A dual approach to partial hedging

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based on a *work in progress* with
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Framework

- **Controlled wealth process**: $y \in \mathbb{R}$, $Z \in \mathscr{H}^2$,

$$Y_t^{y,Z} = y - \int_0^t \mathfrak{f}(s, Y_s^{y,Z}, Z_s)\,\mathrm{d}s + \int_0^t Z_s\,\mathrm{d}W_s, \qquad (1)$$

- **BSDEs**: For any $\xi \in \mathcal{L}^2(\mathcal{F}_T)$, there exist unique $\mathcal{Y}_0, \mathcal{Z}$ such that $Y_T^{\mathcal{Y}_0,\mathcal{Z}} = \xi$. Notation $\mathcal{Y}_0[\xi], \mathcal{Z}[\xi]$
- **This talk**: mostly linear case i.e. $f(s, y, z) = a_t y + b_t z$, a,b bounded processes. Then setting $\mathrm{d}\Gamma_t = \Gamma_t(a_t\,\mathrm{d}t + b_t^\top\,\mathrm{d}W_t)$ and $\Gamma_0 = 1$, one has

$$y = \mathbb{E}\Big[\Gamma_T Y_T^{y,Z}\Big]$$

- **Financial interpretation**:
  $\rightarrow$ Γ_T is the pricing kernel. The market is 'linear'.
  $\rightarrow$ the asset price process is denoted by X (for numerics in markovian framework solution of a SDE),
  $\rightarrow$ European option with terminal payoff ξ (e.g. $g(X_\cdot)$ or $g(X_T)$)
  $\rightarrow$ replication price $\mathcal{Y}_0 = \mathbb{E}[\Gamma_T \xi]$ and Z (almost) the Delta of the option.

Quantile hedging [FL99]

- **Quantile hedging price** of payoff $\xi \geqslant 0$: Given $p \in [0,1]$,

$$\mathcal{V}_{\mathrm{QH}}(p) := \inf\{y \geqslant 0 \,|\, \exists Z \in \mathscr{H}^2,\ \mathbb{P}(Y_T^{y,Z} \geqslant \xi) \geqslant p \text{ and } Y_T^{y,Z} \geqslant 0\} \tag{2}$$

- **Basic properties**:
  $\rightarrow$ $\mathcal{V}_{\mathrm{QH}}(1) =$ replication price
  $\rightarrow$ $\mathcal{V}_{\mathrm{QH}}$ non-decreasing in p
  $\rightarrow$ $\mathcal{V}_{\mathrm{QH}} = 0$ for $p \leqslant p_{\min} := \mathbb{P}[\xi = 0]$ indeed: $0 = Y^{0,0}$
- **Well studied problem** [FL99, BET10, BBC16]
- **Reformulation**: set $\mathcal{P}_p = \{P \in \mathcal{L}^2(\mathcal{F}_T) | P \in [0,1], \mathbb{E}[P] = p\}$

$$\mathcal{V}_{\mathrm{QH}}(p) := \inf\{y \geqslant 0 \,|\, \exists P \in \mathcal{P}_p, \exists Z \in \mathscr{H}^2,\ Y_T^{y,Z} \geqslant \xi \mathbf{1}_{\{P>0\}}\} \tag{3}$$

Proof of QH problem reformulation

Sketch of the "classical" proof The random variable P is seen as the terminal condition of a controled martingale : $P^{p,\alpha}$ for some $\alpha \in \mathscr{H}^2$.

$$\Gamma(p) := \{y \geqslant 0 \ : \ \exists Z, \text{ s.t. } \mathbb{P}[Y_T^{y,Z} \geqslant \xi] \geqslant p \text{ and } Y_T^{y,Z} \geqslant 0\}$$

$$\bar{\Gamma}(p) = \left\{ y \geqslant 0 : \exists \, (Z, \alpha) \text{ s.t. } Y_T^{y,Z} \geqslant \xi \mathbf{1}_{\{P_T^{p,\alpha} > 0\}} \right\} .$$

