Weak solutions to the master equation of potential mean field games

Seminar on SDEs and related topics

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1. Mean field games and related master equation

Background

• Originates from macroeconomics: games with many many heterogeneous rational agents

∘ associate cost functional with each player

∘ interactions through aggregated quantity: say price...

∘ each agent has nearly infinitesimal impact but the sum of all forms a compromise!

∘ advocate for games with a continuum of players

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  ◦ advocate for games with a continuum of players

- Paradigm for games with a continuum of players [Aumann, 64]

  ◦ taken from mean-field approach in statistical mechanics $\rightsquigarrow$ players interact one with the others through the theoretical distribution of the population

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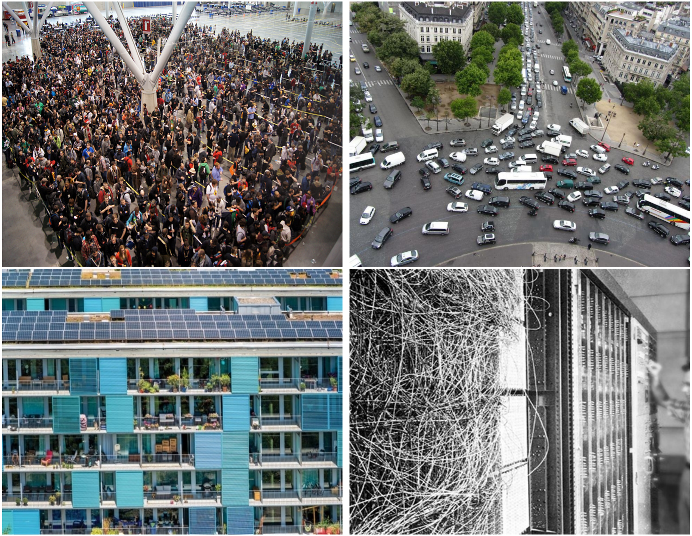
    ◦ taken from Von Neumann and Morgenstern:

    > *this is a well known phenomenon in many branches of the exact and physical sciences that very great numbers are often easier to handle than those of medium size. This is of course due to the excellent possibility of applying the laws of statistics and probabilities in the first case*

Typical examples...



Matching problem of MFG

• Define a compromise for the whole population as the solution of a

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(1) fix a flow of probability measures $(\mu_t)_{0\le t\le T}$ (with values in $\mathcal{P}_2(\mathbb{R}^d)$)

(2) solve the stochastic optimal control problem in the environment $(\mu_t)_{0\le t\le T}$

$$dX_t = \alpha_t dt + dW_t$$

∘ with $X_0 = \xi$ being fixed on some set-up $(\Omega, \mathbb{F}, \mathbb{P})$ with a d-dimensional B.M.

∘ with cost $J(\boldsymbol{\alpha}) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\big)dt\Big]$

∘ example: $f(x, \mu) = \int_{\mathbb{R}^d} h(x-y)d\mu(y)$

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(3) let $(X_t^{\star,\mu})_{0\leq t\leq T}$ be the unique optimizer (under nice assumptions)
$\rightsquigarrow$ find $(\mu_t)_{0\leq t\leq T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\mu}), \quad t \in [0, T]$$

- [Lasry-Lions], [Caines-Huang-Malhamé]...

PDE point of view

- Value function in environment $(\mu_t)_{0\le t\le T}$

$$u(t,x) = \inf_{\alpha \text{ processes}} \mathbb{E}\Big[g(X_T,\mu_T) + \int_t^T \Big(f(X_s,\mu_s) + \frac{1}{2}|\alpha_s|^2\Big)ds | X_t = x\Big]$$

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  - u solution Backward HJB

$$\Big(\partial_t u + \tfrac{1}{2}\Delta_x u\Big)(t,x) + \underbrace{\inf_{\alpha\in\mathbb{R}^d}\big[\alpha\cdot\partial_x u(t,x) + \tfrac{1}{2}|\alpha|^2 + f(x,\mu_t)\big]}_{\text{standard Hamiltonian in HJB}} = 0$$

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$$\Big(\partial_t u + \tfrac{1}{2}\Delta_x u\Big)(t,x) - \tfrac{1}{2}|\partial_x u(t,x)|^2 + f(x,\mu_t) = 0$$

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  - Pay attention that u depends on $(\mu_t)_t$!

