Limit theorems for additive functionals of the fractional Brownian motion

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Fractional Brownian motion

The fractional Brownian motion (fBm) $B = (B_t^H, t \geq 0)$ is a zero mean Gaussian process with covariance function given by

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

$H \in (0,1)$ is called the Hurst parameter. For $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a Brownian motion.

- *Stationary increments*: $\mathbb{E}[(B_t^H - B_s^H)^2] = |t-s|^{2H}$.
- *Regularity*: For any $\gamma < H$, with probability one, the trajectories $t \to B_t^H(\omega)$ are Hölder continuous of order γ:

$$|B_t^H(\omega) - B_s^H(\omega)| \leq G_{\gamma,T}(\omega)|t-s|^{\gamma}, \quad s,t \in [0,T].$$

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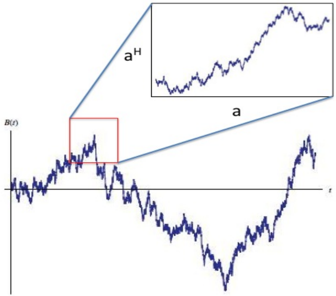
$$|B_t^H(\omega) - B_s^H(\omega)| \leq G_{\gamma,T}(\omega)|t-s|^{\gamma}, \quad s,t \in [0,T].$$

- *Self-similarity*: For all $a > 0$, the process

$$\{a^{-H} B^H_{at}, t \geq 0\}$$

is a fractional Brownian motion with Hurst parameter H.



- *Correlated increments*:
  (i) For $H \neq \frac{1}{2}$, the fBm B^H has correlated increments:

$$\begin{aligned}\rho_H(n) &= \mathbb{E}(B_1^H(B_{n+1}^H - B_n^H)) \\ &= \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right) \\ &\sim H(2H-1)n^{2H-2},\end{aligned}$$

  as $n \to \infty$.

  (ii) If $H > \frac{1}{2}$, then $\rho_H(n) > 0$ and $\sum_n \rho_H(n) = \infty$ (*long memory*).

  (iii) If $H < \frac{1}{2}$, then $\rho_H(n) < 0$ (*intermittency*) and $\sum_n |\rho_H(n)| < \infty$.

- *Local nondeterminism:*

  For any $0 = s_0 < s_1 < \cdots < s_n < \infty$ and $u_1, \ldots, u_n \in \mathbb{R}$,

$$\mathrm{Var}\Big(\sum_{i=1}^{n} u_i \big(B^H_{s_i} - B^H_{s_{i-1}}\big)\Big) \geq k_H \sum_{i=1}^{n} u_i^2 (s_i - s_{i-1})^{2H}.$$

Local time

Let B^H be a d-dimensional fBm with Hurst parameter H.

- The local time of the fBm B^H is formally defined as

$$L_t(x) = \int_0^t \delta(B_s^H - x)ds,$$

  for $t \geq 0$ and $x \in \mathbb{R}^d$.

- The local time is the density of the occupation measure:

$$\int_0^t f(B_s)ds = \int_{\mathbb{R}^d} f(x)L_t(x)dx.$$

- If $H < \frac{1}{d}$ there exists a version of the local time which is continuous in (t, x) (Geman-Horowitz '80). In fact,

$$\mathbb{E}[L_t(0)] = \int_0^t (2\pi s n^{2H})^{-\frac{d}{2}} ds < \infty \quad \Leftrightarrow \quad H < \frac{1}{d}$$

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First order limit result

Theorem

Assume $H < \frac{1}{d}$. Let $f \in L^1(\mathbb{R}^d)$. Then, for all $t \geq 0$,

$$n^{Hd} \int_0^t f(n^H(B_s^H - \lambda))ds \to L_t(\lambda) \int_{\mathbb{R}^d} f(y)dy,$$

in $L^2(\Omega)$.

Proof:

We simply write

$$\begin{aligned} n^{Hd} \int_0^t f(n^H(B_s^H - \lambda))ds &= n^{Hd} \int_{\mathbb{R}^d} f(n^H(x - \lambda))L_t(x)dx \\ &= \int_{\mathbb{R}^d} f(y)L_t(n^{-H}y + \lambda)dy. \end{aligned}$$

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Second order limit result

What happens if $\int_{\mathbb{R}^d} f(y)dy = 0$?

