UNIVERSITÀ DEGLI STUDI DI PADOVA

Coupling methods in stochastic control

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Based joint works with **A.Cecchin, M.Cirant, A.Durmus, K.Eichinger, A.Porretta**

Motivation

Stochastic control

Stochastic basis

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0})$ a filtered probability space with a d-dimensional Brownian motion

Set of admissible controls

$$\mathrm{Adm}_T = \left\{ (u_t)_{t\in[0,T]} \ \mathcal{F}_t\text{-prog.meas. with } \mathbb{E}\Big[\int_0^T |u_s|^2 ds\Big] < +\infty \right\}$$

Stochastic control

Controlled dynamics

$$\mathrm{d}X^u_s = [b(X^u_s) + u_s]\,\mathrm{d}s + \sigma(X^u_s)\cdot \mathrm{d}B_s \qquad X^u_t = x$$

Action functional and control problem

$$J^{T,g}_{t,x}(u) = \mathbb{E}\left[\int_t^T \ell(X^u_s, u_s)\,\mathrm{d}s + g(X^u_T)\right] \qquad \varphi^{T,g}_t(x) = \inf_{u\in \mathrm{Adm}_T} J^{T,g}_{t,x}(u)$$

Markovian controls

Markov control policies

- Markov control policies

$$\mathscr{M}_T = \{\alpha : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d\}$$

- Controlled state

$$\mathrm{d}X_t^\alpha = [b(X_s^\alpha) + \alpha_s(X_s^\alpha)]\mathrm{d}s + \sigma(X_s^\alpha) \cdot \mathrm{d}B_s \qquad X_t^\alpha = x$$

- X^α coincides with X^{u^α} with $u_t^\alpha = \alpha_t(X_t^\alpha)$

Equivalence of formulations

Under mild assumptions

$$\varphi_t^{T,g}(x) = \inf_{\alpha \in \mathscr{M}_T} J_{t,x}^{T,g}(u^\alpha)$$

Ergodic control problem

Ergodic controls

- $\mathscr{M}_\infty = \{\alpha : \mathbb{R}^d \longrightarrow \mathbb{R}\}$ is the set of ergodic Markov policies
- X^α_∞ is the invariant distribution for the Markov process X^α

$$\mathrm{d}X^\alpha_s = [b(X^\alpha_s) + \alpha(X^\alpha_s)]\mathrm{d}s + \mathrm{d}B_s$$

Ergodic control problem

$$J_\infty(\alpha) = \mathbb{E}[\ell(X^\alpha_\infty, \alpha(X^\alpha_\infty))] \qquad \inf_{\alpha \in \mathscr{M}_\infty} J_\infty(\alpha)$$

Convergence to equilibrium

The Meta-Theorem (informal)

Let $\alpha^T \in \mathscr{M}_T$ be optimal and α^∞ be the optimal ergodic policy. Then, there exist $C, \lambda \in (0, +\infty)$ independent of T such that

$$W_1(\mathscr{L}(X_t^{\alpha^T}), \mathscr{L}(X^{\alpha^\infty})) \lesssim C\left(e^{-\lambda t} + e^{-\lambda(T-t)}\right) \quad \forall t \in [0,T]$$

holds for a large class of initial conditions and terminal costs.

- Manifestation of the exponential turnpike phenomenon
- Exponential estimates for $|\alpha_t^T - \alpha^\infty|$

Exponential convergence to equilibrium

Ergodicity of uncontrolled diffusions

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t) \cdot \mathrm{d}B_t$$

Monotonicity of the drift

$$\langle b(x) - b(\hat{x}), x - \hat{x} \rangle \leq -\kappa_b(|x - \hat{x}|)|x - \hat{x}|^2, \qquad \liminf_{r \to +\infty} \kappa_b(r) > 0.$$

Lyapunov condition

There exist $\lambda, C > 0,\ L < +\infty$ s.t.

$$b(x) \cdot x \leq -\lambda |x|^2 + C, \qquad Db(x) \leq L$$

What induces ergodicity in control?

For simplicity consider a separable cost

$$\ell(x,u) = l(u) + V(x)$$

Drift-induced ergodicity

- b is asymptotically monotone
- l strongly confining in u
- V does not oscillate too much or does not grow too fast

Ergodicity of b

Optimal control policy is "small"

- V has small oscillations
- l strongly confining

"Choosing a big control u to minimize V is not effective as it costs more than what it can achieve"

Optimal drift is a small perturbation of b

- b is ergodic
- α^T is small

$\Rightarrow b + \alpha^T$ is ergodic

Cost-induced ergodicity

Monotonicity of ∇V

- ∇V satisfies a Lyapunov condition, or ∇V asymptotically monotone
- l not too confining (but still convex)
- b not too explosive (or absent)

Propagation of monotonicity from cost to control

"Choosing a big control u to minimize V is not too expensive and pays off as an optimal strategy"

$$\alpha_t^T(\,\cdot\,) \approx -\nabla V(\,\cdot\,)$$

Some questions

Objectives

- Can we show that the turnpike estimates

$$W_1(\mathscr{L}(X_t^{\alpha^T}), \mathscr{L}(X^{\alpha^\infty})) \lesssim C(e^{-\lambda t} + e^{-\lambda(T-t)}) \quad \forall t \in [0,T]$$

  holds in each of the two scenarios?