- Let $y \in \bar{\Gamma}$, then $\exists Z, \alpha$ s.t. $Y_T^{y,Z} \geqslant \xi \mathbf{1}_{\{P_T^{p,\alpha}) > 0\}}$.
  In particular $\{P_T^{p,\alpha} > 0\} \subset \{Y_T^{y,Z} \geqslant \xi\}$ which implies
  $\mathbb{P}[Y_T^{y,Z} \geqslant \xi] \geqslant \mathbb{E}\left[\mathbf{1}_{\{P_T^{p,\alpha} > 0\}}\right] \geqslant \mathbb{E}[P_T^{p,\alpha}] \geqslant p$.
- Let now $y \in \Gamma$ and denote $p_0 := \mathbb{P}[Y_T^{y,Z} \geqslant \xi] \, (\geqslant p)$. We also have $Y_T^{y,Z} \geqslant 0$.
  For some α (martingale representation theorem), $\mathbf{1}_{\{Y_T^{y,Z} \geqslant \xi\}} = P_T^{p_0,\alpha} \geqslant P_T^{p,\alpha}$.
  Restrict $P^{p,\alpha}$ to $[0,1]$ by stopping and check that
  $\mathbf{1}_{\{Y_T^{y,Z} \geqslant \xi\}} \geqslant P_T^{p,\alpha}$ is equivalent to $Y_T^{y,Z} \geqslant \xi \mathbf{1}_{\{P_T^{p,\alpha} > 0\}}$.

Known solution

- From the stochastic target formulation,

$$\mathcal{V}_{\mathrm{QH}}(p) = \inf\{y \geqslant 0 \,|\, \exists \alpha \ s.t. \ P^{p,\alpha} \in \mathcal{P}_p, Z \in \mathscr{H}^2, \ Y_T^{y,Z} \geqslant \xi \mathbf{1}_{\{P_T^{p,\alpha} > 0\}}\} \tag{4}$$

  one obtains

$$\mathcal{V}_{\mathrm{QH}}(p) = \inf_{\alpha} \mathbb{E}\Big[\Gamma_T \xi \mathbf{1}_{\{P_T^{p,\alpha} > 0\}}\Big]$$

- One proves (convexification)

$$\mathcal{V}_{\mathrm{QH}}(p) = \inf_{\alpha} \mathbb{E}[\Gamma_T \xi P_T^{p,\alpha}]$$

- Markovian framework: this is the starting point for PDE methods. One can also use duality in the p-variable (see below!)
- Numerical methods available - problem is challenging! [BBC16, BCR21]

PnL matching

- [BNV12] considers **finitely many quantile constraints**

$$\mathbb{P}(Y_T^{y,Z} \geqslant \xi + \gamma_i) \geqslant q_i \tag{5}$$

Solve the problem in the Markovian case using PDE techniques (and imposing constraints on Z)

- A **natural extension** would be to consider a probability law μ and the following constraints:

$$\mathbb{P}(Y_T^{y,Z} \geqslant \xi + \gamma) \geqslant \mu([\gamma, +\infty)) \tag{6}$$

$\hookrightarrow$ "PnL hedging with given probability law"

- **Problem formulation**:

$$\mathcal{V}_{\mathrm{WH}}(\mu) = \inf\{y \in \mathbb{R} \,|\, \exists Z \in \mathscr{H}^2, \mathbb{P}(Y_T^{y,Z} \geqslant \xi + \gamma) \geqslant \mu([\gamma, +\infty))\,,\ \forall \gamma\} \tag{7}$$

- Notations. First order stochastic dominance between ν and μ:

$$\nu \succeq \mu \Leftrightarrow \nu([\gamma, +\infty)) \geqslant \mu([\gamma, +\infty))\,,\ \forall \gamma \in \mathbb{R}.$$

$\hookrightarrow$ Observe that in (7), one imposes $\mathcal{L}(Y_T^{y,Z} - \xi) \succeq \mu$.