Theorem (Hu-N.-Xu '14)

Suppose $\frac{1}{d+2} < H < \frac{1}{d}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies $\int_{\mathbb{R}^d} |f(y)|(1 + |y|^{\frac{1}{H}-d})dy < \infty$ and $\int_{\mathbb{R}^d} f(y)dy = 0$. Then,

$$n^{\frac{Hd+1}{2}} \int_0^t f(n^H B_s^H) ds \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(0)}$$

in $C([0,\infty))$, as $n \to \infty$, where $\widetilde{W}$ is a Brownian motion independent of B^H and

$$\|f\|^2_{\frac{1}{H}-d} = -\int_{\mathbb{R}^{2d}} f(x)f(y)|x-y|^{\frac{1}{H}-d} dxdy.$$

and

$$C_{H,d} = \frac{2^{1-1/(2H)}}{(1-Hd)\pi^{d/2}} \Gamma\left(\frac{Hd+2H-1}{2H}\right).$$

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$$n^{\frac{Hd+1}{2}} \int_0^t f(n^H B_s^H)ds \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(0)}$$

in $C([0,\infty))$, *as* $n \to \infty$, *where* $\widetilde{W}$ *is a Brownian motion independent of* B^H *and*

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- The proof is based on the method of moments and uses the local nondeterministic property.
- By self-similarity, we can replace $n^H B_s^H$ by B_{sn}^H and making the change of variable $ns \to u$, we obtain

$$n^{\frac{Hd-1}{2}} \int_0^{nt} f(B_u^H) du \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(0)}.$$

- When $d = 1$ we need $H > \frac{1}{3}$. In the particular case $d = 1$ and $H = \frac{1}{2}$, we obtain

$$n^{-\frac{1}{4}} \int_0^{nt} f(B_u^{\frac{1}{2}}) du \xrightarrow{\mathcal{L}} \sqrt{2} \|f\|_1 \widetilde{W}_{L_t(0)},$$

which was proved by Papanicolau-Stroock-Varadhan '77 using martingale methods.

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Suppose $d = 1$

Questions:

- What happens if $H \leq \frac{1}{3}$?
- Can we obtain a second order limit result when $\int_{\mathbb{R}} f(y)dy \neq 0$?

Second order limit result for $\int_{\mathbb{R}} f(y)dy \neq 0$:

Theorem (Jaramillo-Nourdin-N.-Peccati '22)

Suppose $H > \frac{1}{3}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies $\int_{\mathbb{R}} |f(y)|(1+|y|)dy < \infty$. Then

$$n^{\frac{H+1}{2}} \left(\int_0^t f(n^H(B_s^H - \lambda))ds - n^{-H} L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) \xrightarrow{f.d.d.} \sqrt{C_{H,f}}\, \widetilde{W}_{L_t(\lambda)},$$

as $n \to \infty$, where f.d.d means convergence in law of the finite-dimensional distributions, $\widetilde{W}$ is a Brownian motion independent of B^H, and $C_{H,f}$ is a constant depending on H and f.

- The proof is based on an integral representation of the local time based on Malliavin calculus and Clark-Ocone formula.

Second order limit result for $\int_{\mathbb{R}} f(y)dy \neq 0$:

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Integral representation of fBm

We can assume that

$$B_t^H = \int_0^t K_H(t,s)dW_s,$$

where W is a standard Brownian motion and

$$K_H(t,s) = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

if $H > \frac{1}{2}$ and

$$K_H(t,s) = \left[\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}\right]^{\frac{1}{2}} \times \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du\right],$$

if $H < \frac{1}{2}$.

Malliavin calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $h_i \in \mathfrak{H} = L^2(\mathbb{R}_+)$, $W(h_i) = \int_0^\infty h_i(t)dW_t$ and $f \in C_b^\infty(\mathbb{R}^n)$.

- If $F \in \mathcal{S}$ we define its *derivative* by

$$D_s F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i(s).$$

DF is a random variable with values in $\mathfrak{H}$.

- Sobolev spaces: For $p \geq 1$, $\mathbb{D}^{k,p} \subset L^p(\Omega)$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|DF\|_{k,p} = \sum_{j=0}^{k} \left(\mathbb{E}(\|D^j F\|^p_{\mathfrak{H}^{\otimes j}})\right)^{1/p}.$$

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Clark-Ocone formula

For any random variable $F \in \mathbb{D}^{1,2}$ we have

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dW_t,$$

where $\{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by the Brownian motion W.

Stochastic integral representation of the local time:

Applying the Clarck-Ocone formula we can show that

$$L_t(\lambda) = \mathbb{E}[L_t(\lambda)] + \int_0^t \mathbb{E}[D_r L_t(\lambda)|\mathcal{F}_r] dW_r$$

$$= \int_0^t p_{s^{2H}}(\lambda) ds + \int_0^t \left(\int_r^t p'_{\mu_{r,s}}(B_{r,s} - \lambda) K_H(s,r) ds \right) dW_r,$$

where

$$B_{r,s} = \int_0^r K_H(s,\theta) dW_\theta, \qquad \mu_{r,s} = \int_r^s K_H^2(s,\theta) d\theta,$$

and $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$.

