Instantaneous blowup of parabolic SPDEs

Eulalia Nualart
(Universitat Pompeu Fabra, Barcelona)

Joint work with Mohammud Foondun (Strathclyde) and Davar Khoshnevisan (Utah)

International Seminar on SDEs and Related Topics, March 8th 2024

The Osgood condition for ODEs

- **Assumption 1 :** $b : \mathbf{R} \to \mathbf{R}_+$ is nonegative, locally Lipschitz and nondecreasing

- Consider the ODE

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s, \quad t \geq 0, \quad a \geq 0.$$

This equation admits a unique solution up to its blow up time

$$T := \sup\{t > 0 : |X_t| < \infty\} = \int_a^\infty \frac{1}{b(s)}\,\mathrm{d}s.$$

We say that the solution blows up in finite time if $T < \infty$.

The Osgood condition for integral equations

- **The Osgood condition** : for some $a > 0$

$$\int_a^\infty \frac{1}{b(s)}\,\mathrm{d}s < \infty.$$

- **Assumption 2 :** $g : [0, \infty) \to \mathbf{R}$ is continuous and

$$\limsup_{t\to\infty} \inf_{0\le h\le 1} g(t+h) = \infty.$$

Theorem (León-Villa'11)

Suppose that Assumptions 1 and 2 hold. The solution to

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + g(t), \quad a \geq 0,$$

blows up in finite time if and only if the Osgood condition holds.

Proof (recall $X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + g(t)$)

- Assume $T < \infty$. Set $M := \sup_{0 \le s \le T} |g(s)|$. For $t \in [0, T]$,

$$X_t \le a + M + \int_0^t b(X_s)\,\mathrm{d}s.$$

Let

$$Y_t = a + M + 1 + \int_0^t b(Y_s)\,\mathrm{d}s.$$

Then $X_t \le Y_t$ on $[0, T]$.

So Y_t will also blow up by time T and b satisfies the Osgood condition.

Proof (recall $X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + g(t)$)

- Suppose $T = \infty$. Let $t_n \to \infty$. Then, for $t \in [0, 1]$,

$$X_{t+t_n} \geq a + \int_{t_n}^{t+t_n} b(X_s)\,\mathrm{d}s + g(t+t_n)$$
$$\geq a + \int_0^t b(X_{s+t_n})\,\mathrm{d}s + \inf_{0\leq h\leq 1} g(h+t_n),$$

This means that $X_{t+t_n} \geq Z_t$ where

$$Z_t = \frac{1}{2}\left(a + \inf_{0\leq h\leq 1} g(h+t_n)\right) + \int_0^t b(Z_s)\,\mathrm{d}s.$$

In particular,

$$\int_{\frac{1}{2}(a+\inf_{0\leq h\leq 1} g(h+t_n))}^{\infty} \frac{1}{b(s)}\,\mathrm{d}s > 1.$$

But from Assumption 2, we can find $t_n \to \infty$ such that

$$\frac{1}{2}(a + \inf_{0\leq h\leq 1} g(h+t_n)) \to \infty.$$

This contradicts the Osgood condition.

SDEs driven by Bifractional Brownian motion

Introduced by Houdré-Villa'03 is defined as a centered Gaussian process $(B_t^{H,K})_{t\geq 0}$ with covariance

$$R^{H,K}(t,s) = 2^{-K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}),$$

where $H \in (0,1)$ and $K \in (0,1]$. $B_t^{H,1}$ is a fBm.

Theorem (León-Villa'11)

Suppose that Assumptions 1 holds. Then the solution to

$$X_t = a + \int_0^t b(X_s)\,\mathrm{d}s + B_t^{H,K}, \quad a \geq 0,$$

blows up in finite time almost surely if and only if the Osgood condition holds.

Proof (of $\inf_{h\in[0,1]} B^{H,K}_{t_n+h} \to \infty$)

- One first shows that for $\psi_{H,K}(t) := t^{HK}\sqrt{2\log\log t}$, $t > e$, a.s.

$$\sup_{s,t\in[n,n+2]} \frac{|B^{H,K}_t - B^{H,K}_s|}{\psi_{H,K}(n)} \longrightarrow 0, \quad \text{as } n \to \infty. \tag{1}$$

In fact,

$$\mathrm{E}\left[\sum_{n=1}^{\infty} \sup_{s,t\in[n,n+2]} \frac{|B^{H,K}_t - B^{H,K}_s|^p}{\psi_{H,K}(n)^p}\right] \le \sum_{n=1}^{\infty} \frac{A_p 2^{pHK}}{\psi_{H,K}(n)^p} < \infty.$$