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- Need for a PDE characterization of $(\mathcal{L}(X_t^{\star,\boldsymbol{\mu}}))_t$
  - Dynamics of $X^{\star,\boldsymbol{\mu}}$ at equilibrium

$$dX_t^{\star,\boldsymbol{\mu}} = -\partial_x u(t, X_t^{\star,\boldsymbol{\mu}})dt + dW_t$$

  - Law $(X_t^{\star,\boldsymbol{\mu}})_{0\le t\le T}$ satisfies Fokker-Planck (FP) equation

$$\partial_t \mu_t = \mathrm{div}_x\big(\partial_x u(t,x)\mu_t\big) + \tfrac{1}{2}\Delta_x \mu_t$$

- End-up with forward-backward system in ∞ dimension

Master equation

• A lot on the MFG forward-backward system [Cardaliaguet and Porretta] $\leftrightarrow$ McKean Vlasov [FBSDE]

• A key point of view $\rightsquigarrow$ regard MFG system as the characteristics of a PDE on space of probability measures

∘ look for $v(t, x, m)$, with $m \in \mathcal{P}(\mathbb{R}^d)$ such that

$$u_t(x) = v(t, x, m_t)$$

∘ $v(t, x, m)$ is the (???) equilibrium value of the game with (t, m) as initial state

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∘ formal proof is to expand $v(t, x, m_t)$ by chain rule and to identify with the expansion of $u_t(x)$

$$\begin{aligned} & d[v(s, X_s^\star, \mathcal{L}(X_s^\star))] \\ & = \Big[f\Big(X_s^\star, \partial_x v(s, X_s^\star, \mathcal{L}(X_s^\star))\Big) + \frac{1}{2}\big|\partial_x v(s, X_s^\star, \mathcal{L}(X_s^\star))\big|^2\Big] ds \\ & \quad + \partial_x v(t, X_s^\star, \mathcal{L}(X_s^\star)) dB_s \end{aligned}$$

∘ $X_t^\star \sim m$

Master equation

- Look for $v(t, x, m)$, with $m \in \mathcal{P}(\mathbb{R}^d)$ such that

$$u_t(x) = v(t, x, m_t)$$

- Equation for $v(t, x, m)$ with $x \in \mathbb{R}^d$ and $m \in \mathcal{P}\left(\mathbb{R}^d\right)$

$$\begin{aligned}
&\partial_t v(t, x, m) - \tfrac{1}{2} \left|\partial_x v(t, x, m)\right|^2 + \tfrac{1}{2}\Delta_x v(t, x, m) + f(x, m) \\
&\quad - \tfrac{1}{2} \int \partial_x v(t, y, m) \partial_\mu v(t, x, m)(y) dm(y) \\
&\quad + \tfrac{1}{2} \int \mathrm{Tr}\Big(\partial_y \partial_\mu v(t, x, m)(y)\Big) dm(y) = 0
\end{aligned}$$

  - definition of the derivative

$$\partial_\mu v(t, x, m)(y) = \partial_y \frac{\delta v}{\delta m}(t, x, m)(y)$$

$$\frac{\delta v}{\delta m}(t, x, m)(y) = \lim_{\varepsilon \searrow 0} \frac{v\Big(t, x, (1-\varepsilon)m + \varepsilon\delta_y\Big) - v(t, x, m)}{\varepsilon}$$

Master equation

- Equation for $v(t,x,m)$ with $x \in \mathbb{R}^d$ and $m \in \mathcal{P}\left(\mathbb{R}^d\right)$

$$
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&\quad -\tfrac{1}{2}\int \partial_x v(t,y,m)\partial_\mu v(t,x,m)(y)dm(y) \\
&\quad +\tfrac{1}{2}\int \mathrm{Tr}\Big(\partial_y \partial_\mu v(t,x,m)(y)\Big)dm(y) = 0
\end{aligned}
$$

$\rightsquigarrow$ blue part is HJB; red terms are mean field

◦ definition of the derivative

$$
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$$

$$
\frac{\delta v}{\delta m}(t,x,m)(y) = \lim_{\varepsilon \searrow 0} \frac{v\Big(t,x,(1-\varepsilon)m + \varepsilon\delta_y\Big) - v(t,x,m)}{\varepsilon}
$$

- Smoothness in x is ok (thanks to Δ_x) but smoothness in m is an issue: infinite dimensional hyperbolic system

Bibliography

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$$\int_{\mathbb{R}^d} (f(x,\mu) - f(x,\mu'))d(\mu - \mu')(x) \geq 0$$

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∘ research on master equation in weaker setting (Bertucci, Mou & Zhang, Gangbo & Meszaros...)