Solving PnL hedging problem

- One shows: $\mathcal{V}_{\mathrm{WH}}(\mu) = \inf\{y \in \mathbb{R} \,|\, \exists \chi \sim \nu \succeq \mu, \exists Z \in \mathscr{H}^2, Y_T^{y,Z} - \xi \geqslant \chi\}$
- Look first at problem with $\chi \sim \mu$

$$\mathcal{V}_{\mathrm{OT}}(\mu) = \inf_{\chi \sim \mu} \mathbb{E}[\Gamma_T(\xi + \chi)] = \mathcal{V}_{\mathrm{QH}}(1) + \inf_{\chi \sim \mu} \mathbb{E}[\Gamma_T \chi] \tag{8}$$

$\hookrightarrow$ Optimal transport problem with explicit solution

$$\mathcal{V}_{\mathrm{OT}}(\mu) = \mathbb{E}[\Gamma_T \xi] - \frac{1}{2}\mathbb{E}\left[(\Gamma_T)^2\right] - \frac{1}{2}\int x^2 \mu(\mathrm{d}x) + \frac{1}{2}\mathcal{W}_2^2(\mathcal{L}(-\Gamma_T), \mu).$$

- Moreover, replication result: $\mathcal{V}_{\mathrm{OT}}(\mu) = \mathbb{E}[\Gamma_T(\xi + \chi^\star)]$, with

$$\chi^\star = N_\mu^{-1} \circ N_{\mathcal{L}(-\Gamma_T)}(-\Gamma_T),$$

assuming $\mathcal{L}(\Gamma_T)$ is abs. cont. (N_μ stands for the c.d.f. of the law μ)
- conclusion: One proves $\mathcal{V}_{\mathrm{WH}}(\mu) = \mathcal{V}_{\mathrm{OT}}(\mu)$ 'the constraint is saturated'
- we shall use the above result in the numerics.

the two problems as one?

Set $\mu = (1-p)\delta_\ell + p\delta_0$ $(\ell < 0)$

- Recall:

$$\mathcal{V}_{\mathrm{QH}}(p) = \inf\{y \in \mathbb{R} \,|\, \exists Z \in \mathscr{H}^2,\ \mathbb{P}(Y_T^{y,Z} \geqslant \xi) \geqslant p \text{ and } Y_T^{y,Z} \geqslant 0\}$$
$$\mathcal{V}_{\mathrm{WH}}(\mu) = \inf\{y \in \mathbb{R} \,|\, \exists Z \in \mathscr{H}^2, \mathbb{P}(Y_T^{y,Z} \geqslant \xi + \ell) \geqslant 1\,, \mathbb{P}(Y_T^{y,Z} \geqslant \xi) \geqslant p\}$$

- The super-replication constraint are not the same (whatever ℓ).
- The solutions are different. In one case the constraint is saturated: the law μ is attained. In the other case there is a lower limit threshold $p_{\min} = \mathbb{P}(\xi = 0)$ in some cases.
- Is it possible to have a common framework for both problem?

Generic problem formulation

- Take a $(\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable random function

$$\Omega \times \mathbb{R} \ni (\omega, \gamma) \mapsto G(\omega, \gamma) \in \mathbb{R}$$

  such that $\gamma \mapsto G(\gamma)$ is non-decreasing and left-continuous.

- Introduce, for $\mu \in \mathcal{P}(\mathbb{R})$, the set of super weak hedging price by

$$\mathfrak{H}(\mu) := \{y \in \mathbb{R} \,|\, \exists Z \in \mathscr{H}^2, \mathbb{P}(Y_T^{y,Z} \geqslant G(\gamma)) \geqslant \mu([\gamma, +\infty)), \forall \gamma \in \mathbb{R}\}. \quad (9)$$

- We now define the *weak hedging price* as

$$\mathcal{V}_{\mathrm{WH}}(\mu) := \inf \mathfrak{H}(\mu) \;, \quad \text{for } \mu \in \mathcal{P}(\mathbb{R}). \quad (10)$$

