Propagation of Monotonicity for Mean Field Game Master Equations

Jianfeng ZHANG (USC)

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Outline

1. Introduction
2. Stochastic calculus on Wasserstein space
3. Propagation of (anti-)monotonicity condition for MFG
4. MFGC and propagation of (anti-)monotonicity condition

N-player games

- A large system : $i = 1, \cdots, N$,

$$X_t^i = x_i + \int_0^t b(X_s^i, \alpha_s^i, \nu_s^{\vec{\alpha}})ds + \int_0^t \sigma(\cdots)dB_s^i;$$

$$\mu_s^{\vec{\alpha}} := \frac{1}{N}\sum_{i=1}^N \delta_{X_s^i}, \quad \nu_s^{\vec{\alpha}} := \frac{1}{N}\sum_{i=1}^N \delta_{(X_s^i, \alpha_s^i)}.$$

- $J_i(0, \vec{x}, \vec{\alpha}) := \boldsymbol{E}\Big[G(X_T^i, \mu_T^{\vec{\alpha}}) + \int_0^T F(X_s^i, \alpha_s^i, \nu_s^{\vec{\alpha}})ds\Big]$

- Nash equilibrium $\vec{\alpha}^*$:

$$J_i(0, \vec{x}, \vec{\alpha}^*) \le J_i(0, \vec{x}, \vec{\alpha}^{*,-i}, \alpha^i), \quad \forall \alpha^i, \forall i.$$

- The goal : $N \to \infty$?

Mean field game (of control)

- The population :

$$X_t^\xi = \xi + \int_0^t b(X_s^\xi, \alpha_s, \nu_s^\alpha) ds + \int_0^t \sigma(\cdots) dB_s;$$

$$\mu_s^\alpha := \mathcal{L}_{X_s^\xi}, \quad \nu_s^\alpha := \mathcal{L}_{(X_s^\xi, \alpha_s)}.$$

- The individual player : using α',

$$X_t^x = x + \int_0^t b(X_s^x, \alpha'_s, \nu_s^\alpha) ds + \int_0^t \sigma(\cdots) dB_s; ;$$

$$J(0, \xi, \alpha; x, \alpha') := \boldsymbol{E}\Big[G(X_T^x, \mu_T^\alpha) + \int_0^T F(X_s^x, \alpha'_s, \nu_s^\alpha) ds\Big].$$

- Mean field game (MFG) : b, σ, F depend on μ^α

  Mean field game of control (MFGC) : b, σ, F depend on ν^α

Mean field equilibrium

- Mean field equilibrium α^* :

$$J(0,\xi,\alpha^*;x,\alpha^*) \leq J(0,\xi,\alpha^*;x,\alpha'), \quad \forall \alpha', \ \forall x.$$

- Caines-Huang-Malhame (2006), Lasry-Lions (2007)
- Lions (2008), Cardaliaguet (2010), Bensoussan-Frehse-Yam (2013), Carmona-Delarue (2018), ...

The master equation

- When MFE α^* is unique, introduce the value function

$$V(t,x,\mu) := J(t,\xi,\alpha^*;x,\alpha^*), \quad \mu := \mathcal{L}_\xi$$

- Our goal is to study the master equation for V.
    - ◇ V serves as the decoupling field of the mean field game system, either in PDE/SPDE form or in FBSDE form.

- The local (in time) wellposedness is relatively easy

- Our goal is the global wellposedness, which typically requires certain monotonicity condition

- Ok to include common noise, but so far, all the works assume σ is not controlled. For simplicity, $\sigma \equiv 1$.

Literature for global wellposedness

- Buckdahn-Li-Peng-Rainer (2017)
  - ◇ linear equation, not MFG, so monotonicity is not required
- Chassagneux-Crisan-Delarue (2014), Carmona-Delarue (2018), Cardaliaguet-Delarue-Lasry-Lions (2019)
  - ◇ $b(x, \alpha, \nu) = \alpha, \quad F = F_0(x, \mu) + F_1(x, \alpha)$ (separable)
  - ◇ Lasry-Lions monotonicity
- Mou-Z. (2019), Bertucci (2021), Cardaliaguet-Souganidis (2021)
  - ◇ same as above, but weak solutions
- Gangbo-Meszaros (2020), Bensoussan-Graber-Yam (2020)
  - ◇ similarly as above, but under displacement monotonicity

Literature for global wellposedness (cont.)