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• How to go beyond montonicity? $\rightsquigarrow$ potential games

∘ MFG solutions are critical points of a mean field control problem: control problem with trajectories taking values in $\mathcal{P}(\mathbb{R}^d)$

∘ solve first HJ equation and say that MFG system is the derivative of the HJ equation

∘ identify the solution of the master equation with the derivative of the value function (finite dimension: [Kruzhkov, Lions])

2. Mean field control: optimal control of McKean Vlasov equations

Potential mean field game

- Mean field game

$$dX_t = \alpha_t dt + dB_t$$

$$J(\alpha) = \mathbb{E}\int_0^T \left(f\left(X_t, \mu_t\right) + \tfrac{1}{2}\left|\alpha_t\right|^2\right)dt + \mathbb{E}g\left(X_T, \mu_T\right)$$

  - α_t = control, μ= environment

- No monotonicity but potential (below same for g/G)

$$f(x,\mu) = \frac{\delta F}{\delta m}(\mu)(x) = \lim_{\varepsilon \searrow 0} \frac{F\left((1-\varepsilon)\mu + \varepsilon\delta_x\right) - F(\mu)}{\varepsilon}$$

  - flat derivative (defined up to a constant: centred w.r.t. m)

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  - flat derivative (defined up to a constant: centred w.r.t. m)

- Derives from a mean field control problem!

Mean field control problem

- Same state dynamics but ≠ optimization

$$dX_t = \alpha_t dt + dB_t$$

$$J(\alpha) = \int_0^T \left(F(\mu_t) + \frac{1}{2}\mathbb{E}(|\alpha_t|^2) \right) dt + G(\mu_T)$$

◦ $\mu_t = \mathcal{L}(X_t)$ (law of X_t)

◦ this is a McKean-Vlasov probem: the fixed point condition is fixed before the optimization step and not after!

Mean field control problem

• PDE formulation

$$\partial_t \mu_t(x) + \mathrm{div}_x(\alpha_t(x)\mu_t(x)) - \frac{1}{2}\Delta_x \mu_t(x) = 0$$

$$J(\alpha) = \int_0^T \left(F(\mu_t) + \frac{1}{2}\int_{\mathbb{R}^d} |\alpha_t(x)|^2 \right) dt + G(\mu_T)$$

∘ Pontryagin principle ⇒ MFG system is the first-order condition [Lasry-Lions], [Briani-Cardaliaguet]

Mean field control problem

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- Value function $V(t,m) \rightsquigarrow$ minimal cost when $\mathcal{L}(X_t) = m$
- HJ equation

$$\begin{cases} \partial_t V(t,m) - \frac{1}{2}\int \left|\partial_\mu V(t,m)(y)\right|^2 dm(y) \\ \qquad + \frac{1}{2}\int \mathrm{Tr}\left(\partial_y \partial_\mu V(t,m)(y)\right) dm(y) + F(m) = 0 \\ V(T,m) = G(m) \end{cases}$$

- No smooth solution in general setting; theory of viscosity solutions still in development $\rightsquigarrow$ main difficulty: Δ, see [Pham and co-authors, Conforti and co-authors]

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- Objective: write a generalized form ('a.e. derivatives') and uniqueness for semi-concave solutions

Using Fourier analysis

• Provide a notion of generalized solution to HJB based on finite dimensional Fourier approximations

∘ use periodic setting and get discretization with respect to Fourier coefficients

$$\mathcal{P}\left(\mathbb{R}^d\right) \Rightarrow \mathcal{P}\left(\mathbb{T}^d\right)$$

and

$$\phi(m), \quad m \in \mathbb{T}^d \ \Rightarrow \ \phi\left(\left(\widehat{m}^k\right)_k\right), \ \hat{m}^k = \int_{\mathbb{T}^d} e^{i2\pi k\cdot x} dm(x)$$

∘ restrict functions to measures m such that

$$\widehat{m}^k = 0 \quad \text{if} \quad |k| \geq N, \quad \text{for some } N \geq 1$$

• Good point because those measures are stable by the heat equation which is the characteristic equation of the operator

$$\left(\phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}\right) \mapsto \frac{1}{2} \int_{\mathbb{T}^d} \mathrm{Tr}\left(\partial_y \partial_\mu \phi(m)(y)\right) dm(y)$$

