Reflected SDEs in non-smooth time-dependent domains. Applications to PDEs with Neumann boundary condition.

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International Seminar on SDEs and Related Topics.

Outline

Outline

Reflected SDEs.

Framework:

- $d \geqslant 1$, $D' \subset \mathbb{R}^{1+d}$ bounded open connected.
- For $T > 0$, $D = D' \cap \big([0,T] \times \mathbb{R}^d\big)$.
- Slices $D_t = \{x : (t,x) \in D\} \neq \emptyset$.

Reflected stochastic differential equation: for all $0 \leqslant t \leqslant T$,

$$\begin{cases} X_t = X_0 + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dW_r + \Lambda_t, \\ X_t \in \overline{D_t}, \\ \Lambda_t = \int_0^t \gamma_r d|\Lambda|_r, \end{cases}$$

where

- $|\Lambda|$ = total variation of Λ,
- γ_r = direction of the reflection.

Literature (a non-exhaustive review).

Time-independent domains ($D_t = D$, $\forall t$) $\rightsquigarrow$ large literature

- Tanaka (1979) SDEs with reflecting boundary condition in convex regions.
- Lions and Sznitman (1984) SDEs with reflecting boundary conditions.
- Saisho (1987) SDEs for multi-dimensional domain with reflecting boundary.
- Costantini (1992) The Skorohod oblique reflection problem in domains with corners and application to SDEs.
- Dupuis and Ishii (2008) SDEs with oblique reflections on nonsmooth domains.
- Pardoux and Rascanu (2014) SDEs, BSDEs, PDEs. Chapter 4.

Literature (a non-exhaustive review).

Time-dependent domains

- Burdzy, Chen & Sylvester (2004) The heat equation and reflected Brownian motion in time-dependent domains.
- Costantini, Gobet & El Karoui (2006) Boundary sensitivities for diffusion processes in time dependent domains.
- Nyström and Önskog (2010). The Skorohod oblique reflection problem in time-dependent domains.
- Lundström and Önskog (2019). Stochastic and partial differential equations on nonsmooth time-dependent domains.

Two remarks:

- Smooth domains or smooth directions of reflection ($\gamma(s,x) \in \mathcal{C}^{1,2}$).
- Weak solution for the reflected SDE.

A different approach.

In our case:

- Non-smooth time-dependent domain, but with convex slides.
- Normal reflection.
- Existence of a strong solution as limit of solutions of penalized SDEs.
- Intermediate step: regularization of the domains.

Penalization for time-independent domains:

- Lions, Menaldi & Sznitman (1981) Construction de processus de diffusion réfléchis par pénalisation du domaine,.
- Bahlali, Maticiuc & Zalinescu (2013) Penalization method for a nonlinear Neumann PDE via weak solution of reflected SDEs.
- Ren and Wu (2019) Probabilistic approach for nonlinear partial differential equations and stochastic partial differential equations with Neumann boundary conditions.

Outline

Probabilistic setting.

Given:

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space,
- a k-dimensional Brownian motion $W = (W_t)_{0 \leqslant t \leqslant T}$,
- $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant T}$: completed filtration generated by W,
- $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times k}$ continuous

Assumptions:

(C1) Lipschitz continuity w.r.t. x:

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leqslant C|x - x'|.$$

(C2) Linear growth:

$$|b(t, x)| + |\sigma(t, x)| \leqslant C(1 + |x|).$$

Time regularity of the domain.

Framework: $d \geqslant 1$, $T > 0$, $D' \subset \mathbb{R}^{1+d}$ bounded open connected subset.

$$D = D' \cap \big([0, T] \times \mathbb{R}^d\big), \quad D_t = \{x : (t, x) \in D\} \neq \emptyset \quad \text{and convex.}$$

$N(t, x)$: cone of unit inward normal vectors at a boundary point $x \in \partial D_t$.

Distance to the domain:

$$d(t, x) := \inf_{y \in D_t} |y - x| = |x - \pi(t, x)|, \quad \forall t \in [0, T], \forall x \in \mathbb{R}^d.$$

Assumptions:

(C3) Reference point $(P_t, 0 \leqslant t \leqslant T)$ (Itô process with bounded parameters b^P, σ^P)

$$\forall t \in [0, T], \quad B(P_t, r_t) \subset D_t, \qquad \inf_{t \in [0,T]} r_t \geq R_\star > 0.$$

(C4) Time regularity: $d(\cdot, x) \in \mathcal{W}^{1,p}([0, T], [0, \infty))$, with $p > 2$, uniformly in x.