- Let ω such that both LIL and (1) hold. Then

$$\begin{aligned}\inf_{h\in[0,1]} B^{H,K}_{t+h} &\ge B^{H,K}_t + \inf_{h\in[0,1]}\left(-|B^{H,K}_{t+h} - B^{H,K}_t|\right)\\ &\ge \frac{B^{H,K}_t}{\psi_{H,K}(t)}\psi_{H,K}(t) - \sup_{h\in[0,1]} \frac{|B^{H,K}_{t+h} - B^{H,K}_t|}{\psi_{H,K}([t])}\psi_{H,K}([t]).\end{aligned}$$

Finally, LIL and (1) conclude the proof.

The stochastic heat equation on $[0, 1]$

$$\left|\begin{array}{l}\dfrac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + b(u(t, x)) + \sigma \dot{W}(t, x), \quad x \in [0, 1],\ t > 0, \\ u(0, x) = u_0(x),\end{array}\right.$$

- homogeneous Dirichlet boundary conditions, $\sigma > 0$, $\dot{W}$ space-time white noise
- $u_0(x)$ nonnegative and continuous function

- If b is locally Lipschitz then there exists a unique local random field solution which is a jointly measurable and adapted space-time process satisfying

$$\begin{aligned} u(t, x) = \int_0^1 p(t, x, y) u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 p(t-s, x, y) b(u(s, y))\,\mathrm{d}y\,\mathrm{d}s \\ + \sigma \int_0^t \int_0^1 p(t-s, x, y) W(\mathrm{d}y\,\mathrm{d}s), \end{aligned}$$

for all $t \in (0, T)$, where $T = \sup\{t > 0 : \sup_{x\in[0,1]} |u(t,x)| < \infty\}$ and $p(t,x,y)$ is the Dirichlet heat kernel on $[0, 1]$.

Proof (of unique local random field solution)

- For each $N \geq 1$, let

$$b_N(x) := \mathbf{1}_{\{|x| \leq N\}} b(x) + \mathbf{1}_{\{|x| > N\}} b(N) + \mathbf{1}_{\{|x| < -N\}} b(-N)$$

and obtain a unique global solution $(u_N(t,x))_{(t,x) \in \mathbf{R}^+ \times [0,1]}$ where b is replaced by b_N. Moreover, $u_N(t,x)$ is almost surely continuous in (t,x).

- Let

$$\tau_N := \inf \left\{ t > 0 : \sup_{x \in [0,\,1]} |u_N(t,\, x)| > N \right\},$$

By the local property of the stochastic integral, for each $N \geq \|u_0\|_\infty$, we have a unique local random field solution $u(t,x) = u_N(t,x)$ for all $x \in [0,1]$ and $t \in [0,\, \tau_N)$. In particular, $u(t,x)$ is almost surely continuous in (t,x).

- Moreover, $\tau_N \leq \tau_{N+1}$. Denote $\tau_\infty = \lim_{N \to \infty} \tau_N$. It is easy to show that $\tau_\infty = T$ where recall $T = \sup\{t > 0 : \sup_{x \in [0,1]} |u(t,x)| < \infty\}$.

Blow up results

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \quad x \in [0,1],\ t > 0.$$

Theorem (Bonder-Groisman'09)

If b is nonnegative, locally Lipschitz, convex, and satisfies the Osgood condition, then the solution blows up in finite time.

Theorem (Dalang-Khoshnevisan-Zhang'19)

If b is locally Lipschitz and $|b(x)| = O(|x| \log |x|)$ as $|x| \to \infty$, then there exists a global solution.

Observe that if $b(x) \sim |x|(\log |x|)^{\delta}$, as $x \to \infty$, the Osgood condition holds iff $\delta < 1$.

Thus, Dalang-Khoshnevisan-Zhang'19 result shows that the Osgood condition is optimal.

Blow up result= converse of Bonder-Groisman

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \quad x \in [0,1],\ t > 0.$$

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. If the solution blows up in finite time with positive probability then b satisfies the Osgood condition.

Observe that Bonder-Groisman's Theorem shows that if $b(u) = u^{1+\eta}$, with $\eta > 0$, then there is no global solution no matter how small the initial condition is.

When $\sigma = 0$, for any $\eta > 0$ one can construct nontrivial global solutions by taking u_0 small enough.

