Strong approximation of the CIR process: A never ending story?

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Cox-Ingersoll-Ross process

scalar stochastic differential equation

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{|X_t|}dW_t, \quad t \in [0,T], \qquad X_0 = x_0$$

with

(i) $\kappa, \theta \geq 0$, $\sigma, x_0 > 0$,

(ii) Brownian motion $W = (W_t)_{t\in[0,T]}$

Prototype of a **square-root diffusion**; applications in biology, finance, chemistry, ...

Feller, Two Singular Diffusion Problems, 1951

Cox, Ingersoll, Ross, A Theory of the Term Structure of Interest Rates, 1985

$$dX_t = \kappa\,(\theta - X_t)\,dt + \sigma\sqrt{|X_t|}dW_t$$

The map

$$\mathbb{R} \ni x \mapsto \sqrt{|x|} \in [0,\infty)$$

is not Lipschitz continuous at $x = 0$

Thus: Standard theory not applicable

What's the problem? Gronwall fails!

$$|g(t)| \leq c_1 + c_2 \int_0^t \sqrt{|g(s)|}ds, \quad t \geq 0$$

$$\not\Rightarrow$$

$$|g(t)| \leq c_1 \exp(c_2 t), \quad t \geq 0$$

$$dX_t = \kappa\,(\theta - X_t)\,dt + \sigma\sqrt{|X_t|}dW_t$$

Existence of a unique strong solution $X = \Phi(W)$ by results of Skorokhod (1961) and Yamada, Watanabe (1971)

Moreover:

$$\mathbf{P}(X_t \geq 0,\ t \in [0, T]) = 1$$

and

$$\mathbf{P}(X_t > 0,\ t \in [0, T]) = 1 \quad \text{if} \quad \nu \geq 1$$

where

$$\nu = \frac{2\kappa\theta}{\sigma^2} \qquad\qquad \text{(Feller index)}$$

Thus $|X_t| = X_t^+ = X_t$

The numerical problem

Strong approximation of X_T by

$$\varphi_n(W_{T/n}, W_{2T/n}, \ldots, W_T)$$

with

$$\varphi_n : \mathbb{R}^n \to \mathbb{R} \qquad \text{measurable}$$

Error criterion (L^1-error):

$$e(\varphi_n) = \mathbf{E}|X_T - \varphi_n(W_{T/n}, W_{2T/n}, \ldots, W_T)|$$

Goals

- determine the behaviour of $e(\varphi_n)$ for particular φ_n
- determine the best possible error, i.e., behaviour of

$$e_{min}(n) = \inf_{\varphi_n} e(\varphi_n)$$

Contributions

Deelstra, Delbaen, 1998; Alfonsi, 2005; Higham, Mao, 2005; Alfonsi, 2005; Bossy, Diop, 2007; Berkaoui, Bossy, Diop, 2008; Gyöngy, Rásonyi, 2011; Dereich, N, Szpruch, 2012; Alfonsi, 2013; N, Szpruch, 2014; Hutzenthaler, Jentzen, Noll, 2014; Hefter, Herzwurm, 2017,2018; Chassagneux, Jacquier, Mihaylov, 2016; Bossy, Quinteros, 2018; Hefter, Herzwurm, Müller-Gronbach, 2019; Hefter, Jentzen, 2019; Cozma, Reisinger, 2020; Kelly, Lord, Maulana, 2022; Mickel, N, 2022+

Other approximation/simulation problems (not covered here): exact simulation, ε-strong approximation, weak approximation, ...

Part of numerics for SDEs under non-standard assumptions; recent breakthrough results[1] e.g. for piecewise Lipschitz drift by *Müller-Gronbach, Yaroslavtseva, 2020+; Dareiotis, Gerencsér, Lê, 2022+; Rauhögger, Müller-Gronbach, Yaroslavtseva, 2023+*

[1] Euler scheme & minimal error

Phase I: the Euler scheme

$\varphi_n^E(W_{T/n}, W_{2T/n}, \ldots, W_T) = x_n$ where

$$x_{k+1} = x_k + \kappa(\theta - x_k)\frac{T}{n} + \sigma\sqrt{|x_k|}\left(W_{\frac{(k+1)T}{n}} - W_{\frac{kT}{n}}\right), \quad k \geq 0$$

convergence with logarithmic rate (using Y-W approx of $|\cdot|$):

$$e(\varphi_n^E) \preceq \log(n)^{-1}$$

Deelstra, Delbaen, 1998; Higham, Mao, 2005; Lord et al. 2009; Gyöngy, Rásonyi, 2011; ...