About the condition (C3).

- D' connected open set of $\mathbb{R}^{1+d}$: existence of a continuous path P s.t. $P_t \in D_t$.
- D_t convex and bounded:
  - P_t = centroid of D_t
  - $t \mapsto P_t$ continuous due to the time regularity condition (C4) and the fact that vol $(D_t) \neq 0$. See Schneider (Convex Bodies).
- Reynolds' derivation theorems applied to the centroids $\rightsquigarrow t \mapsto P_t$ of class $\mathcal{C}^1$. But requires more regularity conditions.

Key properties:

- Regularity of the trajectory of P: $P_t = P_0 + \int_0^t b_s^P ds + \int_0^t \sigma_s^P dW_s$.
- Condition $R_\star > 0$.

Remark: if

$$B(P_0, r_0) \subset \bigcap_{t \in [0,T]} D_t,$$

then $P_t = P_0$ and $r_t = r_0$ for all t.

About the condition (C4).

Morrey's inequality: existence of $1/2 < \alpha < 1$ and $K \in [0,\infty)$ s.t.

$$|d(s,x) - d(t,x)| \leqslant K|s-t|^{\alpha}.$$

$\rightsquigarrow$ Hölder regularity.

For

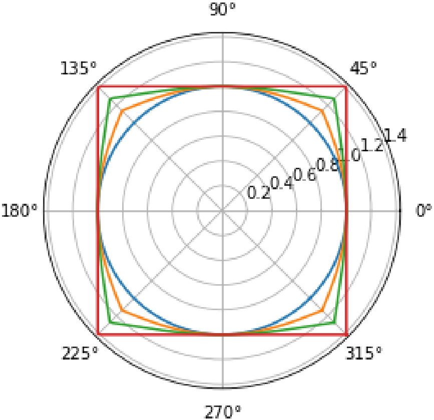
$$\ell(r) = \sup_{\substack{s,t\in[0,T] \\ |s-t|\leqslant r}} \sup_{x\in\overline{D_s}} \inf_{y\in\overline{D_t}} |x-y| = \sup_{\substack{s,t\in[0,T] \\ |s-t|\leqslant r}} \sup_{x\in\overline{D_s}} d(x,D_t),$$

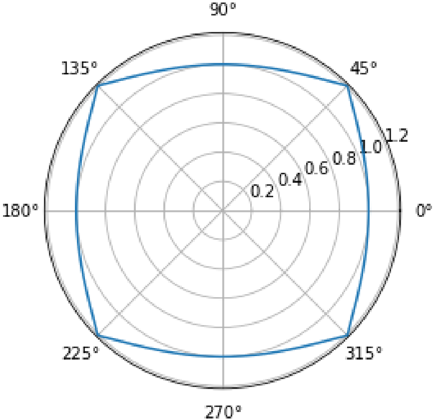
Hölder condition equivalent to: $\ell(r) \leqslant Kr^{\alpha}$.

Stronger condition (C4): $d(\cdot,x)$ is Lipschitz continuous, uniformly w.r.t. x ($p=\infty$). Iff $\ell(r) \leq Kr$.

Example of a family of convex subsets of $\mathbb{R}^2$.

$$D_0 = B(0,1), \quad D_1 = \{(x,y), |x| \leq 1, \ |y| \leq 1\}.$$





Here: $P_t = 0$, $\ell(r) \leqslant (\sqrt{2} - 1)r$.

Reflected SDEs.

Definition

Let D be a time-dependent domain. For $x \in \overline{D}_0$, a strong solution of the reflected SDE in D with initial condition x and normal reflection, is an $\mathbb{F}$-adapted stochastic process $(X_t, \Lambda_t)_{t\in[0,T]}$ s.t. $\mathbb{P}$-almost surely, for all $t \in [0, T]$,

$$\begin{cases} X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \Lambda_t; \\ X_t \in \overline{D_t}; \\ |\Lambda|_t = \int_0^t \mathbf{1}_{\{X_r \in \partial D_r\}} d|\Lambda|_r < \infty; \\ \Lambda_t = \int_0^t \gamma_r d|\Lambda|_r, \ \gamma_r \in N(r, X_r) \ d|\Lambda| - \text{a.s., when } X_r \in \partial D_r. \end{cases}$$

$N(t, x)$: cone of unit inward normal vectors at a boundary point $x \in \partial D_t$.

Penalized SDEs and main result.