Finite dimensional restrictions

- For $\phi : \mathcal{P}\left(\mathbb{T}^d\right) \to \mathbb{R}$, restrict to

$$\mathcal{O}_N = \left\{ m \in \mathcal{P}(\mathbb{T}^d) : \hat{m}^k = 0, \ \max_{j=1,\cdots,d} |k_j| \geq N \right\}$$

  - Bochner-Herglotz: $\exists$ density f_N s.t. $f_N * m \in \mathcal{O}_N$ for $m \in \mathcal{P}(\mathbb{T}^d)$

- If ϕ is Lipschitz for total variation

$$| \phi(m_1) - \phi(m_2) | \leq C \int_{\mathbb{T}^d} |m_1(x) - m_2(x)| dx \leq C \Big(\sum_{\max_{j=1,\cdots,d} |k_j| < N} |\widehat{m}_1^k - \widehat{m}_2^k|^2 \Big)^{1/2}$$

  - restriction of ϕ is a.e. differentiable on $\mathcal{O}_N$
  - identification of $(\widehat{\delta\phi/\delta m})^k$ for $\max_{j=1,\cdots,d} |k_j| < N$

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} \phi\Big((\widehat{m}^k + \varepsilon \widehat{r}^k)_k \Big) = \sum_{\max_{j=1,\cdots,d} |k_j| < N} (\widehat{\delta\phi/\delta m})^{-k} \widehat{r}^k$$

$$\rightsquigarrow \partial_{\hat{m}^k / \overline{\hat{m}^k}} \phi = (\widehat{\delta\phi/\delta m})^{-k}$$

Application to HJ

• Standard property: value function V is time-space Lipschitz (for even coarser topologies)

• Write HJB at a point (t, m) $[0, T] \times \mathcal{O}_N$ at which the restriction of V is differentiable

$\rightsquigarrow$ we have $\partial_{\widehat{m}^k} V(t, m)$ for $\max_{j=1,\cdots,d} |k_j| < N$

• Philosophy: rewrite the terms as derivatives w.r.t. Fourier coefficients

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$$\rightsquigarrow \tfrac{1}{2}\int_{\mathbb{T}^d} \mathrm{Tr}\left(\partial_y\partial_\mu V(t,m)(y)\right) dm(y) = \tfrac{1}{2}\int_{\mathbb{T}^d} \Delta_y \frac{\delta}{\delta m} V(t,m)(y) dm(y)$$

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$$\frac{1}{2} \int_{\mathbb{T}^d} \mathrm{Tr}\left(\partial_y \partial_\mu V(t, m)(y)\right) dm(y) = - \sum_{|k_j|<N} 2\pi^2 |k|^2 \widehat{m}^k \partial_{\widehat{m}^k} V(t, m)$$

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$\rightsquigarrow -\frac{1}{2}\int_{\mathbb{T}^d} \left|\partial_\mu V(t, m)(y)\right|^2 dm(y) \Rightarrow$ ALL the Fourier modes of $\partial_\mu V$

$$-\frac{1}{2}\int_{\mathbb{T}^d} \left|\partial_\mu V(t, m)(y)\right|^2 dm(y) \simeq -\frac{1}{2}\int_{\mathbb{T}^d} \Big|\sum_{|k_j|<N} k \partial_{\widehat{m}^k} V(t, m) e^{i2\pi k\cdot y}\Big|^2 dm(y)$$

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  ◦ By Pontryagin principle $\partial_{\widehat{m}^k} V(t, m) = u_0^{-k} \Rightarrow$ known decay with k

Reformulation of HJ

- General formulation (for V Lipschitz, not being necessarily the value function)
  - require for any N and a.e. $(t,m) \in [0,T] \times \mathcal{O}_N$

$$\partial_t V(t,m) - \sum_{|k_j|<N} 2\pi^2 |k|^2 \widehat{m}^k \partial_{\widehat{m}^k} V(t,m)$$
$$- \frac{1}{2} \int \Big| \sum_{|k_j|<N} k \partial_{\widehat{m}^k} V(t,m) e^{i2\pi k\cdot y} \Big|^2 dm(y) + F(m) = O(\eta_N)$$

$\rightsquigarrow$ η_N converges to 0 and notation $O(\cdot)$ is (almost) uniform on $[0,T] \times \mathcal{O}_N$

- Result 1: The value function satisfies the general formulation
  - use DPP...
- Question: uniqueness?