Extension to multiplicative noise and fractional Laplacian in a ball

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x), \quad x \in B_1(0),\ t > 0, \\
u(0,x) = u_0(x),
\end{cases}
$$

- $\mathcal{L}$ is the fractional Laplacian in $B_1(0) \subset \mathbf{R}^d$ and F is white in time with a general spatial correlation
- σ is a locally Lipschitz function satisfying $\frac{1}{K} \leq \sigma(x) \leq K$ for all $x \in \mathbf{R}$ and some $K > 0$.

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. If the solution blows up in finite time with positive probability, then b satisfies the Osgood condition.

The stochastic heat equation on **R**

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x) \quad x \in \mathbf{R},\ t > 0, \\ u(0,x) = u_0(x). \end{cases}$$

Theorem (Foondun-Nualart'20)

Suppose that Assumption 1 holds. Then, if b satisfies the Osgood condition, then almost surely, there is no global solution.

This Theorem shows that if $b(u) = u^{1+\eta}$, with $\eta > 0$, then there is no global solution meaning that there is no Fujita exponent in the stochastic setting.
Recall that when $\sigma = 0$ and $x \in \mathbf{R}^d$, if $\eta > 2/d$, one can construct nontrivial global solutions when u_0 is small enough (Fujita'66).

The Theorem is also true for fractional Laplacian and $\mathbf{R}^d$ with a general spatial correlation given by a Riesz kernel.

The stochastic heat equation on **R**

- The mild formulation writes as

$$u(t,\,x) = \int_{\mathbf{R}} G(t,\,x,\,y) u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbf{R}} G(t-s,\,x,\,y) b(u(s,\,y))\,\mathrm{d}y\,\mathrm{d}s + \sigma g(t,x)$$

  where $G(t,\,x,\,y)$ is the heat kernel and

$$g(t,x) := \int_0^t \int_{\mathbf{R}} G(t-s,\,x,\,y) W(\mathrm{d}y\,\mathrm{d}s).$$

- For a fixed $x \in \mathbf{R}$, the process $(g(t,x), t \geq 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ multiplied by a constant (Lei-D.Nualart'09).

Theorem (Foondun-Nualart'20)

A.s. there exists $t_n \to \infty$ such that

$$\inf_{h\in[0,1],x\in[0,1]} g(t_n + h, x) \to \infty \quad \textit{as} \quad n \to \infty.$$

Proof (of $\inf_{h\in[0,1],x\in[0,1]} g(t_n+h,x)\to\infty$)

- Using an improvement of the classical Garsia's lemma obtained in Dalang-khosnevisan-Nualart'07 we show that for all $p\geq 2$ and integer $n\geq 1$,

$$\mathrm{E}\left[\sup_{s,t\in[n,n+2],x,y\in[0,1]} |g(t,x)-g(s,y)|^p\right]\leq A_p 2^{p/4}.$$

- As a consequence, a.s.

$$\sup_{s,t\in[n,n+2],x,y\in[0,1]} \frac{|g(t,x)-g(s,y)|}{\psi_{\frac12,\frac12}(n)}\longrightarrow 0, \quad \text{as } n\to\infty.$$

- Fix $x_0\in[0,1]$ and write

$$\begin{aligned}\inf_{h\in[0,1],x\in[0,1]} g(t+h,x) &\geq g(t,x_0)+\inf_{h\in[0,1],x\in[0,1]}(-|g(t+h,x)-g(t,x_0)|)\\ &\geq \frac{g(t,x_0)}{\psi_{\frac12,\frac12}(t)}\psi_{\frac12,\frac12}(t)-\sup_{h\in[0,1],x\in[0,1]}\frac{|g(t+h,x)-g(t,x_0)|}{\psi_{\frac12,\frac12}([t])}\psi_{\frac12,\frac12}([t]).\end{aligned}$$

- Using the LIL for bifBm, we conclude the proof.

Proof (of the sufficiency of the Osgood condition)

- Assume that there is a global solution a.s. Let $t_n \to \infty$. Then

$$u(t+t_n,\, x) \geq \int_{\mathbf{R}} G(t+t_n,\, x,\, y) u_0(y)\,\mathrm{d}y + \sigma g(t+t_n, x) + \int_0^t \int_{\mathbf{R}} G(t-s,\, x,\, y) b(u(s+t_n,\, y))\,\mathrm{d}y\,\mathrm{d}s.$$

- There exists $t_n \to \infty$ such that $g(t+t_n, x) > 0$ for all $x \in (0,\, 1)$ and $t \in [0, 1]$, and thus $u(t+t_n,\, x) > 0$ as well.