For all $n \geqslant 1$, $\forall t \in [0, T]$,

$$X_t^n = x + \int_0^t b(r, X_r^n) dr + \int_0^t \sigma(r, X_r^n) dW_r - n \int_0^t (X_r^n - \pi(r, X_r^n)) dr,$$
$$= x + \int_0^t b(r, X_r^n) dr + \int_0^t \sigma(r, X_r^n) dW_r + \Lambda_t^n$$

Total variation of Λ^n

$$|\Lambda^n|_t = n \int_0^t d(r, X_r^n) dr.$$

Theorem

Under our setting, the penalization scheme converges to a pair of processes (X, Λ). The limiting pair (X, Λ) provides the unique solution to the normally reflected SDE with initial condition $x \in \overline{D}_0$ and coefficients b and σ.

Scheme for the proof.

Step 1, a priori estimate:

$$\sup_{n\geqslant 1} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X_t^n|^{2q} + \sup_{0\leqslant t\leqslant T} |\Lambda^n|_t^q\right] < \infty.$$

Apply Itô's formula to $|X_t^n - P_t|^2$ (condition (`C3`)).

Step 2, uniform control of the distance: there exists C s.t.

$$\forall n \geqslant 1, \quad \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} d(t, X_t^n)^p\right] \leqslant \frac{C}{n^{\frac{p-2}{2}}},$$

and

$$\forall n \geqslant 1, \quad \mathbb{E}\left[\int_0^T d(t, X_t^n)^p dt\right] \leqslant \frac{C}{n^{\frac{p}{2}}}.$$

Regularization of the domain (see later).

Scheme for the proof.

Step 3, Cauchy sequence:

$$\lim_{m,n\to\infty} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X_t^n - X_t^m|^q\right] = 0 \quad \lim_{m,n\to\infty} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\Lambda_t^n - \Lambda_t^m|^q\right] = 0.$$

Itô's formula:

$$|X_t^n - X_t^m|^2 \leqslant M_t^{m,n} + H_t^{m,n} + \int_0^t c_{b,\sigma}|X_s^n - X_s^m|^2 ds,$$

with $M^{m,n}$ local martingale and

$$\begin{aligned} H_t^{m,n} =& 2(m+n)\int_0^t d(s,X_s^n)d(s,X_s^m)ds \\ & \leqslant 4 \sup_{0\leqslant t\leqslant T} d(t,X_t^n) \times m \int_0^t d(s,X_s^m)ds = 4\left(\sup_{0\leqslant t\leqslant T} d(t,X_t^n)\right)|\Lambda^m|_t \end{aligned}$$

Scheme for the proof.

Step 3, Cauchy sequence:

$$\lim_{m,n\to\infty} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |X^n_t - X^m_t|^q\right] = 0 \quad \lim_{m,n\to\infty} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\Lambda^n_t - \Lambda^m_t|^q\right] = 0.$$

Define (X, Λ) as the limit:

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \Lambda_t.$$

Step 4, minimality condition: (adaptation of Gégout-Petit & Pardoux)

$$|\Lambda|_t = \int_0^t \mathbf{1}_{\{X_r \in \partial D_r\}} d|\Lambda|_r,$$

and existence of a vector field γ s.t. $\gamma_s \in N(s, X_s)$ for $d|\Lambda|$-almost $s \in [0, T]$ with

$$\Lambda_t = \int_0^t \gamma_s d|\Lambda|_s.$$

Step 5, uniqueness.

Outline

Step 2, uniform control of the distance.

There exists C s.t.

$$\forall n \geqslant 1, \quad \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} d(t, X_t^n)^p\right] \leqslant \frac{C}{n^{\frac{p-2}{2}}},$$

and

$$\forall n \geqslant 1, \quad \mathbb{E}\left[\int_0^T d(t, X_t^n)^p dt\right] \leqslant \frac{C}{n^{\frac{p}{2}}}.$$

- Idea of the proof: apply Itô's formula with X^n and

$$\varphi(t, y) := (d(t, y))^p = |y - \pi(t, y)|^p.$$

- Problem: d and φ not regular !
- Tool: replace d by a smooth approximation d_ϵ and establish a uniform control of $(d_\epsilon(t, X_t^n))^p$.

Approximation of the domain: the result.