- Bayraktar-Cohen (2018), Bertucci-Lasry-Lions (2019)
  - ◇ Finite state MFG, under Lasry-Lions monotonicity
- Gomes-Voskanyan (2013), Carmona-Delarue (2018), Cardaliaguet-Lehalle (2018), Kobeissi (2020)
  - ◇ MFGC, under Lasry-Lions monotonicity
  - ◇ They are about mean field game system, not about master equation
  - ◇ To our best knowledge, there is no work on MFGC master equations

Our works

- Gangbo-Meszaros-Mou-Z. (2021)
  - ◇ MFG master equation with non-separable F, under displacement monotonicity
- Mou-Z. (2022a)
  - ◇ MFG master equation under anti-monotonicity
- Mou-Z. (2022b)
  - ◇ MFGC master equation under Lasry-Lions monotonicity, displacement monotonicity, anti-monotonicity

Our road map

appropriate conditions on data

(a prior) propagation of (anti-)monotonicity of V

(a prior) Lipschitz continuity of V under W_1

Global wellposedness of master equation

Convergence of the N-player game

Outline of the talk

- Stochastic calculus on Wasserstein space
- Propagation of (anti-)monotonicity condition for MFG
- MFGC and propagation of (anti-)monotonicity condition

- For simplicity
  - ◇ $d = 1$
  - ◇ No common noise

Outline

Wasserstein derivatives

- $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{R})$, Wasserstein distance W_2
- For $U : \mathcal{P}_2 \to \mathbb{R}$, linear derivative $\delta_\mu U : \mathcal{P}_2 \times \mathbb{R} \to \mathbb{R}$
  - ◇ $\lim_{\varepsilon \to 0} \dfrac{U(\mu + \varepsilon(\nu - \mu)) - U(\mu)}{\varepsilon} = \int_{\mathbb{R}} \delta_\mu U(\mu, \tilde{x})[\nu(d\tilde{x}) - \mu(d\tilde{x})]$
  - ◇ Lions derivative $\partial_\mu U(\mu, \tilde{x}) = \partial_{\tilde{x}} \delta_\mu U(\mu, \tilde{x})$
- For $V : (t, x, \mu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$,
  - ◇ $\partial_t V, \partial_x V, \partial_{xx} V$ are standard
  - ◇ $\delta_\mu V(t, x, \mu, \tilde{x})$ and higher order derivatives in obvious sense

Ito formula

- Let $V(t,x,\mu)$ be smooth, and $dX_t = b_t dt + \sigma_t dB_t$

$$dV(t,X_t,\mathcal{L}_{X_t}) = \partial_t V dt + \partial_x V dX_t + \frac{1}{2}\partial_{xx} V \sigma_t^2 dt$$
$$+\tilde{\boldsymbol{E}}\Big[\frac{1}{2}\partial_{\tilde{x}\mu} V(t,X_t,\mathcal{L}_{X_t},\tilde{X}_t)\tilde{\sigma}_t^2 + \partial_\mu V(t,X_t,\mathcal{L}_{X_t},\tilde{X}_t)\tilde{b}_t \Big| \mathcal{F}_t^X\Big] dt$$

  - ◇ $(\tilde{X},\tilde{b},\tilde{\sigma})$ is an independent copy of (X,b,σ)
  - ◇ Buckdahn-Li-Peng-Rainer (2017)
    Chassagneux-Crisan-Delarue (2014)

Hamiltonian

- MFG : assuming $b = \alpha$,

$$dX_t = \alpha_t^* dt + dB_t;$$
$$V_0 = \boldsymbol{E}\Big[G(X_T, \mathcal{L}_{X_T}) + \int_0^T F(X_t, \alpha_t^*, \mathcal{L}_{X_t})dt\Big]$$

- Hamiltonian $\quad H(x, \mu, p) := \inf_a \Big[ap + F(x, a, \mu)\Big]$
  - ◇ $\partial_{pp} H < 0$
  - ◇ $a^* = \partial_p H(t, x, \mu, p)$
- Separability :

$$F = F_0(x, \mu) + F_1(x, a) \Longrightarrow H = F_0(x, \mu) + H_1(x, p)$$

Master equation

- DPP + Ito formula $\Longrightarrow$ PDE
- Master equation for $V(t,x,\mu)$

$$
\begin{aligned}
&0 = \partial_t V + \frac{1}{2}\partial_{xx} V + H(x,\mu,\partial_x V(t,x,\mu)) \\
&+\tilde{\mathbb{E}}\Big[\frac{1}{2}\partial_{\tilde{x}\mu} V(t,x,\mu,\tilde{\xi}) + \partial_\mu V(t,x,\mu,\tilde{\xi})\ \partial_p H(\tilde{\xi},\mu,\partial_x V(t,\tilde{\xi},\mu))\Big]; \\
&V(T,x,\mu) = G(x,\mu).
\end{aligned}
$$