- For fixed $x \in (0,\, 1)$ and $t \in [0, 1]$,

$$\begin{aligned}\int_0^t \int_{\mathbf{R}} G(t-s,\, x,\, y) b(u(s+t_n,\, y))\,\mathrm{d}y\,\mathrm{d}s &\geq \int_0^t b\left(\inf_{y\in(0,\,1)} u(s+t_n,\, y)\right) \int_{(0,\,1)} G(t-s,\, x,\, y)\,\mathrm{d}y\,\mathrm{d}s \\ &\geq \int_0^t b\left(\inf_{y\in(0,\,1)} u(s+t_n,\, y)\right) \mathrm{d}s,\end{aligned}$$

as $G(t,\, x,\, y) \geq \frac{c}{t^{1/2}}$ whenever $|x-y| \leq t^{1/2}$.

Proof (of the sufficiency of the Osgood condition)

- Set $Y_t := \inf_{y\in(0,1)} u(t+t_n, y)$.
- We have shown that

$$Y_t \geq \inf_{0\leq h\leq 1, x\in(0,1)} \left\{ \int_{\mathbf{R}} G(h+t_n, x, y)u_0(y)\,\mathrm{d}y + \sigma g(h+t_n, x) \right\} + \int_0^t b(Y_s)\,\mathrm{d}s.$$

- Using the last Theorem, we conclude that the Osgood condition cannot hold.

Everywhere and instantaneous blow up

$$\left|\begin{array}{l} \dfrac{\partial u(t,\, x)}{\partial t} = \Delta u(t,\, x) + b(u(t,\, x)) + \sigma(u(t\,,x))\dot{W}(t,\, x) \quad x \in \mathbf{R},\ t > 0, \\ u(0,\, x) = u_0(x). \end{array}\right. \tag{2}$$

- $\sigma : \mathbf{R} \to (0, \infty)$ is bounded, globally Lipschitz and bounded away from the origin.
- u_0 is bounded.

We say that a random field solution $u = \{u(t\,,x)\}_{t\geq 0, x\in \mathbf{R}}$ blows up everywhere and instantaneously if for any $t > 0$ and every $x \in \mathbf{R}$,

$$u(t\,,x) = \infty \quad \text{a.s.}.$$

Everywhere and instantaneous blow up

Theorem (Foondun-Khoshnevisan-Nualart'23)

If the Osgood condition holds then u blows up everywhere and instantaneously almost surely.

We start by making sense of the solution to (2) by a truncation argument : let

$$b^{(n)} = b \wedge n.$$

Every $b^{(n)}$ is Lipschitz continuous and $b^{(n)} \leq b^{(m)}$ when $n \leq m$.

We replace b by $b^{(n)}$ with the same σ and u_0, then (2) will have a solution $u^{(n)}$.

By a standard comparison theorem for SPDEs , we have $u^{(n)}(t\,,x) \leq u^{(m)}(t\,,x)$ for all $t > 0$ and $x \in \mathbf{R}$ when $n \leq m$.

Therefore, $u(t\,,x) = \lim_{n\to\infty} u^{(n)}(t\,,x)$ and it satisfies the mild formulation of (2).

Elements of Malliavin calculus and ergodicity

The following Poincaré inequality holds for all F, G in $\mathbb{D}^{1,2}$

$$|\mathrm{Cov}(F,G)| \leq \int_0^\infty dr \int_{-\infty}^\infty dz \, \|D_{r,z}F\|_2 \|D_{r,z}G\|_2.$$

We say that a predictable random field $Z = \{Z(t,x)\}_{(t,x)\in(0,\infty)\times\mathbf{R}}$ is spatially mixing if the random field $x \to Z(t,x)$ is weakly mixing for every $t > 0$.

By Chen-Khoshnevisan-Nualart-Pu'21-22, this holds if and only if for all integers $k \geq 1$, real numbers $t > 0$ and $\xi^1, \dots, \xi^k$, and Lipschitz-continuous functions $g_1, \dots, g_k : \mathbf{R} \to \mathbf{R}$ satisfying $g_j(0) = 0$ and $\mathrm{Lip}(g_j) = 1$ for every $j = 1, \dots, k$,

$$\lim_{|x|\to\infty} \mathrm{Cov}[\mathcal{G}(x), \mathcal{G}(0)] = 0,$$

where

$$\mathcal{G}(x) = \prod_{j=1}^{k} g_j(Z(t, x+\xi^j)), \quad x \in \mathbf{R}.$$

Moreover, if the process $x \to Z(t,x)$ is stationary and weakly mixing for all $t > 0$, then it is ergodic.