Example of PnL matching

- Discrete case for μ, 'generic' G: For $n \geqslant 1$ and $0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \in L^2(\mathcal{F}_T)$, we define

$$G(\omega, i) := \xi_i(\omega), \omega \in \Omega, 1 \leqslant i \leqslant n,$$

  and $p_1, \ldots, p_n > 0$ such that $\sum_{i=1}^n p_i = 1$, we set $\mu = \sum_{i=1}^n p_i \delta_i$.
  $\hookrightarrow$ Here, $\gamma_i = i$ simply.
- General case for μ, structure on G: Let $\xi \in \mathcal{L}^2(\mathcal{F}_T)$ and $\mu \in \mathcal{P}_2(\mathbb{R})$. Set $G(\omega, \gamma) = \xi(\omega) + \gamma$, for $(\omega, \gamma) \in \bar{\Omega}$

Monge Problem

Let $\mu \in \mathcal{P}_4(\mathbb{R})$. Assume some mild structural conditions on G and let Ψ denotes the right continuous inverse of G. We have the equivalent formulations

- Weak stochastic target version:

$$\mathcal{V}_{\mathrm{WH}}(\mu) = \inf \tilde{\mathfrak{H}}(\mu), \text{ with } \tilde{\mathfrak{H}}(\mu) := \{y \in \mathbb{R} | \exists Z \in \mathscr{H}^2, \mathcal{L}(\Psi(Y_T^{y,Z})) \succeq \mu\}, \quad (11)$$

$\hookrightarrow$ *weak hedging problem* is an example of a more generic problem ([BER15]).

- 'Lifted' version:

$$\mathcal{V}_{\mathrm{WH}}(\mu) = \inf\{y \in \mathbb{R} | \exists \chi \sim \nu \succeq \mu, \exists Z \in \mathscr{H}^2, \Psi(Y_T^{y,Z}) \geqslant \chi\}, \quad (12)$$

- 'Monge' version: Denote $\mathcal{T}_+(\mu) = \{\chi \in \mathcal{L}^4(\mathcal{F}_T) \,|\, \mathcal{L}(\chi) \succeq \mu\}$, then

$$\mathcal{V}_{\mathrm{WH}}(\mu) = \widehat{\mathcal{V}}_{\mathrm{RM}} := \inf_{\chi \in \mathcal{T}_+(\mu)} \mathcal{Y}_0[G(\chi)], \quad (13)$$

$\hookrightarrow$ Linear case: denote $H(\omega, \gamma) := \Gamma_T(\omega) G(\omega, \gamma)$.

$$\widehat{\mathcal{V}}_{\mathrm{RM}}(\mu) = \inf_{\chi \in \mathcal{T}_+(\mu)} \mathbb{E}[\Gamma_T G(\chi)] = \inf_{\chi \in \mathcal{T}_+(\mu)} \int H(\omega, \chi(\omega))\,\mathrm{d}\mathbb{P}(\omega),$$

$$= \inf_{\chi \in \mathcal{T}_+(\mu)} \int\int H(\omega,\gamma)\delta_{\chi(\omega)}(\,\mathrm{d}\gamma)\,\mathrm{d}\mathbb{P}(\omega)$$

Kantorovich problem

- Linear case: $C^{\mathrm{r}}(\mathbb{P},\mu)$ is the set of coupling between $\mathbb{P}$ and $\nu \geq \mu$, then

$$\mathcal{V}_{\mathrm{KP}}(\mu) = \inf_{\Pi \in C^{\mathrm{r}}(\mathbb{P},\mu)} \int \Gamma_T(\omega) G(\omega,\gamma)\,\mathrm{d}\Pi(\omega,\gamma).$$