◇ Optimal control : $\alpha_t^* = \partial_p H(X_t, \mathcal{L}_{X_t}, \partial_x V(t, X_t, \mathcal{L}_{X_t}))$

◇ Non-local equation, comparison principle fails

Monotonicity conditions

- Lasry-Lions monotonicity condition

$$\boldsymbol{E}\Big[G(\xi_1,\mathcal{L}_{\xi_1})+G(\xi_2,\mathcal{L}_{\xi_2})-G(\xi_1,\mathcal{L}_{\xi_2})-G(\xi_2,\mathcal{L}_{\xi_1})\Big]\geq 0;$$

$$\text{or equivalently}\quad \tilde{\boldsymbol{E}}\Big[\partial_{x\tilde{x}}\delta_\mu G\big(\xi,\mathcal{L}_\xi,\tilde{\xi}\big)\ \eta\tilde{\eta}\Big]\geq 0,\quad \forall\xi,\eta$$

- Displacement monotonicity condition

$$\tilde{\boldsymbol{E}}\Big[\partial_{x\tilde{x}}\delta_\mu G\big(\xi,\mathcal{L}_\xi,\tilde{\xi}\big)\ \eta\tilde{\eta}+\partial_{xx}G(\xi,\mathcal{L}_\xi)\eta^2\Big]\geq 0,\quad \forall\xi,\eta \qquad (1)$$

- For potential games : $G(x,\mu)=\delta_\mu\mathcal{G}(\mu,x)$ for some $\mathcal{G}:\mathcal{P}_2\to\mathbb{R}$
  - ◇ Lasry-Lions monotonicity means : $\mu\in\mathcal{P}_2\to\mathcal{G}(\mu)$ is convex
  - ◇ Displacement monotonicity means : $\xi\in\mathbb{L}^2\to\mathcal{G}(\mathcal{L}_\xi)$ is convex

A motivating example

- Lasry-Lions : $G(x,\mu) = (x - m_\mu)^2$, where $m_\mu := \int x\mu(dx)$

$$\partial_{x\tilde{x}}\delta_\mu G(x,\mu,\tilde{x}) = -2,$$

$$\tilde{\boldsymbol{E}}\left[\partial_{x\tilde{x}}\delta_\mu G\left(\xi,\mathcal{L}_\xi,\tilde{\xi}\right)\ \eta\tilde{\eta}\right] = -2|\boldsymbol{E}[\eta]|^2 \leq 0.$$

- Carmona-Cooney-Graves-Lauriere (2019) :

$$G(x,\mu) = (x - m_\mu)^2 + (x - x_0)^2$$

- Displacement anti-monotonicity :

$$G(x,\mu) = (x - m_\mu)^2 - (x - x_0)^2$$

Outline

Characteristics

- Given ξ, η,

$$X_t = \xi + \int_0^t \partial_p H(X_s, \mu_s, \partial_x V(s, X_s, \mu_s))ds + B_t,$$

$$\mu_t := \mathcal{L}_{X_t};$$

$$\delta X_t := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[X_t^{\xi+\varepsilon\eta} - X_t^{\xi}] = \eta + \int_0^t [\cdots]ds + \int_0^t [\cdots]dB_s$$

The main idea

- For Lasry-Lions monotonicity, want

$$I_{LL}(t) := \tilde{\boldsymbol{E}}\Big[\partial_{x\mu} V(t, X_t, \mu_t, \tilde{X}_t)\ \delta X_t \delta \tilde{X}_t\Big] \geq 0.$$