Ergodicity of stochastic convolutions

Let $Z = \{Z(t,x)\}_{(t,x)\in(0,\infty)\times\mathbf{R}}$ be a predictable random field satisfying

$$c_1 \leq \inf_{(t,x)\in(0,\infty)\times\mathbf{R}} Z(t\,,x) \leq \sup_{(t,x)\in(0,\infty)\times\mathbf{R}} Z(t\,,x) \leq c_2.$$

Set $I_Z(0\,,x) = 0$ and consider the stochastic convolution

$$I_Z(t\,,x) = \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)Z(s\,,y)\, W(\mathrm{d}s\,\mathrm{d}y) \qquad \text{for every } t>0.$$

Theorem

Assume that $x \to Z(t\,,x)$ is stationary and $Z(t,x) \in \mathbb{D}^{1,k}$, for all $k \geq 2$, $t>0$, $x \in \mathbf{R}$. Assume that for all $T>0$ and $k \geq 2$, there exists $C_{T,k}>0$ such that for $t \in (0,T)$ and $x \in \mathbf{R}$ and for a.e. $(r,z) \in (0,t) \times \mathbf{R}$,

$$\|D_{r,z}Z(t,x)\|_k \leq C_{T,k}p_{t-r}(x-z)p_r(z).$$

Then for all $t>0$, the process $x \to Z(t\,,x)$ is ergodic and the process $x \to I_Z(t\,,x)$ is stationary and ergodic.

Spatial growth of stochastic convolutions

Theorem

Assume the hypotheses of the preceding theorem. Choose and fix $c_2 > c_1 > 0$. Then, there exists $\eta > 0$ such that

$$\limsup_{c\to\infty} \inf_{t\in(a,a+(\eta a)^2)} \inf_{x\in(0,\eta a)} I_Z(t\,,c+x) = \infty \quad a.s.,$$

for every $a > 0$.

We show that

$$\inf_{a>0} \mathrm{P}\left\{\limsup_{c\to\infty} \inf_{t\in(a,a+\varepsilon^4)} \inf_{x\in(c,c+\varepsilon^2)} I_Z(t\,,x) > M\left(\frac{a}{\pi}\right)^{1/4}\right\} > 0,$$

uniformly for all $M \geq M_0$. Ergodicity implies that

$$\mathrm{P}\left\{\limsup_{c\to\infty} \inf_{t\in(a,a+\varepsilon^4)} \inf_{x\in(c,c+\varepsilon^2)} I_Z(t\,,x) > M\left(\frac{a}{\pi}\right)^{1/4}\right\} = 1,$$

uniformly for all $M \geq M_0$ and $a > 0$. We finally send $M \to \infty$.

Ergodicity of the solution

Set

$$I(t\,,x)=\int_{(0,t)\times\mathbf{R}}p_{t-s}(y-x)\sigma(u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y)$$

Theorem

Consider the solution to but with b globally Lipschitz and constant initial condition. Then the processes $x\to u(t\,,x)$ and $x\to\mathcal{I}(t\,,x)$ are both stationary and ergodic for all $t>0$.

Recall that $u(t,x)\in\mathbb{D}^{1,k}$, for all $k\geq 2$, $t>0$ and $x\in\mathbf{R}$ and the Malliavin derivative satisfies

$$\begin{aligned}D_{r,z}u(t\,,x)=p_{t-r}(x-z)\sigma(u(r,z))&+\int_{(r,t)\times\mathbf{R}}p_{t-s}(y-x)B_{s,y}D_{r,z}u(s\,,y)\,\mathrm{d}s\,\mathrm{d}y\\&+\int_{(r,t)\times\mathbf{R}}p_{t-s}(y-x)\Sigma_{s,y}D_{r,z}u(s,y)\,W(\mathrm{d}s\,\mathrm{d}y)\end{aligned}$$

a.s.,where B and Σ are a.s. bounded random fields.

An important lower bound

Theorem

Assume b Lipschitz continuous. For every $a > 0$, there exists $\varepsilon > 0$ such that for every $M > \|u_0\|_{L^\infty(\mathbf{R})}$: There exists an a.s.-finite random variable $c = c(a, M) > 0$ such that

$$\inf_{t\in[a+\varepsilon,a+2\varepsilon]} \inf_{x\in(c,c+\sqrt{\varepsilon})} u(t,x) \geq \sup\left\{N > M : \int_{M+\rho}^{N+\rho} \frac{\mathrm{d}y}{b(y)} < \varepsilon\right\} \qquad a.s. \quad [\sup\varnothing = 0],$$

where $\rho := \inf_{x\in\mathbf{R}} u_0(x)$.