Scheme for uniqueness

- General idea: V_1 and V_2 two solutions
  - if classical $\Rightarrow V_1 - V_2$ satisfies

$$\partial_t[V_1 - V_2](t,m) + \frac{1}{2}\int \mathrm{Tr}\Big(\partial_y\partial_\mu[V_1 - V_2](t,m)(y)\Big)dm(y)$$
$$-\int \frac{1}{2}\Big(\partial_\mu V_1 + \partial_\mu V_2\Big)(t,m)(y)\Big(\partial_\mu[V_1 - V_2]\Big)(t,m)(y)dm(y) = 0$$

  - solve Fokker-Planck equation/McKean-Vlasov equation

$$\partial_t\mu_t(x) + \mathrm{div}_x\Big[\frac{1}{2}\Big(\partial_\mu V_1 + \partial_\mu V_2\Big)(t,\mu_t)(x)\mu_t(x)\Big]dt - \frac{1}{2}\Delta_x\mu_t(x) = 0$$
$$dX_t = -\frac{1}{2}\Big(\partial_\mu V_1 + \partial_\mu V_2\Big)(t,\mathcal{L}(X_t))(X_t)dt + dB_t$$

  - expand $(V_1 - V_2)(t,\mu_t)$ in time and get

$$[V_1 - V_2](0,\mu_0) = 0$$

- Question: When generalized solutions?

Mollified HJ equation

• Claim: If V is a generalized solution, then mollification that nearly solves HJ in classical sense ⇒ mollification parameters:

$\rightsquigarrow$ N: truncation of Fourier coefficients of m in V

$\rightsquigarrow$ ϱ_N: finite-dimensional mollification kernel on Fourier coefficients up to order N (mollification for a given truncation)

• Statement: If V is generalized solution, then $\exists V^{N,\varrho_N} \sim V$ such that

$$\partial_t V^{N,\varrho_N}(t,m) - \frac{1}{2}\int \left|\partial_\mu V^{N,\varrho_N}(t,m)(y)\right|^2 dm(y)$$
$$+ \frac{1}{2}\int \mathrm{Tr}\left(\partial_y \partial_\mu V^{N,\varrho_N}(t,m)(y)\right) dm(y) + F(m) = O(\eta_N) + \phi_t^{N,\varrho_N}(m)$$

∘ $\eta_N \to 0$ (independently of ϱ_N)

∘ $\phi_t^{N,\varrho_N}(m)$ depends on $\widehat{m}^k$ for $\max_{j=1,\cdots,d}|k_j| < N$ and in L^1

$$\int |\phi_t^{N,\varrho_N}(m)| d\left(\hat{m}^k\right)_{|k_j|<N} \to 0 \quad \text{as } \varrho_N \to \delta_0 \quad (N \text{ fixed})$$

Back to uniqueness

• Proceed as before and now compare $V_1^{N,\varrho_N} - V_2^{N,\varrho_N} \Rightarrow$ Fokker-Planck equation

$$\partial_t \mu_t(x) + \text{div}_x\Big[\tfrac{1}{2}\Big(\partial_\mu V_1^{N,\rho_N} + \partial_\mu V_2^{N,\rho_N}\Big)(t,\mu_t)(x)\mu_t(x)\Big]dt - \tfrac{1}{2}\Delta_x \mu_t(x) \sim 0$$

∘ with solutions in $\mathcal{O}_N$
then

$$\Big(V_1^{N,\rho_N} - V_2^{N,\rho_N}\Big)(0,\mu_0) = O(\eta_N) + \int_0^T \phi_t^{N,\rho_N}(\mu_t)dt$$

∘ difficult term is $\phi_t^{N,\rho_N}(\mu_t)$: $\phi_t^{N,\rho_N}(\cdot) \to 0$ in L^1 (taking as inputs $(\hat{m}_k)_{|k_j|<N}$)

∘ need to consider $\big(\widehat{\mu}_t^k\big)_{|k_j|<N}$:

If $\big(\widehat{\mu}_0^k\big)_{|k_j|<N}$ are chosen randomly according to a bounded density

$\Rightarrow$ the density of $\big(\widehat{\mu}_t^k\big)_{|k_j|<N}$ remains bounded (indep. of ϱ_N)