- Notice that $V(T, \cdot) = G$, and thus $I_{LL}(T) \geq 0$

- Suffices to have

$$I'_{LL}(t) \leq 0 \quad \text{and thus} \quad I_{LL}(t) \geq 0$$

- Ito formula + master equation to analyze $I'_{LL}(t)$

The key calculation

- The non-separable case

$$I'_{LL}(t) = \tilde{E}\Big[H_{pp}(X_t)\Big|\tilde{E}_{\mathcal{F}_t}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t + \frac{H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t}{2H_{pp}(X_t)}\Big]\Big|^2$$

$$-\partial_{xx}V(X_t)H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t$$

$$-H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t - \frac{\Big|\tilde{E}_{\mathcal{F}_t}\Big[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\Big]\Big|^2}{4H_{pp}(X_t)}\Big].$$

- The separable case : $H = H_0(x,\mu) + H_1(x,p)$, and $H_{p\mu} = 0$,

$$I'_{LL}(t) = \tilde{E}\Big[H_{pp}(X_t)\Big|\tilde{E}_{\mathcal{F}_t}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t\Big]\Big|^2 - H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t\Big]$$

  ◇ Need : H_0 is Lasry-Lions monotone

Displacement monotonicity

$$I_D'(t) := \frac{d}{dt}\tilde{\boldsymbol{E}}\left[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta X_t\delta\tilde{X}_t + \partial_{xx}V(X_t)|\delta X_t|^2\right]$$

$$= \tilde{\boldsymbol{E}}\Bigg[H_{pp}(X_t)\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\Big[V_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t + V_{xx}(X_t)\delta X_t + \frac{H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t}{2H_{pp}(X_t)}\Big]\Big|^2$$

$$-H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t - H_{xx}(X_t)|\delta X_t|^2 - \frac{\left|\tilde{E}_{\mathcal{F}_t}\left[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\right]\right|^2}{4H_{pp}(X_t)}\Bigg].$$

- Our monotonicity condition

$$\tilde{\boldsymbol{E}}\left[H_{x\mu}(\xi,\tilde{\xi})\eta\tilde{\eta} + H_{xx}(\xi)\eta^2 + \frac{\left|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\left[H_{p\mu}(\xi,\tilde{\xi})\tilde{\eta}\right]\right|^2}{4H_{pp}(\xi)}\right] \geq 0 \qquad (2)$$

- The separable case : $H = H_0(x,\mu) + H_1(x,p)$, and $H_{pu} = 0$,
  - ◇ Need : H_0 is displacement monotone

Displacement anti-monotonicity

- Recall $I_D(t) := \tilde{\boldsymbol{E}}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta X_t\delta\tilde{X}_t + \partial_{xx}V(X_t)|\delta X_t|^2\Big]$

- Denote

$$J_D(t) := c_1\boldsymbol{E}\Big[\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_T}\big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t\big]\Big|^2 + c_2\Big|\partial_{xx}V(X_t)\delta X_t\Big|^2\Big] \geq 0.$$

- Anti-monotonicity : for some appropriate constants $c_1, c_2 > 0$,

$$I_D(t) + J_D(t) \leq 0 \quad \text{or equivalently} \quad I_D(t) \leq -J_D(t) \leq 0.$$

  - ◇ See more general condition in the paper

- $I_D(t)$ and $J_D(t)$ involve different orders of $\partial_{xx}V$ and $\partial_{x\mu}V$, so we need precise estimates on their bounds, in particular, $\partial_{xx}V < 0$.

Propagation of anti-monotonicity

• Assume G satisfies : $I_D(T) + J_D(T) \leq 0$

• Assume H satisfies certain condition which gives us appropriate estimates on $\partial_{xx}V$ (and $\partial_{x\mu}V$)

• The crucial estimation :

$$I'_D(t) + J'_D(t) \geq 0 \quad \Longrightarrow \quad I_D(t) + J_D(t) \leq 0, \ \forall t.$$

• In the separable case, we may consider Lasry-Lions anti-monotonicity in the same manner :

$$\tilde{\boldsymbol{E}}\left[\partial_{x\mu}V(X_t, \tilde{X}_t)\delta X_t \delta \tilde{X}_t + c\left|\tilde{\boldsymbol{E}}_{\mathcal{F}_T}\left[\partial_{x\mu}V(X_t, \tilde{X}_t)\delta \tilde{X}_t\right]\right|^2\right] \leq 0.$$

Outline

MFGC

- Optimal trajectories : given $\{\nu_t\}$,

$$X_t = \xi + \int_0^t b(X_s, \alpha_s^*, \nu_s)ds + B_t; \quad \mu_t := \mathcal{L}_{X_t};$$