Proposition

Let $\epsilon > 0$, there exists a smooth time-dependent domain D^ϵ s.t.:

(i) *D^ϵ_t is an open convex and ∂D^ϵ_t is $\mathcal{C}^\infty$-smooth for every $t \in [0, T]$.*

(ii) *The projection map $\pi_\epsilon(t, \cdot)$ onto D^ϵ_t, is 1-Lipschitz and differentiable with*

$$\|\partial_x \pi_\epsilon(t, x)\| \leqslant 1.$$

(iii) *The map $(t, x) \mapsto d_\epsilon(t, x) := d(x, D^\epsilon_t)$ is of class $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$ and for any x, $t \mapsto d_\epsilon(t, x)$ belongs to $\mathcal{W}^{1,p}([0, T], [0, \infty))$, uniformly w.r.t. x and ϵ.*

(iv) *D_t and D^ϵ_t satisfy:*

$$h(D_t, D^\epsilon_t) \leqslant \epsilon, \ \forall t \in [0, T].$$

Hausdorff distance between F and G:

$$h(F, G) = \max(\sup\{d(y, F) : y \in G\}, \sup\{d(y, G) : y \in F\}).$$

Announced in Nyström and Olofsson (2016) Reflected BSDE of Wiener-Poisson type in time-dependent domains.

Approximation of the domain: the proof.

Regularization of the distance: define the smooth mollification g_δ of d

$$g_\delta(t,y) := \phi_\delta * d(t,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \phi_\delta(t-s, y-x) d(s,x) dx ds.$$

Smooth function with:

- $x \mapsto g_\delta(t,x)$ convex ;
- $|g_\delta(t,x) - d(t,x)| \leqslant \max(K\delta^\alpha, \delta)$;
- $\|\partial_t g_\delta(\cdot,x)\|_{L^p} \leqslant \|\partial_t d(\cdot,x)\|_{L^p}$.

New domain: for $\eta > 0$

$$D^{\delta,\eta} := \left\{(t,x) \in [0,T] \times \mathbb{R}^d \ : \ g_\delta(t,x) < \eta\right\}.$$

- Open and bounded in $\mathbb{R}^{1+d}$.
- $\partial D^{\delta,\eta} = \{(t,x) : g_\delta(t,x) = \eta\}$ smooth hypersurface whenever η regular value of g_δ.

Approximation of the domain: the proof.

Regularization of the distance: define the smooth mollification g_δ of d

$$g_\delta(t,y) := \phi_\delta * d(t,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \phi_\delta(t-s, y-x) d(s,x) dx ds.$$

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- $\partial D^{\delta,\eta} = \{(t,x) : g_\delta(t,x) = \eta\}$ smooth hypersurface whenever η regular value of g_δ.

Regularity of the slices $D_t^{\delta,\eta} = \{x, \ (t,x) \in D^{\delta,\eta}\}$

- Convexity: orthogonal projection of x on $D_t^{\delta,\eta}$ = $\pi_{\delta,\eta}(t,x)$ and

$$x - \pi_{\delta,\eta}(t,x) = d_{\delta,\eta}(t,x) \vec{\mathbf{n}}_t^{\delta,\eta}(\pi_{\delta,\eta}(t,x)).$$

Approximation of the domain: the proof.

Regularization of the distance: define the smooth mollification g_δ of d

$$g_\delta(t,y) := \phi_\delta * d(t,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \phi_\delta(t-s, y-x) d(s,x) dx ds.$$

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- Convexity: orthogonal projection of x on $D_t^{\delta,\eta}$ = $\pi_{\delta,\eta}(t,x)$ and

$$x - \pi_{\delta,\eta}(t,x) = d_{\delta,\eta}(t,x) \vec{\mathbf{n}}_t^{\delta,\eta}(\pi_{\delta,\eta}(t,x)).$$

Lemma

There exists a constant $\delta^* = \delta^*(\eta) > 0$ such that if $0 < \delta \leqslant \delta^*$, then every point $(t,x) \in [0,T] \times \mathbb{R}^d$ satisfying $g_\delta(t,x) = \eta$ verifies $|\nabla_x g_\delta(t,x)| \geqslant 1/2$.

- The slice $D_t^{\delta,\eta} = \{x \in \mathbb{R}^d, \ (t,x) \in D^{\delta,\eta}\}$ has a smooth boundary.
- $d_{\delta,\eta}(t,\cdot)$ is smooth.

Approximation of the domain: the proof.