Need for semi-concavity

• Need to control the contraction of the Fourier coefficients of the solution to MKV equation

◦ semi-concavity comes in

$$V(t, \mathcal{L}(X+Y)) + V(t, \mathcal{L}(X-Y)) - 2V(t, \mathcal{L}(X)) \leq C\mathbb{E}[|Y|^2]$$

• Result 2: Uniqueness of Lipschitz, semi-concave generalized solutions such that

$$\partial_{\widehat{m}^k} V(t, m)$$

decreases fast enough at points where the derivative exists

$\rightsquigarrow$ covers the value function

3. Back to mean field games

Back to the master equation

- Recall the master equation

$$\partial_t v(t,x,m) - \frac{1}{2}|\partial_x v(t,x,m)|^2 + \frac{1}{2}\Delta_x v(t,x,m) + f(x,m)$$
$$- \frac{1}{2}\int \partial_x v(t,y,m)\partial_\mu v(t,x,m)(y)dm(y)$$
$$+ \frac{1}{2}\int \mathrm{Tr}\Big(\partial_x\partial_\mu v(t,x,m)(y)\Big)dm(y) = 0$$

- Conservative form

$\rightsquigarrow$ if v smooth then $\bar{v} = v - \int v dm$ solves

$$\partial_t \bar{v}(t,x,m) - \frac{1}{2}\frac{\delta}{\delta m}\Big[\int \big|\partial_y \bar{v}(t,y,m)\big|^2 dm(y)\Big](x)$$
$$+ \frac{1}{2}\frac{\delta}{\delta m}\Big\{\int \mathrm{Tr}\left(\partial_y^2 \bar{v}(t,y,m)\right) dm(y)\Big\}(x) + \frac{\delta F}{\delta m}(x) = 0$$

$\rightsquigarrow$ guess: $\bar{v}(t,x,m) = \frac{\delta V}{\delta m}(t,m)(x)$

Generalized solution

- Bounded measurable function

$$\mathcal{Z} : [0,T] \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}^d,$$

such that the system of Fourier coefficients

$$\widehat{\mathcal{Z}}^k : [0,T] \times \mathbb{P}(\mathbb{T}^d) \ni (t,m) \mapsto Z\widehat{(t,m,\cdot)}^{-k},$$

satisfies for $N \geq 1$, for $\max_{j=1,\cdots,d} |k_j| < N$, a.e. on $\mathcal{O}_N$

$$\partial_t \widehat{\mathcal{Z}}^k(t,m) - 2\pi^2 \partial_{\widehat{m}^k} \int_{\mathbb{T}^d} \Big| \sum_{|k_j|<N} j \widehat{\mathcal{Z}}^j(t,m) e_j(y) \Big|^2 dm(y)$$

$$- \sum_{|k_j|<N} 2\pi^2 |j|^2 \partial_{\widehat{m}_k} \widehat{\mathcal{Z}}^j(t,m) \widehat{m}^j + \widehat{f}^{-k}(m,\cdot) + \partial_{\widehat{m}^k}(\eta_N(t,m)) = 0$$

with $\eta_N \to 0$ (almost) uniformly on $[0,T] \times \mathcal{O}_N$

Uniqueness

• Claim: Uniqueness of generalized functions that satisfy a weak one-sided Lipschitz property

◦ weak solutions derive from a potential on $\mathcal{O}_N$ and this potential is semi-concave

• Some care is needed to aggregate all the equations (for any truncation N)

◦ ∃ probability measure $\mathbb{P}$ on $\mathcal{P}(\mathbb{T}^d)$ with full support such that uniqueness holds a.e. under $\mathbb{P}$ for solutions $\mathcal{Z}$ such that

$$\widehat{\mathcal{Z}}^k(t, m * f_N) \rightharpoonup \widehat{\mathcal{Z}}^k(t, m)$$

weakly (when test with respect to bounded functions depending on a finite number of Fourier coefficients)

◦ derivative of HJB satisfies the requirements

• Short look at construction of $\mathbb{P}$: $\mathbb{P}$ is the weak limit of

$$d\mathbb{P}_N(m) = 1_{\mathcal{O}_N}(m)\frac{1}{Z_N}\exp\Big(-\sum_{|k_j|<N}|k|^{2pd}|\widehat{m}^k|^2\Big)\prod_{|k_j|<N} d\widehat{m}^k$$