Proof of everywhere and instantaneous blow up

Fix $M > \|u_0\|_{L^\infty(\mathbf{R})}$ such that

$$\int_{M+\rho}^{\infty} \frac{\mathrm{d}y}{b(y)} < \varepsilon.$$

Then, the construction of u and the preceding theorem together yield ε such that

$$\begin{aligned}\inf_{t\in[a+\varepsilon,a+2\varepsilon]} \inf_{x\in(c,c+\sqrt{\varepsilon})} u(t\,,x) &\geq \inf_{t\in(a+\varepsilon,a+2\varepsilon)} \inf_{x\in(c,c+\sqrt{\varepsilon})} u^{(n)}(t\,,x)\\ &\geq \sup\left\{N > M :\ \int_{M+\rho}^{N+\rho} \frac{\mathrm{d}y}{b^{(n)}(y)} < \varepsilon\right\} \qquad \text{a.s.}\end{aligned}$$

Proof of everywhere and instantaneous blow up

Let $n \uparrow \infty$ to see from the monotone convergence theorem that

$$\inf_{t \in [a+\varepsilon, a+2\varepsilon]} \inf_{x \in (c, c+\sqrt{\varepsilon})} u(t\,,x) \geq \sup \left\{ N > M : \int_{M+\rho}^{N+\rho} \frac{\mathrm{d}y}{b(y)} < \varepsilon \right\} = \infty \qquad \text{a.s.}$$

This proves that the blowup time is a.s. $\leq a + 2\epsilon(a)$ and that the solution blows up everywhere in a random interval of the type $(c\,, c + \sqrt{\varepsilon})$.

This proves that for every $t > 0$ there is a.s. a random closed interval $I(t) \subset (0, \infty)$ and and a non-random closed interval $\tilde{I}(t) = [a + \epsilon, a + 2\epsilon] \subset (0, t)$ such that

$$\inf_{(s,x) \in \tilde{I}(t) \times I(t)} u(s\,,x) = \infty \qquad \text{a.s.}$$

Proof of everywhere and instantaneous blow up

By the monotone convergence theorem,

$$\int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)b^{(n)}(u^{(n)}(s,y))\,ds\,dy \geq \int_{\tilde{I}(t)\times I(t)} p_{t-s}(y-x)b^{(n)}(u^{(n)}(s,y))\,ds\,dy \uparrow \infty$$

Moreover,

$$\sup_n E\left(\sup_{(t,x)\in K}\left|\int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(u^{(n)}(s,y))\,W(ds\,dy)\right|^2\right) < \infty,$$

for every compact set $K \subset \mathbb{R}_+ \times \mathbb{R}$. Therefore,

$$\liminf_{n\to\infty} \sup_{(t,x)\in K} \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(u^{(n)}(s,y))\,W(ds\,dy) < \infty,$$

by Fatou's lemma. This proves that

$$\inf_{(t,x)\in K} u(t,x) = \infty$$

for all compact sets $K \subset \mathbb{R}_+ \times \mathbb{R}$, which concludes the proof.

References

Blow up SDEs :

1. León, J.A, and Villa, J. (2011), An Osgood criterion for integral equations with applications to stochastic differential equations with an additive noise, *Statist. Probab. Lett.*

Blow up Heat :

1. Bonder, J.-F. and Groisman, P. (2009), Time-space white noise eliminates global solutions in reaction-diffusion equations, *Physica D. Nonlinear Phenomena*.
2. Dalang, R. C., Khoshnevisan, D. and Tusheng, Z. (2019), Global solutions to stochastic reaction–diffusion equations with super-linear drift and multiplicative noise *Ann. Probab*.
3. Foondun, M. and Nualart, E. (2020), The Osgood condition for stochastic partial differential equations *Bernoulli*.

Ergodicity Heat :

1. Chen, L., Khoshnevisan, D., Nualart, D. and Pu, F. (2021) Spatial ergodicity for SPDEs via Poincaré-type inequalities *Electr. J. Probab.*
2. Chen, L., Khoshnevisan, D., Nualart, D. and Pu, F. (2022) Spatial ergodicity and central limit theorems for parabolic Anderson model with delta initial condition *J. Functional Analysis*