Regularization of the distance: define the smooth mollification g_δ of d

$$g_\delta(t,y) := \phi_\delta * d(t,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \phi_\delta(t-s, y-x) d(s,x) dx ds.$$

New domain: for $\eta > 0$

$$D^{\delta,\eta} := \left\{ (t,x) \in [0,T] \times \mathbb{R}^d \ : \ g_\delta(t,x) < \eta \right\}.$$

Regularity of the slices $D_t^{\delta,\eta} = \{x, \ (t,x) \in D^{\delta,\eta}\}$

- $d_{\delta,\eta}(t,\cdot)$ is smooth.

Time regularity: $t \mapsto d_{\delta,\eta}(t,x)$ is of class $\mathcal{C}^1$ with

$$\partial_t d_{\delta,\eta}(t,y) = \begin{cases} \dfrac{\partial_t g_\delta(t,\pi(t,y))}{|\nabla_x g_\delta(t,\pi(t,y))|} & \text{if } y \notin \overline{D_t} \\ 0 & \text{if } y \in \overline{D_t}. \end{cases}$$

Hausdorff distance: $h(D_t, D_t^{\delta,\eta}) \leqslant 2\eta$.

Using the approximation of the domain.

$(t,x) \mapsto d_\epsilon(t,x) := d(x, D_t^\epsilon)$ is of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$

- Itô formula to $\varphi_\epsilon(t,y) := (d_\epsilon(t,y))^p = |y - \pi_\epsilon(t,y)|^p$.

$$\varphi_\epsilon(t, X_t^n) = \varphi_\epsilon(0, X_0^n) + \int_0^t \partial_t \varphi_\epsilon(s, X_s^n) ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), b(s, X_s^n) \rangle ds$$
$$+ \frac{1}{2} \int_0^t \sigma^\top(s, X_s^n) \partial_{xx} \varphi_\epsilon(s, X_s^n) \sigma(s, X_s^n) ds$$
$$- n \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), X_s^n - \pi(s, X_s^n) \rangle ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), \sigma(s, X_s^n) dW_s \rangle.$$

Young's inequality:

$$\int_0^t \partial_t \varphi_\epsilon(s, X_s^n) ds \leqslant c_1 n \int_0^t d_\epsilon(s, X_s^n)^p ds + \frac{C_1}{n^{p-1}} \int_0^t |(\partial_t d_\epsilon(s, X_s^n))^+|^p ds.$$

Using the approximation of the domain.

$(t,x) \mapsto d_\epsilon(t,x) := d(x, D^\epsilon_t)$ is of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$

- Itô formula to $\varphi_\epsilon(t,y) := (d_\epsilon(t,y))^p = |y - \pi_\epsilon(t,y)|^p$.

$$
\begin{aligned}
\varphi_\epsilon(t, X^n_t) &= \varphi_\epsilon(0, X^n_0) + \int_0^t \partial_t \varphi_\epsilon(s, X^n_s) ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), b(s, X^n_s) \rangle ds \\
&\quad + \frac{1}{2} \int_0^t \sigma^\top(s, X^n_s) \partial_{xx} \varphi_\epsilon(s, X^n_s) \sigma(s, X^n_s) ds \\
&\quad - n \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), X^n_s - \pi(s, X^n_s) \rangle ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), \sigma(s, X^n_s) dW_s \rangle.
\end{aligned}
$$

Similarly

$$
\int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), b(s, X^n_s) \rangle ds \leqslant c_2 n \int_0^t d_\epsilon(s, X^n_s)^p ds + \frac{C_2}{n^{p-1}} \int_0^t |b(s, X^n_s)|^p ds,
$$

$$
\frac{1}{2} \int_0^t (\sigma^\top \partial_{xx} \varphi_\epsilon \sigma)(s, X^n_s) ds \leqslant c_3 n \int_0^t d_\epsilon(s, X^n_s)^p ds + \frac{C_3}{n^{\frac{p-2}{2}}} \int_0^t |\sigma(s, X^n_s)|^p ds.
$$

Using the approximation of the domain.

$(t,x) \mapsto d_\epsilon(t,x) := d(x, D_t^\epsilon)$ is of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$

- Itô formula to $\varphi_\epsilon(t,y) := (d_\epsilon(t,y))^p = |y - \pi_\epsilon(t,y)|^p$.

$$\begin{aligned}\varphi_\epsilon(t, X_t^n) &= \varphi_\epsilon(0, X_0^n) + \int_0^t \partial_t \varphi_\epsilon(s, X_s^n) ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), b(s, X_s^n) \rangle ds \\ &+ \frac{1}{2} \int_0^t \sigma^\top(s, X_s^n) \partial_{xx} \varphi_\epsilon(s, X_s^n) \sigma(s, X_s^n) ds \\ &- n \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), X_s^n - \pi(s, X_s^n) \rangle ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), \sigma(s, X_s^n) dW_s \rangle.\end{aligned}$$

Lemma

There exists a constant $c \geqslant 0$, s.t.