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&= \int_0^t p_{s^{2H}}(\lambda)ds + \int_0^t \left(\int_r^t p'_{\mu_{r,s}}(B_{r,s}-\lambda)K_H(s,r)ds\right) dW_r,
\end{aligned}
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where

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and $p_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)}$.

Proof:

(i) One one hand we have

$$\mathbb{E}[L_t(\lambda)] = E\left[\int_0^t \delta(B_s^H - \lambda)ds\right] = \int_0^t p_{s^{2H}}(\lambda)ds.$$

(ii) On the other hand, for $r \leq t$,

$$\begin{aligned}\mathbb{E}[D_r L_t(\lambda)|\mathcal{F}_r] &= \lim_{\epsilon\to 0}\mathbb{E}\left[D_r \int_0^t p_\epsilon(B_s^H - \lambda)ds\middle|\mathcal{F}_r\right] \\ &= \lim_{\epsilon\to 0}\mathbb{E}\left[\int_r^t K_H(s,r)\mathbb{E}[p'_\epsilon(B_s^H-\lambda)|\mathcal{F}_r]ds\right],\end{aligned}$$

because

$$D_r B_s^H = D_r \int_0^s K_H(s,\theta)dW_\theta = K_H(s,r)\mathbf{1}_{[0,s]}(r).$$

(iii) Making the decomposition, for $r \leq s$,

$$B_s^H = \int_0^r K_H(s,\theta)dW_\theta + \int_r^s K_H(s,\theta)dW_\theta,$$

and taking into account that $B_{r,s} = \int_0^r K_H(s,\theta)dW_\theta$ is $\mathcal{F}_r$-measurable and $\int_r^s K_H(s,\theta)dW_\theta$ is independent of $\mathcal{F}_r$, we obtain

$$\begin{aligned}\mathbb{E}[p'_\epsilon(B_s-\lambda)|\mathcal{F}_r] &= \mathbb{E}\left[p'_\epsilon\left(B_{r,s} + \int_r^s K_H(s,\theta)dW_\theta - \lambda\right)\middle|\mathcal{F}_r\right] \\ &= p'_{\epsilon+\mu_{r,s}}\left(B_{r,s}-\lambda\right),\end{aligned}$$

where $\mu_{r,s} = \int_r^s K_H^2(s,\theta)d\theta$.
Therefore, letting $\epsilon \to 0$,

$$\mathbb{E}[D_r L_t(\lambda)|\mathcal{F}_r] = \int_r^t K_H(s,r)p'_{\mu_{r,s}}\left(B_{r,s}-\lambda\right)ds.$$

Sketch of the proof of the theorem:

(i) We want to show that

$$Z_t^{(n)}(f) \xrightarrow{f.d.d.} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(\lambda)},$$

where

$$Z_t^{(n)}(f) := n^{\frac{H+1}{2}} \left(\int_0^t f(n^H(B_s^H - \lambda))ds - n^{-H} L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right).$$

(ii) Using the local time, we can write

$$Z_t^{(n)}(f) = n^{\frac{1-H}{2}} \int_{\mathbb{R}} f(x)(L_t(n^{-H}x + \lambda) - L_t(\lambda))dx.$$

(ii)i From the representation of the local time, we obtain

$$Z_t^{(n)}(f) = n^{\frac{1-H}{2}} \int_0^t G_{r,t}^{f,n} dW_r + n^{\frac{1-H}{2}} R_t^{(f,n)},$$

where

$$G_{r,t}^{f,n} = \int_{\mathbb{R}} \int_r^t f(x) \left(p'_{\mu_{r,s}}(B_{r,s} - \frac{x}{n^H} - \lambda) - p'_{\mu_{r,s}}(B_{r,s} - \lambda) \right) \times K_H(s,r) ds dx$$

and

$$R_t^{f,n} = \int_{\mathbb{R}} \int_0^t f(x) \left(p_{s^{2H}}(\frac{x}{n^H} + \lambda) - p_{s^{2H}}(\lambda) \right) ds dx.$$

(iv) We have

$$\lim_{n\to\infty} \sup_{0\le t\le T} n^{\frac{1-H}{2}} |R_t^{(f,n)}| = 0.$$

(v) To show the convergence

$$n^{\frac{1-H}{2}} \int_0^t G_{r,t}^{f,n} dW_r \xrightarrow{f.d.d.} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(\lambda)},$$

where $\widetilde{W}$ is a Brownian motion independent of B^H, we use martingale methods.

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where $\widetilde{W}$ is a Brownian motion independent of B^H, we use martingale methods.