Notation:

$$a_n \preceq b_n \;\Leftrightarrow\; \limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$$

General framework (Lord et al. 2009)

$$x_{k+1} = f_1(x_k) + \kappa(\theta - f_2(x_k))\frac{T}{n} + \sigma\sqrt{f_3(x_k)}\left(W_{\frac{(k+1)T}{n}} - W_{\frac{kT}{n}}\right)$$

with suitable functions f_i that are chosen from

$$\begin{aligned} \texttt{id} : \mathbb{R} \to \mathbb{R}, &\quad \texttt{id}(x) = x \\ \texttt{abs} : \mathbb{R} \to [0,\infty), &\quad \texttt{abs}(x) = x^+ \\ \texttt{sym} : \mathbb{R} \to [0,\infty), &\quad \texttt{sym}(x) = |x| \end{aligned}$$

Some schemes from the literature:

Scheme	$f_1(x)$	$f_2(x)$	$f_3(x)$
Absorption (AE)	x^+	x^+	x^+
Symmetrized (SE)	$\|x\|$	$\|x\|$	$\|x\|$
Higham and Mao (HM)	x	x	$\|x\|$
Partial Truncation (PTE)	x	x	x^+
Full Truncation (FTE)	x	x^+	x^+

We will come back to the Euler scheme at the end of the talk!

Phase II: Lamperti-backward Euler scheme

For $\nu \geq 1$ SDE for $U_t = \sqrt{X_t}$ given by

$$dU_t = a(U_t)dt + \frac{\sigma}{2}dW_t$$

with

$$a(x) = \left(\frac{\kappa\theta}{2} - \frac{\sigma^2}{8}\right)\frac{1}{x} - \frac{\kappa}{2}x$$

Now: a is one-sided Lipschitz, so use backward Euler scheme

$\varphi_n^{LbE}(W_{T/n}, W_{2T/n}, \ldots, W_T) = u_n^2$ where

$$\boxed{u_{k+1} = u_k + a(u_{k+1})\frac{T}{n} + \frac{\sigma}{2}\left(W_{\frac{(k+1)T}{n}} - W_{\frac{kT}{n}}\right), \quad k \geq 0}$$

Theorem A Alfonsi 2013; N, Szpruch 2014

If $\nu > 2$ then

$$e(\varphi_n^{LbE}) \preceq n^{-1}$$

Remarks

(i) implicit equation is explicitly solvable

(ii) polynomial error bounds for LbE also available for $\nu \geq 1/2$ *(Dereich, N, Szpruch, 2012; Hutzenthaler, Jentzen, Noll, 2014)*

(iii) Feller index ν controls probability CIR takes values close to zero

What if ν is small ?

Phase II: truncated Milstein scheme

$\varphi_n^{tM}(W_{T/n}, W_{2T/n}, \ldots, W_T) = x_n$ where

$$x_{k+1} = \Theta\left(x_k, \frac{T}{n}, W_{\frac{(k+1)T}{n}} - W_{\frac{kT}{n}}\right), \qquad k \geq 0$$

with

$$\Theta(x,t,w) = \left(\left|\max\left(\sqrt{\sigma^2/4 \cdot t}, \sqrt{\max(\sigma^2/4 \cdot t, x)} + \sigma/2 \cdot w\right)\right|^2 + \left(\kappa\theta - \sigma^2/4 - \kappa \cdot x\right) \cdot t\right)^+$$