- Hamiltonian :

$$H(x, \nu, p) := \sup_a \Big[b(x, a, \nu)p + F(x, a, \nu) \Big],$$
$$a^* = I(x, \nu, p).$$

- Optimal control :

$$\alpha_t^* = I(X_t, \nu_t, Z_t), \quad Z_t := \partial_x V(t, X_t, \mu_t).$$

- Fixed point :

$$\nu_t = \mathcal{L}_{(X_t, \alpha_t^*)} = \mathcal{L}_{(X_t, I(X_t, \nu_t, Z_t))} \Longrightarrow \nu_t = \psi(\mathcal{L}_{(X_t, Z_t)})$$

An example

- The separable case :

$$b = b_0(x, a, \mu) + b_1(x, \nu), \quad F = F_0(x, a, \mu) + F_1(x, \nu)$$

- Hamiltonian :

$$H_0(x, \mu, p) := \sup_a \Big[b_0(x, a, \mu) p + F_0(x, a, \mu) \Big],$$
$$a^* = I(x, \mu, p).$$

- Fixed point :

$$\nu_t = \mathcal{L}_{(X_t, \alpha_t^*)} = \mathcal{L}_{(X_t, I(X_t, \mu_t, Z_t))} =: \psi(\mathcal{L}_{(X_t, Z_t)})$$

- See the paper for examples in non-separable case

MFGC master equation

- Master equation for $V(t, x, \mu)$

$$
\begin{aligned}
0 &= \partial_t V + \frac{1}{2}\partial_{xx} V + \hat{H}(x, \mathcal{L}_{(\xi,\ \partial_x V(t,\xi,\mu))}, \partial_x V(t, x, \mu)) \\
&+ \tilde{E}\Big[\frac{1}{2}\partial_{\tilde{x}\mu} V(t, x, \mu, \tilde{\xi}) \\
&\quad + \partial_\mu V(t, x, \mu, \tilde{\xi})\ \hat{I}(\tilde{\xi}, \mathcal{L}_{(\xi,\ \partial_x V(t,\xi,\mu))}, \partial_x V(t, \tilde{\xi}, \mu))\Big]; \\
V(T, x, \mu) &= G(x, \mu).
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{H}(x, \mathcal{L}_{(\xi,\zeta)}, p) &:= H(x, \psi(\mathcal{L}_{(\xi,\zeta)}), p), \\
\hat{I}(x, \mathcal{L}_{(\xi,\zeta)}, p) &:= I(x, \psi(\mathcal{L}_{(\xi,\zeta)}), p)
\end{aligned}
$$

The "algorithm"

- Step 0. Compute H_0, I, and ψ
- Step 1. Solve the master equation
- Step 2. Solve the McKean-Vlasov SDE : $\mu_t = \mathcal{L}_{X_t}$,

$$X_t = \xi + \int_0^t \hat{b}\Big(X_s, \hat{I}(X_s, \mathcal{L}_{(X_s,\ Z_s)}, Z_s), \mathcal{L}_{(X_s,\ Z_s)}\Big) ds + B_t,$$

$$\text{where} \quad \mu_t = \mathcal{L}_{X_t}, \quad Z_t = \partial_x V(t, X_t, \mu_t)$$

- Step 3. MFE : $\alpha_t^* = \hat{I}(X_t, \mathcal{L}_{(X_t,\ Z_t)}, Z_t)$

Propagation of displacement monotonicity

- Main condition : for any ξ, η,

$$\boldsymbol{E}\left\{\hat{H}_{x\nu_1}(\xi,\tilde{\xi})\eta\tilde{\eta}+\hat{H}_{xx}(\xi)|\eta|^2+\frac{\left|\tilde{\mathbb{E}}\left[[\hat{H}_{p\nu_1}(\xi,\tilde{\xi})+\hat{H}_{x\nu_2}(\xi,\tilde{\xi})]\tilde{\eta}\right]\right|^2}{4[\hat{H}_{pp}(\xi)+\|\hat{H}_{p\nu_2}(\xi,\cdot)\|_\infty]}\right\}\geq 0.$$

  ◇ This is the same as before, when $\hat{H}_{\nu_2} = 0$.

- Theorem. Assume above and other technical conditions. If G is displacement monotone, then $V(t,\cdot)$ is displacement monotone for all t.

- Similarly we may derive conditions for Lasry-Lions monotonicity and anti-monotonicity

Thank you very much for your attention !