- When does the drift win over the cost?
- When does the cost win over the drift?
- Critical situations where the two mechanisms coexist and cooperate?

Pontryagin's maximum principle

Stochastic Maximum principle

The optimal Markov policy is

$$\alpha_t^T(x) = -\nabla l^\star(\nabla \varphi_t^{T,g}(x)).$$

Setting

$$X_t = X_t^{\alpha^T}, \quad Y_t = \nabla \varphi_t^{T,g}(X_t), \quad Z_t = \nabla^2 \varphi_t^{T,g}(X_t)$$

we have

$$\begin{cases} \mathrm{d}X_s = [b(X_s) - \nabla l^\star(Y_s)]\mathrm{d}s + \mathrm{d}B_s \\ \mathrm{d}Y_s = -[Db^T(X_s)\cdot Y_s + \nabla V(X_s)]\mathrm{d}s + Z_s \cdot \mathrm{d}B_s \\ X_t = x, \quad Y_T = \nabla g(X_T) \end{cases}$$

A “toy” model and some answers

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The model

Example

- For the cost, we choose

$$l(u) = \frac{|u|^{q^\star}}{q^\star}, \quad V(x) = \frac{\lambda_f}{n}|x|^n,$$

for some $n, q^\star \geq 1$ and $\lambda_f \in \mathbb{R}$

- For the drift

$$b(x) = -\lambda_b |x|^{2m-2} x$$

for some $m \geq 0,\ \lambda_b \in \mathbb{R}$

Goal: study the interplay between $n, m, q^\star, \lambda_f, \lambda_b$

A (not so) toy model

Stochastic maximum principle

$$\begin{cases} \mathrm{d}X_s = -\lambda_b |X_s|^{2m-2} X_s - |Y_s|^{q-1} Y \mathrm{d}s + \mathrm{d}B_s \\ \mathrm{d}Y_s = \lambda_b (2m-1) |X_s|^{2m-2} Y_s - \lambda_f |X_s|^{n-2} X_s \mathrm{d}s + Z_s \cdot \mathrm{d}B_s \\ X_t = x, \quad Y_T = \nabla g(X_T) \end{cases}$$

HJB equation

$$\begin{cases} \partial_t \varphi_t(x) + \frac{1}{2}\Delta \varphi_t(x) - \frac{1}{q} |\nabla \varphi_t(x)|^q - \lambda_b |x|^{2m-2} x \cdot \nabla \varphi_t(x) + \frac{\lambda_f}{n} |x|^n = 0 \\ \varphi_T \equiv g \end{cases}$$

Preliminary considerations

Drift-induced ergodicity

- Expected for $\lambda_b > 0$, large m, $q^\star$ and n not too large, regardless of the sign of λ_f

Cost-induced ergodicity

- Expected for $\lambda_f > 0$, large n, and m, $q^\star$ not too large regardless of the sign of λ_b

Cooperation

When $\lambda_f > 0, \lambda_b > 0$

Drift-induced ergodicity

Theorem (Cirant, C. Porretta)

Assume that $\lambda_b > 0$ and

$$q^\star(2m - 1) > n$$

Then, we have

$$|\nabla \varphi_t(x)| \lesssim C(|x|^{n/q} + 1)$$

As a corollary, the optimal drift

$$b(\cdot) - |\alpha_t^T(\cdot)|^{q-2}\alpha_t^T(\cdot)$$

satisfies a Lyapunov condition.

Proof idea

A monotonicity bound

1. Introduce the process

$$V_s = |Y_s|^q - \frac{\lambda_f}{n}|X_s|^n$$

2. There exists $\tau > 0$ such that

$$d\mathbb{E}[V_s] \geq \tau\,\mathbb{E}[V_s]\mathrm{d}s - C$$

3. Integrate and use boundary conditions

$$|\nabla\varphi_t(x)| - \lambda|x|^n = V_t \leq e^{-\tau(T-t)}V_T + \frac{C}{\tau}$$

Propagation of a Lyapunov condition

Theorem (Cirant, C. Porretta)

Assume that $\lambda_f > 0$ and

$$q^\star(2m-1) < n$$

Then $\nabla\varphi$ satisfies a Lyapunov condition

$$\nabla\varphi_t(x)\cdot x \geq \lambda|x|^{1+n/q} - C$$

As a corollary, the optimal drift satisfies a Lyapunov condition.