(i) $|\pi(t,y) - \pi_\epsilon(t,y)| \leqslant c \min(\sqrt{\epsilon^2 + \epsilon d_\epsilon(t,y)}; \sqrt{\epsilon}(1 + d_\epsilon(t,y)); \sqrt{\epsilon^2 + \epsilon d(t,y)})$,

(ii) $|\pi(t,y) - \pi_\epsilon(t,y)| \leqslant c \sqrt{\epsilon} \sqrt{d_\epsilon(t,y)}$ *whenever* $d_\epsilon(t,y) > \epsilon$.

Using the approximation of the domain.

$(t,x) \mapsto d_\epsilon(t,x) := d(x, D^\epsilon_t)$ is of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$

- Itô formula to $\varphi_\epsilon(t,y) := (d_\epsilon(t,y))^p = |y - \pi_\epsilon(t,y)|^p$.

$$
\begin{aligned}
\varphi_\epsilon(t, X^n_t) &= \varphi_\epsilon(0, X^n_0) + \int_0^t \partial_t \varphi_\epsilon(s, X^n_s) ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), b(s, X^n_s) \rangle ds \\
&+ \frac{1}{2} \int_0^t \sigma^\top(s, X^n_s) \partial_{xx} \varphi_\epsilon(s, X^n_s) \sigma(s, X^n_s) ds \\
&- n \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), X^n_s - \pi(s, X^n_s) \rangle ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), \sigma(s, X^n_s) dW_s \rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&- n \int_0^t \langle \partial_x \varphi_\epsilon(s, X^n_s), X^n_s - \pi_\epsilon(s, X^n_s) + \pi_\epsilon(s, X^n_s) - \pi(s, X^n_s) \rangle ds \\
&\quad \leqslant -pn \int_0^t \varphi_\epsilon(s, X^n_s) ds + c_4 n \int_0^t d_\epsilon(s, X^n_s)^p ds + C_4 n^{\frac{1}{2p}} \epsilon^p + C_5 n \epsilon^{p - \frac{1}{2}}.
\end{aligned}
$$

Using the approximation of the domain.

$(t,x) \mapsto d_\epsilon(t,x) := d(x, D_t^\epsilon)$ is of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$

- Itô formula to $\varphi_\epsilon(t,y) := (d_\epsilon(t,y))^p = |y - \pi_\epsilon(t,y)|^p$.

$$\varphi_\epsilon(t, X_t^n) = \varphi_\epsilon(0, X_0^n) + \int_0^t \partial_t \varphi_\epsilon(s, X_s^n) ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), b(s, X_s^n) \rangle ds$$
$$+ \frac{1}{2} \int_0^t \sigma^\top(s, X_s^n) \partial_{xx} \varphi_\epsilon(s, X_s^n) \sigma(s, X_s^n) ds$$
$$- n \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), X_s^n - \pi(s, X_s^n) \rangle ds + \int_0^t \langle \partial_x \varphi_\epsilon(s, X_s^n), \sigma(s, X_s^n) dW_s \rangle.$$

Burkholder-Davis-Gundy inequality and Gronwall's lemma:

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} d_\epsilon(t, X_t^n)^p \right] \leqslant (c^p + cn^{\frac{1}{2p}}) \epsilon^p + cn \epsilon^{p - \frac{1}{2}} + \frac{C}{n^{\frac{p-2}{2}}}.$$

Hausdorff distance $h(D_t, D_t^\epsilon) < \epsilon$.

Outline

Our context.

Semilinear PDE:

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma^\top(t,x)\partial_x u(t,x)) = 0, \quad (t,x) \in D^{\mathrm{o}},$$

with terminal Cauchy condition $u(T,x) = h(x)$ for $x \in \overline{D_T}$,
and Neumann boundary condition

$$\frac{\partial u}{\partial \vec{\mathbf{n}}}(t,x) + \psi(t,x,u(t,x)) = 0, \quad (t,x) \in \partial D,$$

where:

- $D^{\mathrm{o}} = D' \cap \left([0,T) \times \mathbb{R}^d\right)$
- $\partial D = \left(\overline{D'} \setminus D'\right) \cap \left([0,T) \times \mathbb{R}^d\right)$
- and
$$\mathcal{L}\varphi(t,x) = \frac{1}{2}\operatorname{Tr}((\sigma\sigma^\top)(t,x)\partial^2_{xx}\varphi(x)) + b^\top(t,x)\partial_x\varphi(x).$$

Neumann boundary condition.