Idea: add

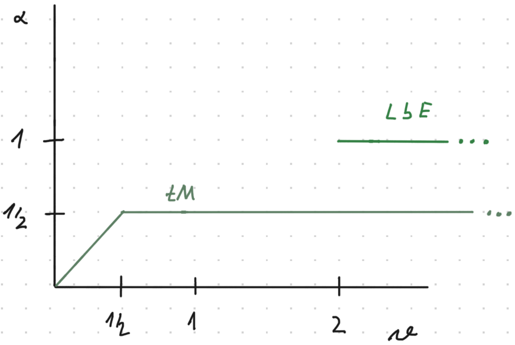
$$\frac{\sigma^2}{4}\left(\left(W_{\frac{(k+1)T}{n}} - W_{\frac{kT}{n}}\right)^2 - \frac{T}{n}\right)$$

to Euler scheme and truncate small values

Theorem B Hefter, Herzwurm, 2018

$$e(\varphi_n^{tM}) \preceq n^{-\min\{\nu,\frac{1}{2}\}+\varepsilon}$$

for all $\varepsilon > 0$



Phase III: minimal error

Theorem C Hefter, Jentzen; Hefter, Herzwurm, Müller-Gronbach, 2019

There exists $c = c_{\kappa,\theta,\sigma,x_0,T} > 0$ such that

$$e_{min}(n) \geq c \cdot n^{-\min\{1,\nu\}}$$

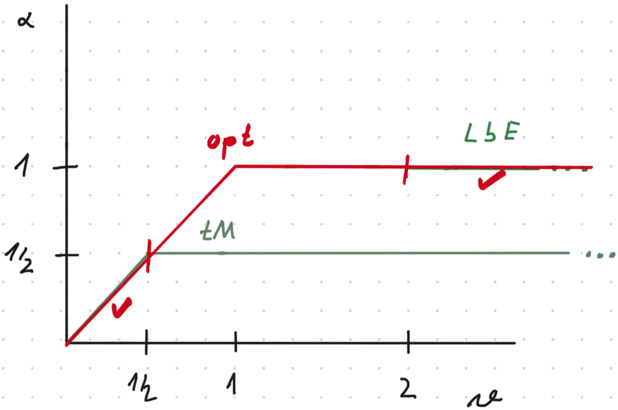
Remarks

(i) strong approximation of CIR arbitrarily hard if $\nu \to 0$

(ii) SDEs with sub-polynomial optimal rates:
Jentzen, Müller-Gronbach, Yaroslavtseva, 2016; ...

(iii) upper bounds for minimal error

$$e_{min}(n) \preceq \begin{cases} n^{-1} & \text{for} \quad \nu > 2 \\ n^{-\frac{1}{2}} & \text{for} \quad \nu \in (1,2] \cup \{\frac{1}{2}\} \\ n^{-\min\{\nu,\frac{1}{2}\}+\varepsilon} & \text{for} \quad \nu \in (0,1] \setminus \{\frac{1}{2}\} \end{cases}$$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Phase IV: back to the Euler scheme

$$\varphi_n^E(W_{T/n}, W_{2T/n}, \ldots, W_T) = x_n \text{ where}$$

$$x_{k+1} = x_k + \kappa(\theta - x_k)\frac{T}{n} + \sigma\sqrt{|x_k|}\left(W_{\frac{(k+1)T}{n}} - W_{\frac{kT}{n}}\right), \quad k \geq 0$$

Why?

- simple scheme, but mathematically challenging
- often (still) preferred in applications; in particular if CIR is part of a system of SDEs

In the meanwhile:

Bossy, Diop, 2007 and *Berkaoui, Bossy, Diop, 2008*:
L^p-convergence order $\frac{1}{2}$ for SE under an explicit condition on ν (for more general drift)

Cozma, Reisinger, 2020:
L^p-convergence order $\frac{1}{2}$ for FTE for $2 \leq p < \nu - 1$ and $\nu > 3$

