(B^H_r + z)\, dz = T(z), \qquad T : \mathbb{R} \to \mathcal{C}^{1+H(\alpha-1)}([0,1]; L_p(\Omega))$$

is "Lipschitz" continuous

For the real estimate we consider something like

$$A_{s,t} = \mathbb{E}^s \int_s^t f(\Xi_{s,r} + Z_s) - f(\Xi_{s,r})\, dr$$

$$\Xi_{s,r} \approx X_r$$

$$\Xi_{s,r} \approx X_s + f(X_s)(r-s) + \sigma(X_s)B^H_{s,t}$$

# The Young case $H \in (1/2, 1)$

**Lemma**

*Let $X$ and $Y$ be strong solutions with initial conditions anf drifts $(x_0, f^X)$ and $(y_0, f^Y)$, respectively. Then for all $C \geq 1$, we have*

$$\|X_{\cdot\wedge\tau} - Y_{\cdot\wedge\tau}\|_{\mathscr{C}_p^{1/2}}$$
$$\leq N^{C^{2-\gamma}}\Big(|x_0 - y_0| + \big(\mathbb{P}([B^H]_{\mathcal{C}^{H-}|\mathbb{F}} \geq C)\big)^{1/2p} + \|f^X - f^Y\|_{\mathcal{C}_x^0}\Big),$$

*$N$ and $\gamma \in (0, 2)$ depend only on structural constants.*

Consequences:

1) stability with respect to the drift
2) Hölder dependence on initial condition
3) 1) $\implies$ existence of strong a solution
4) 2)+... $\implies$ existence of regular semiflow $\hat{X}_t^{s,x}$
5) The existence of the regular semiflow implies path-by-path uniqueness by an argument of [Shaposhnikov; 16]

# The rough case $H \in (1/3, 1/2)$

$$dX_t = f(X_t)\,dt + \sigma(X_t)\,dB_t^H, \qquad X_0 = x.$$

Integral can not be defined as a Young integral. Rough path setting.

Notation:

- $\mathcal{R}^\beta$ denotes the collection of rough paths $(Z, \mathbb{Z}) \in \mathcal{C}^\beta \times \mathcal{C}_2^{2\beta}$
- For $\gamma \in (\beta, 2\beta]$, $\mathcal{D}_Z^\gamma$ denotes the space of controlled paths $(Y, Y')$ with

$$Y_{s,t} = Y'_s Z_{s,t} + R^Y_{s,t}, \qquad Y' \in \mathcal{C}^{\gamma-\beta}, \qquad R^Y \in \mathcal{C}_2^\gamma$$

$$[(Y, Y')]_{\mathcal{D}_Z^\gamma} = [Y']_{\mathcal{C}^{\gamma-\beta}} + [R^Y]_{\mathcal{C}_2^\gamma}$$

If $\gamma + \beta > 1$, then

$$\int_0^t Y\,dZ_r := \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} Y_u Z_{u,v} + Y'_u \mathbb{Z}_{u,v}$$

# The rough case $H \in (1/3, 1/2)$

## Assumption

*$f \in \mathcal{C}_x^\alpha$ for some $\alpha > 0$, $\sigma \in \mathcal{C}_x^3$, and $\sigma\sigma^\top \succeq \lambda I$ for some $\lambda > 0$. $\mathbb{B}_{s,t}^H \in \mathcal{F}_t$, and $\mathbb{P}((B^H, \mathbb{B}^H) \in \mathcal{R}^\beta, \forall \beta < H) = 1$.*

## Definition

Given $\omega \in \Omega_H$ and $x_0 \in \mathbb{R}^d$, we say that a function $Y : [0,1] \to \mathbb{R}^d$ is a *solution* if $\big(Y, \sigma(Y)\big) \in \mathcal{D}^\gamma_{B^H(\omega)}([0,1])$ for some $\gamma > 1 - H$ and it satisfies the equation.

## Definition

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# The rough case $H \in (1/3, 1/2)$

- Lipschitz estimates for the drift as $\mathcal{C}_t^H \to \mathcal{C}_t^{1+(\alpha-1)H}$.
- Stability estimates in $L_p(\Omega; \mathcal{D}_{B^H}^{1-H^-})$ (buckling)

$$
\begin{aligned}
\|X - Y\|_{\mathcal{D}_{B^H}^{1-H^-}} &\leq \|drift\|_{\mathcal{D}_{B^H}^{1-H^-}} + \|stoch\|_{\mathcal{D}_{B^H}^{1-H^-}} \\
&\lesssim \|drift\|_{\mathcal{C}_t^{1+(\alpha-1)H}} + \|X - Y\|_{\mathcal{D}_{B^H}^{1-H^-}} \\
&\lesssim \|X - Y\|_{\mathcal{C}_t^H} + \|X - Y\|_{\mathcal{D}_{B^H}^{1-H^-}} \\
&\lesssim \|X - Y\|_{\mathcal{D}_{B^H}^{1-H^-}}
\end{aligned}
$$

- Existence of strong solution and regular semi-flow
- Path-by-path uniqueness

# The rough case $H \in (1/3, 1/2)$

Why not $1 - 1/(2H) < \alpha < 0$?

- for $\alpha < 0$, $f$ is not a function but a distribution, hence, $f(X_s)$ can not be defined.
- This is not really a problem. $\int_0^t f(X_s)\, ds$ can be defined.
- Real problem: For $\alpha < 0$ we do not have $\mathcal{C}^{1+(\alpha-1)H} \subset \mathcal{D}_{B^H}^{1-H}$.

To buckle the equation, we need to find a space $\mathcal{S}$, such that

$$\|drift(X) - drift(X+Z)\|_{\mathcal{S}} \lesssim \|Z\|_{\mathcal{S}}$$

$$\| \int_0^{\cdot} \sigma(X) - \sigma(Y)\, dB^H \|_{\mathcal{S}} \lesssim \|X - Y\|_{\mathcal{S}}$$

In particular $\int_0^{\cdot} \nabla f(B_r^H)\, dr \in \mathcal{S}$. Next goal: Qive meaning to

$$\mathbb{Q}_{s,t} = \int_s^t \int_s^r \nabla f(B_u^H)\, du\, dB^H r$$

# The rough case $H \in (1/3, 1/2)$

**Theorem**

*Assume $\alpha > 1/2 - 1/(2H)$, $f \in \mathcal{C}_x^\alpha$, $\sigma \in \mathcal{C}_x^2$, and $x_0 \in \mathbb{R}^d$. Then there exists a (probabilistically) weak solution.*

$$X_t = x_0 + D_t + \int_0^t \sigma(X_s)\, dB_s^H,$$

For any sequence $(f^n)_{n\in\mathbb{N}} \subset \mathcal{C}_x^\infty$ with $f^n \to f$ in $\mathcal{C}_x^\alpha$ one has almost surely

$$D = \lim_{n\to\infty} \int_0^{\cdot} f^n(X_s)\, ds.$$

# Summary

- For $H > 1/2$: strong existence, path by path uniqueness provided that $\alpha > 1 - 1/(2H)$.
- For $H \in (1/3, 1/2)$: strong existence, path by path uniqueness provided that $\alpha > 0$. Existence of weak solutions for $1/2 - 1/(2H) < \alpha < 0$.


That's all Folks!
Thank you!