Regular case: unique normal at each boundary point thus

$$\frac{\partial u}{\partial \vec{\mathbf{n}}}(t,x) = \langle \partial_x u(t,x), \vec{\mathbf{n}}(t,x) \rangle.$$

Non smooth case: multiple inward normals at a given point. Define:

$$N^-_{t,x}(q) := \inf_{\vec{\mathbf{n}} \in N(t,x)} \langle q, -\vec{\mathbf{n}} \rangle, \quad N^+_{t,x}(q) := \sup_{\vec{\mathbf{n}} \in N(t,x)} \langle q, -\vec{\mathbf{n}} \rangle.$$

Viscosity subsolution $u \in USC$ if

- $u(T,x) \leqslant h(x)$ for any $x \in \overline{D}_T$
- for any $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$ s.t. $(t,x) \in \widetilde{D}$ local maximum of $u - \phi$
  - Either $(t,x) \in D^o$ and

$$-\partial_t \varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,u(t,x),\sigma^\top(t,x)\partial_x \varphi(t,x)) \leqslant 0,$$

  - Or $(t,x) \in \partial D$ and

$$\Big[-\partial_t \varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,u(t,x),\sigma^\top(t,x)\partial_x \varphi(t,x)) \Big]$$

$$\wedge \Big[N^-_{t,x}(\partial_x \varphi(t,x)) - \psi(t,x,u(t,x)) \Big] \leqslant 0.$$

Neumann boundary condition.

Regular case: unique normal at each boundary point thus

$$\frac{\partial u}{\partial \vec{\mathbf{n}}}(t,x) = \langle \partial_x u(t,x), \vec{\mathbf{n}}(t,x) \rangle.$$

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  - Either $(t,x) \in D^{\mathrm{o}}$ and

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  - Or $(t,x) \in \partial D$ and

$$\Big[-\partial_t \varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,u(t,x),\sigma^\top(t,x)\partial_x\varphi(t,x)) \Big] \wedge \Big[N^-_{t,x}(\partial_x \varphi(t,x)) - \psi(t,x,u(t,x)) \Big] \leqslant 0.$$

Neumann boundary condition.

Regular case: unique normal at each boundary point thus

$$\frac{\partial u}{\partial \vec{\mathbf{n}}}(t,x) = \langle \partial_x u(t,x), \vec{\mathbf{n}}(t,x)\rangle.$$

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- $u(T,x) \leqslant h(x)$ for any $x \in \overline{D}_T$
- for any $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$ s.t. $(t,x) \in \widetilde{D}$ local maximum of $u - \phi$
  - Either $(t,x) \in D^{\mathrm{o}}$ and
  $$-\partial_t \varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,u(t,x),\sigma^\top(t,x)\partial_x\varphi(t,x)) \leqslant 0,$$
  - Or $(t,x) \in \partial D$ and
  $$\Big[- \partial_t \varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,u(t,x),\sigma^\top(t,x)\partial_x\varphi(t,x)) \Big] \wedge \Big[N^-_{t,x}(\partial_x \varphi(t,x)) - \psi(t,x,u(t,x)) \Big] \leqslant 0.$$

Assumptions.

(H1) h is continuous and is of polynomial growth.

(H2) (i) $(t,x) \longmapsto f(t,x,y,z)$ and $(t,x) \longmapsto \psi(t,x,y)$ are continuous, uniformly with respect to (y,z) and y respectively.

(ii) $(t,x) \longmapsto f(t,x,0,0)$ and $(t,x) \longmapsto \psi(t,x,0)$ are of polynomial growth.

(iii) f is Lipschitz continuous with respect to (y,z):

$$|f(t,x,y,z) - f(t,x,y',z')| \leqslant C_L(|y-y'| + |z-z'|).$$

(iv) There exists $\beta \leqslant 0$ such that

$$(y-y')(\psi(t,x,y) - \psi(t,x,y')) \leqslant \beta \mid y-y' \mid^2 .$$

For some continuous and non-decreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$

$$|\psi(t,x,y) - \psi(t,x,0)| \leqslant \Phi(|y|).$$

Forward backward system.