A first attempt ($b \equiv 0$)

A first attempt

Show that

$$V_s = -X_s \cdot Y_s + \lambda |X_s|^{1+n/q}$$

satisfies a differential inequality

$$\mathrm{d}\mathbb{E}[V_s] \geq \tau \mathbb{E}[V_s]\mathrm{d}s - C$$

Problem

$$\mathrm{d}V_s = \ldots - \mathrm{Tr}(Z_s)\mathrm{d}s + \ldots$$

- Uniform bounds on Z_s require ∇V of linear growth

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A trick

Idea: include higher order terms in V_s

$$V_s = -X_s \cdot Y_s + \lambda |X|^{1+n/q} + \varepsilon \mathrm{Tr}(Z_s^+)$$

Benefit

$$\mathrm{d}V_s \geq \cdots - Z_s^+ \mathrm{d}s + \cdots + \\ + \varepsilon[|Y_s|^{q-2}(Z_s^+)^2 - \mathrm{Tr}(\nabla^2 f)(X_s)]\mathrm{d}s$$

The critical case $n = (2m-1)q^\star$

Theorem (Cirant, C., Porretta)

- If $\lambda_f > 0$ and $\lambda_b > 0$, cost and drift cooperate for ergodicity
- If $\lambda_f > 0$ and

$$2m|\lambda_b| < (\lambda_f q^\star)^{1/q^\star}$$

the cost wins; $\nabla\varphi$ satisfies a Lyapunov condition

- If $\lambda_b > 0$ and

$$2m\lambda_b > (|\lambda_f| q^\star)^{1/q^\star}$$

the drift wins; $|\nabla\varphi|(x) = o(|x|^{2m})$

Obstructions to drift-cost cooperation

Ergodic drift goes against monotone cost

We want to show a monotonicity estimate

$$\mathrm{d}\mathbb{E}[V_s] \geq \tau\mathbb{E}[V_s]\mathrm{d}s - C$$

for

$$V_s = -X_s \cdot Y_s + \lambda |X_s|^{1+n/q}$$

Problem

$$\mathrm{d}(-X_s \cdot Y_s) = \cdots + [Db^T(X_s) \cdot X_s - b(X_s)] \cdot Y_s \, \mathrm{d}s$$

Problem

$$\mathrm{d}|X_s|^{1+n/q} = \cdots + |X_s|^{n/q-1} b(X_s) \cdot X_s \, \mathrm{d}s$$

A special case: $l(u) = \frac{|u|^2}{2}$

Propagation of asymptotic convexity

Definition

V is asymptotically convex if

$$\langle \nabla V(x) - \nabla V(\hat{x}), x - \hat{x} \rangle \geq \kappa_V(|x - \hat{x}|)|x - \hat{x}|^2$$

with

$$\limsup_{r \to +\infty} \kappa_V(r) > 0, \qquad \int_0^1 r\,\kappa_V(r)\mathrm{d}r > -\infty$$

Theorem (Informal) (Chaintron, C., Eichinger '25)

Let V, g be asymptotically convex. Then $\varphi_t^{T,g}$ is asymptotically convex uniformly in t, T.

Coupling by reflection of FBSDEs

Proof sketch

Monotonicity estimate for

$$\langle Y_t - \hat{Y}_t, \mathrm{e}_t \rangle - g_t(|X_t - \hat{X}_t|), \qquad \mathrm{e}_t = \frac{X_t - \hat{X}_t}{|X_t - \hat{X}_t|}$$

- $(X, Y, Z), (\hat{X}, \hat{Y}, \hat{Z})$ two solutions of the stochastic maximum principle
- $(t, r) \mapsto g_t(r)$ subsolution to 1-d viscous Burger's equation

What is new?

Changing the noise to overcome lack of point wise convexity

- $(X, Y, Z), (\hat{X}, \hat{Y}, \hat{Z})$ are not driven by the same noise
- $\langle Y_t - \hat{Y}_t, \mathrm{e}_t \rangle$ instead of $\langle Y_t - \hat{Y}_t, X_t - \hat{X}_t \rangle$ to profit from noise

Coupling by reflection

$$\mathrm{d}\hat{B}_s = (\mathrm{id} - 2e_s e_s^T)\mathrm{d}B_s$$

The increments of B_s and $\hat{B}_s$ are

- The same in directions orthogonal to $X_s - \hat{X}_s$
- Opposite along the $X_s - \hat{X}_s$ direction

Applications of the quadratic case

Functional inequalities

Generalizations of Prékopa-Leindler and Brascamp-Lieb

Score-based generative models and diffusion models

Optimal dimensional dependence of convergence bounds

Entropic optimal transport

Exponential convergence of Sinkhorn's algorithm

Filtering

Stability of conditional distributions with respect to initial distribution and observation path

What was left out

Mean Field games and mean field control

Developments driving much of the progresses in the field

- **Small interactions**

  [Cecchin et al.'24]

- **Lasry-Lions monotonicity**

  [Cardaliaguet et al. '12] [Caraliaguet et al.,'13] [Caraliaguet Porretta,'19] [Cirant, Porretta'21], …

- **Displacement monotonicity**

  [Cirant, Meszaros '24]

- **Common noise**

  [Cardaliaguet et. al' 25]

Thank you for the attention!!