- Using disintegration $\mathrm{d}\Pi(\omega,\gamma) = \rho^{\Pi}(\omega,\,\mathrm{d}\gamma)\,\mathrm{d}\mathbb{P}(\omega)$, we obtain

$$\begin{aligned}\mathcal{V}_{\mathrm{KP}}(\mu) &= \inf_{\Pi \in C^{\mathrm{r}}(\mathbb{P},\mu)} \int\int \Gamma_T(\omega) G(\omega,\gamma)\rho^{\Pi}(\omega,\,\mathrm{d}\gamma)\,\mathrm{d}\mathbb{P}(\omega)\\ &= \inf_{\Pi \in C^{\mathrm{r}}(\mathbb{P},\mu)} \mathbb{E}\left[\Gamma_T \int G(\gamma)\rho^{\Pi}(\,\mathrm{d}\gamma)\right].\end{aligned}$$

- Non linear version:

$$\mathcal{V}_{\mathrm{KP}}(\mu) := \inf_{\Pi \in C^{\mathrm{r}}(\mathbb{P},\mu)} \mathcal{Y}_0\left[\int G(\gamma)\rho^{\Pi}(\,\mathrm{d}\gamma)\right]. \tag{14}$$

Towards dual formulation

Theorem

μ has discrete and finite support, namely $\mathrm{supp}[\mu] = \{\gamma_1, \ldots, \gamma_d\}$ with $\gamma_1 < \cdots < \gamma_d$. Then,

$$\mathcal{V}_{\mathrm{RM}}(\mu) = \mathcal{V}_{\mathrm{KP}}(\mu).$$

(true in the non-linear setting)

- In the **linear setting**, we work towards a dual formulation by characterising the polar of $C^{\mathrm{r}}(\mathbb{P}, \mu)$ when μ is **discrete with finite support**.
- This gives access to numerical methods.

Quantile hedging with our approach

Let's look at the special case $d = 2$. Namely, $\mu = (1-q)\delta_{\gamma_1} + q\delta_{\gamma_2}$ for $0 < q < 1$.

- For $\mathrm{d}\Pi(\omega,\gamma) = \rho^{\Pi}(\omega, \mathrm{d}\gamma)\,\mathrm{d}\mathbb{P}(\omega) \in C^{\mathrm{r}}(\mathbb{P},\mu)$, we have simply

$$\int G(\gamma)\rho^{\Pi}(\mathrm{d}\gamma) = G(\gamma_1)(1-Q) + G(\gamma_2)Q$$

  with $Q \in \mathfrak{Q}_q$ and $\mathfrak{Q}_q := \{Q \in \mathcal{L}^2(\mathcal{F}_T) | 1 \geqslant Q \geqslant 0, \mathbb{E}[Q] \geqslant q\}$

- Setting $G(\gamma_1) = 0$ and $G(\gamma_2) = \xi$, (this is the quantile hedging problem we started with...)

$$\mathcal{V}_{\mathrm{KP}}(\mu) := \inf_{\Pi \in C^{\mathrm{r}}(\mathbb{P},\mu)} \mathbb{E}\left[\Gamma_T \int G(\gamma)\rho^{\Pi}(\mathrm{d}\gamma)\right] = \inf_{Q \in \mathfrak{Q}_q} \mathbb{E}[\Gamma_T G(\gamma_2) Q]$$

Dual formulation

- In this context, one obtains $\mathcal{V}_{\mathrm{KP}}(\mu) \geqslant \mathcal{V}_{\mathrm{DP}}(\mu)$ where

$$\mathcal{V}_{\mathrm{DP}}(\mu) = \sup_{\theta \geqslant 0} w(\theta) \quad \text{with} \quad w(\theta) := \theta q - \mathbb{E}\big[(\theta - \Gamma_T G(\gamma_2))^+\big] \tag{15}$$