(vi) If we fix $t > 0$, the process

$$M_u^{(n)} = n^{\frac{1-H}{2}} \int_0^u G_{r,t}^{f,n} dW_r, \qquad u \geq 0,$$

with the convention $G_{r,t}^{f,n} = 0$ if $r > t$, is a martingale, that satisfies

$$\langle M^{(n)} \rangle_u \xrightarrow{P} C_{H,d} \|f\|^2_{\frac{1}{H}-d} L_{t \wedge u}(\lambda) \tag{1}$$

and

$$\langle M^{(n)}, W \rangle_u \xrightarrow{P} 0, \tag{2}$$

uniformly in $u \in [0, T]$, for each fixed $T > 0$.

(vii) (1) and (2) imply:

$$M_u^{(n)} \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_{t \wedge u}(\lambda)}, \tag{3}$$

where $\widetilde{W}$ is a Brownian motion independent of W.

(vi) If we fix $t > 0$, the process

$$M_u^{(n)} = n^{\frac{1-H}{2}} \int_0^u G_{r,t}^{f,n} dW_r, \qquad u \geq 0,$$

with the convention $G_{r,t}^{f,n} = 0$ if $r > t$, is a martingale, that satisfies

$$\langle M^{(n)} \rangle_u \xrightarrow{P} C_{H,d} \|f\|^2_{\frac{1}{H}-d} L_{t\wedge u}(\lambda) \tag{1}$$

and

$$\langle M^{(n)}, W \rangle_u \xrightarrow{P} 0, \tag{2}$$

uniformly in $u \in [0, T]$, for each fixed $T > 0$.

(vii) (1) and (2) imply:

$$M_u^{(n)} \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_{t\wedge u}(\lambda)}, \tag{3}$$

where $\widetilde{W}$ is a Brownian motion independent of W.

Proof that (1) *and* (2) *imply* (3)*:*

(viii) Let $W^{(n)}$ be the Brownian motion such that $M_u^{(n)} = W^{(n)}_{\langle M^{(n)}\rangle_u}$, $u \in [0,T]$.

Then, an asymptotic version of Knight's theorem together with (1) and (2) imply

$$(W, W^{(n)}, \langle M^{(n)}\rangle) \xrightarrow{\mathcal{L}} (W, \widetilde{W}, \langle M^{(\infty)}\rangle),$$

where $\widetilde{W}$ is a Brownian motion independent of W and

$$\langle M^{(\infty)}\rangle_u = C_{H,d}\|f\|^2_{\frac{1}{H}-d} L_{t\wedge u}(\lambda).$$

This implies the following convergence in law for each $u \in [0,T]$:

$$M_u^{(n)} = W^{(n)}_{\langle M^{(n)}\rangle_u} \xrightarrow{\mathcal{L}} \widetilde{W}_{\langle M^{(\infty)}\rangle_u}.$$

Proof that (1) *and* (2) *imply* (3)*:*

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Case $H = \frac{1}{3}$

Theorem (Jaramillo-Nourdin-N.-Peccati '22)

Suppose $H = \frac{1}{3}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $\int_{\mathbb{R}} |f(y)|(1 + |y|^2)dy < \infty$. *Then for any* $t > 0$ *and* $\lambda \in \mathbb{R}$ *we have*

$$(\log n)^{-\frac{1}{2}} n^{\frac{1+H}{2}} \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) \xrightarrow{f.d.d.} \sqrt{C_f} \widetilde{W}_{L_t(\lambda)},$$

as $n \to \infty$.

Case $H < \frac{1}{3}$

Theorem (Jaramillo-Nourdin-N.-Peccati '22)

Suppose $H < \frac{1}{3}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $\int_{\mathbb{R}} |f(y)|(1 + |y|^{\nu})dy < \infty$ *for some* $\nu > 1$. *Then for any* $t > 0$ *and* $\lambda \in \mathbb{R}$ *we have*

$$n^H \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) \xrightarrow{L^2(\Omega)} L_t'(\lambda) \int_{\mathbb{R}} yf(y)dy,$$

as $n \to \infty$.

Sketch of the proof:

Set

$$\mathcal{D}_n := n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t(\lambda) \int_{\mathbb{R}} f(x)dx - n^{-H} L_t'(\lambda) \int_{\mathbb{R}} yf(y)dy$$

We have

$$\lim_{n\to\infty} n^{2H} \mathbb{E}[|\mathcal{D}_n|^2] = 0,$$

which follows from

$$n^H \mathcal{D}_n = \int_{\mathbb{R}} f(y) n^H \left(L_t(\frac{y}{n^H} + \lambda) - L_t(\lambda) - \frac{y}{n^H} L_t'(\lambda) \right) dy.$$

Open problems:

- Tightness in the case $H \geq \frac{1}{3}$.
- d-dimensional fBm:
  (i) For $\frac{1}{d+2} \leq H < \frac{1}{d}$ we expect convergence in law.
  (ii) For $H < \frac{1}{d+2}$ we expect convergence in $L^2(\Omega)$ to some derivatives of the local time.

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