For $(t, x) \in \overline{D}$ fixed, reflected SDE:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r + \Lambda_s^{t,x}; \\ X_s^{t,x} \in \overline{D_s}, \\ |\Lambda^{t,x}|_s = \int_t^s \chi_{\{X_r^{t,x} \in \partial D_r\}} d|\Lambda^{t,x}|_r, \\ \Lambda_s^{t,x} = \int_t^s \gamma_r^{t,x} d|\Lambda^{t,x}|_r, \end{cases}$$

Generalized BSDE:

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T \psi(r, X_r^{t,x}, Y_r^{t,x})d|\Lambda^{t,x}|_r - \int_s^T Z_r^{t,x} dW_r$$

Existence and continuity result.

Pardoux & Rascanu: $(Y^{t,x}, Z^{t,x})$ exists and for any $q > 1$

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant T} \mid Y_s^{t,x} \mid^q + \left(\int_t^T ||Z_r^{t,x}||^2 dr\right)^q\right] < \infty.$$

The mapping $(t,x) \in D \mapsto (X^{t,x}, \Lambda^{t,x})$ is continuous and there exists a constant C such that for all (t,x) and (t',x') in D:

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant T} \mid X_s^{t',x'} - X_s^{t,x} \mid^2 + \sup_{0\leqslant s\leqslant T} \mid \Lambda_s^{t',x'} - \Lambda_s^{t,x} \mid^2\right] \leqslant C|x-x'|^2 + C|t-t'|^{\frac{1}{2}}.$$

Remark

We do not know if the total variation $(t,x) \in D \mapsto |\Lambda^{t,x}|_\cdot$:

- is continuous ?
- has an exponential moment: $\mathbb{E} e^{\mu |\Lambda^{t,x}|_T} < \infty$?

except if the domains are smooth.

Our result.

Approximation:

$$\begin{cases} \partial_t u^n(t,x) + \mathcal{L}_n u^n(t,x) + f(t,x,u^n(t,x),\sigma^\top(t,x)\partial_x u^n(t,x)) = 0, \\ u^n(T,x) = h(x), \end{cases}$$

with

$$\mathcal{L}^n \varphi(t,x) = \frac{1}{2} \operatorname{Tr}\left(\sigma(t,x)\sigma(t,x)^\top \partial^2_{xx}\varphi(t,x)\right) \\ + \Big(b(t,x) - n\big(x - \pi(t,x)\big)\Big)^\top \partial_x \varphi(t,x).$$

Theorem

When $\psi = 0$, the function $u : \widetilde{D} \mapsto \mathbb{R}$ defined by $u(t,x) = Y_t^{t,x}$ is a continuous viscosity solution of the PDE with Neumann boundary condition. Furthermore, for all $(t,x) \in \widetilde{D}$,

$$\lim_{n\to\infty} u^n(t,x) = u(t,x).$$

Idea of the proof.

Step 1, continuity: use continuity result for standard BSDEs.
Step 2, viscosity solution: based on

Lemma

The operator $(t,x) \mapsto N^-_{t,x}(q)$ (resp. $(t,x) \mapsto N^+_{t,x}(q)$) is lower semicontinuous (resp. upper semicontinuous).

Step 3, approximation: define

$$\begin{cases} X^{t,x,n}_s = x + \int_t^s b(r, X^{t,x,n}_r)dr + \int_t^s \sigma(r, X^{t,x,n}_r)dW_r + \Lambda^{t,x,n}_s; \\ \Lambda^{t,x,n}_s = -n \int_t^s (X^{t,x,n}_r - \pi(r, X^{t,x,n}_r))dr, \\ |\Lambda^{t,x,n}|_s = \int_0^s n|X^{t,x,n}_r - \pi(r, X^{t,x,n}_r)|dr. \end{cases}$$

and

$$Y^{t,x,n}_s = h(X^{t,x,n}_T) + \int_s^T f(r, X^{t,x,n}_r, Y^{t,x,n}_r, Z^{t,x,n}_r)dr - \int_s^T Z^{t,x,n}_r dW_r.$$

Then $u^n(t,x) = Y^{t,x,n}_t$ and stability result for standard BSDEs.

Conclusion.

Extension when $\psi \neq 0$, for PDE in smooth time-dependent domain:

- Continuity of $(t, x) \mapsto |\Lambda^{t,x}|$.
- $\mathbb{E}[e^{\mu |\Lambda^{t,x}|_s}] \leqslant C(\mu, s)$.

Paper available on HAL (hal-04118563).

In progress:

- Similar results for PDE for non-smooth domains when $\psi \neq 0$?
- Non-convex slices ?

Thank you for your attention !