- Complete characterization: Let $\theta^\star \in \operatorname{argmax} w$, set

$$Q^\star := \mathbf{1}_{\{\tilde{\Gamma}_T G(\gamma_2) < \theta^\star\}} + \frac{q - \mathbb{P}[\Gamma_T G(\gamma_2) < \theta^\star]}{\mathbb{P}[\Gamma_T G(\gamma_2) = \theta^\star]} \mathbf{1}_{\{\Gamma_T G(\gamma_2) = \theta^\star\}} \in \mathfrak{Q}_q \tag{16}$$

we have (recall $G(\gamma_1) = 0$, $G(\gamma_2) = \xi$)

$$\mathcal{V}_{\mathrm{DP}}(\mu) = \mathbb{E}[\Gamma_T Q^\star G(\gamma_2)] .$$

- Moreover, if $\mathbb{P}[\tilde{\Gamma}_T G(\gamma_2) = \theta^\star] = 0$, then

$$\mathcal{V}_{\mathrm{DP}}(\mu) = \mathbb{E}[\Gamma_T G(\chi^\star)] , \text{ with } \chi^\star := \gamma_1 \mathbf{1}_{\{\Gamma_T G(\gamma_2) \geqslant \theta^\star\}} + \gamma_2 \mathbf{1}_{\{\tilde{\Gamma}_T G(\gamma_2) < \theta^\star\}} \in \mathcal{T}_+^{\mathrm{r}}(\mu).$$

(Monge problem has a solution)

Stochastic gradient algorithm

- When duality holds, we have more generally (if $G(\gamma_1) = 0$),

$$\mathcal{V}_{\mathrm{WH}}(\mu) = \sup_{\theta \in \mathbb{R}_+^{d-1}} \sum_{j=1}^{d-1} \theta_j q_{j+1} - \mathbb{E}\left[\max_{1 \leqslant i \leqslant d-1} \left(\sum_{j=1}^{i} \theta_j - \Gamma_T G(\gamma_{i+1}) \right)_+ \right].$$

- One cannot resist to use SGD to compute the optimal θ (which allows to obtain the payoff to replicate)
- The numerical tests are performed in the Black Scholes model for put or call option.
- SGD results are compared to proxy: closed form solution from [FL99] or Optimal Transport approach, depending on the context.

Sanity check

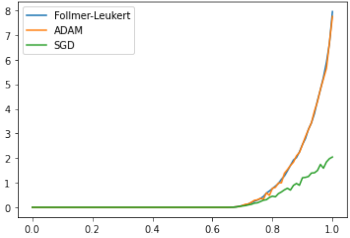
Two quantile constraints (3 points in the support of μ). Computations for Call option.

Quantiles p_1, p_2	γ_1, γ_2	SGD	OT	Computation time of SGD
(0.3,0.5)	(10,20)	17.38	17.48	30.66s
(0.05,0.05)		8.48	8.41	30.31s
(0.05,0.9)		24.41	24.44	29.46s
(0.3,0.5)	(10,100)	42.07	42.19	32.47s
(0.05,0.05)		9.57	9.62	31.40s
(0.05,0.9)		87.89	87.57	30.68s

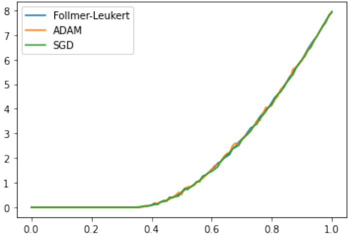
Table: Numerics of measure $\mu = (1 - p_1 - p_2)\delta_0 + p_1\delta_{\gamma_1} + p_2\delta_{\gamma_2}$ with different probabilities and quantiles with SGD algorithm and OT-APPROACH.

Quantile hedging

See also [BFP09].



(a) Put option $(K - S)_+$



(b) Call option $(S - K)_+$

Figure: Comparison of the three methods: SGD algorithm, ADAM optimizer & Exact solution [BCR21, FL99] for put and call options, with parameters $X_0 = 100$, $r = 0$, $\sigma = 0.2$ and $\hat{b} = 0.1$, strike $K = 100$, terminal time $T = 1$.

Conclusion

- We introduce 'weak hedging problem' extending previous formulation of quantile hedging and PnL matching problem.
- It is a special case of a weak stochastic target problem and turns out to be closely related to optimal transport type problem.
- In good cases, dual approach allows to develop alternative numerical methods using SGD algorithm.

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