Theorem D Mickel, N, 2022+

$$e(\varphi_n^E) \preceq n^{-\min\{\frac{\nu}{2},\frac{1}{2}\}+\varepsilon}$$

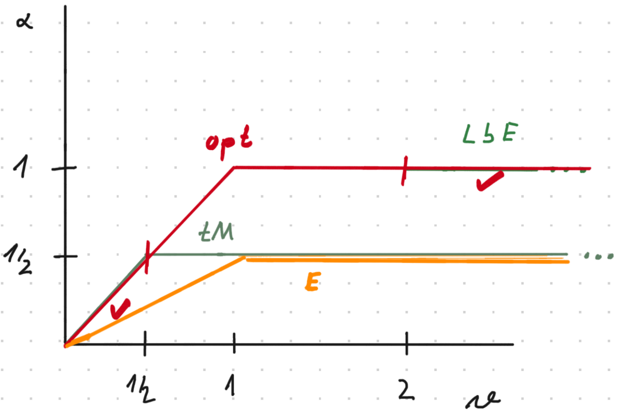
for all $\varepsilon > 0$

Remarks

- $\varphi_n^E = \text{HM}$; result holds also for PTE, FTE
- result holds also for AE, SE for $\nu \geq 1$
- proofs relies on Itō–Tanaka formula, inverse moments of CIR and clever control of local time terms; positivity of scheme not required
- N, Zähle, 2009; Protter et al. 2020:

$$n^{1/2}\left[X_T - \varphi_n^E(W_{T/n}, W_{2T/n}, \ldots, W_T)\right] \xrightarrow{d} \mathcal{Z}_T$$

  for $\nu \geq 1$ via localization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The Euler scheme and the loss of Lipschitzness

Consider

$$dY_t = a(Y_t)dt + \sigma(Y_t)dW_t, \quad t \in [0, T], \qquad Y_0 = y_0 \in \mathbb{R}$$

where

(i) $a : \mathbb{R} \to \mathbb{R}$ globally Lipschitz

(ii) $\sigma : \mathbb{R} \to \mathbb{R}$ Hölder $1-\gamma$, i.e.

$$\sup_{-\infty<x<y<\infty} \frac{|\sigma(x)-\sigma(y)|}{|x-y|^{1-\gamma}} < \infty$$

with $\gamma \in (0, 1/2]$

Theorem E Gyöngy, Rasonyi, 2011

$$e(\varphi_n^E) \preceq \begin{cases} n^{-\frac{1}{2}+\gamma} & \text{for} \quad \gamma \in (0, 1/2) \\ \log(n)^{-1} & \text{for} \quad \gamma = 1/2 \end{cases}$$

An extension

Consider

$$dY_t = a(Y_t)dt + (b(Y_t))^{1-\gamma} dW_t, \quad t \in [0, T], \qquad Y_0 = y_0 \in \mathbb{R}$$

where $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to [0, \infty)$ globally Lipschitz, $\gamma \in (0, 1/2]$

Theorem F Mickel, N, 2023+

If

$$\int_0^T \mathbf{E}\left[\frac{1}{b(Y_t)^{2\gamma}}\right] dt < \infty$$

then

$$e(\varphi_n^E) \preceq n^{-\frac{1}{2}+\varepsilon}$$

for all $\varepsilon > 0$

Remark

- inverse moment condition can be translated into conditions on a, b, γ; work in progress
- covers CIR and Wright-Fisher SDE with non-attainable boundaries, CEV process, ...

What's next?

For CIR:

- can the gap in the upper bound for the minimal error be closed ?
- can the analysis of the Euler scheme be further improved ($\nu < 1$ or L^p-error) ?
- can adapative discretizations help (for $\nu \neq 1/2$) ?
  For $\nu = \frac{1}{2}$ they can, since $X_T = F(W_T, \inf_{s \in [0,T]} W_s)$ (Hefter, Herzwurm, 2017)

More generally: analyze strong approximation of square root diffusions as, e.g., the chemical Langevin equation

$$dX_t = \sum_{k=1}^{d} w_k \lambda_k(X_t) dt + \sum_{k=1}^{d} w_k \sqrt{\lambda_k(X_t)} dW_t^{(k)}$$

with $w_k \in \mathbb{Z}^d$, $\lambda_k : \mathbb{R}^d \to [0, \infty)$, $k = 1, \dots, d$