Estimation of the parameter of the Skew Brownian motion

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Skew Random Walk (SRW)

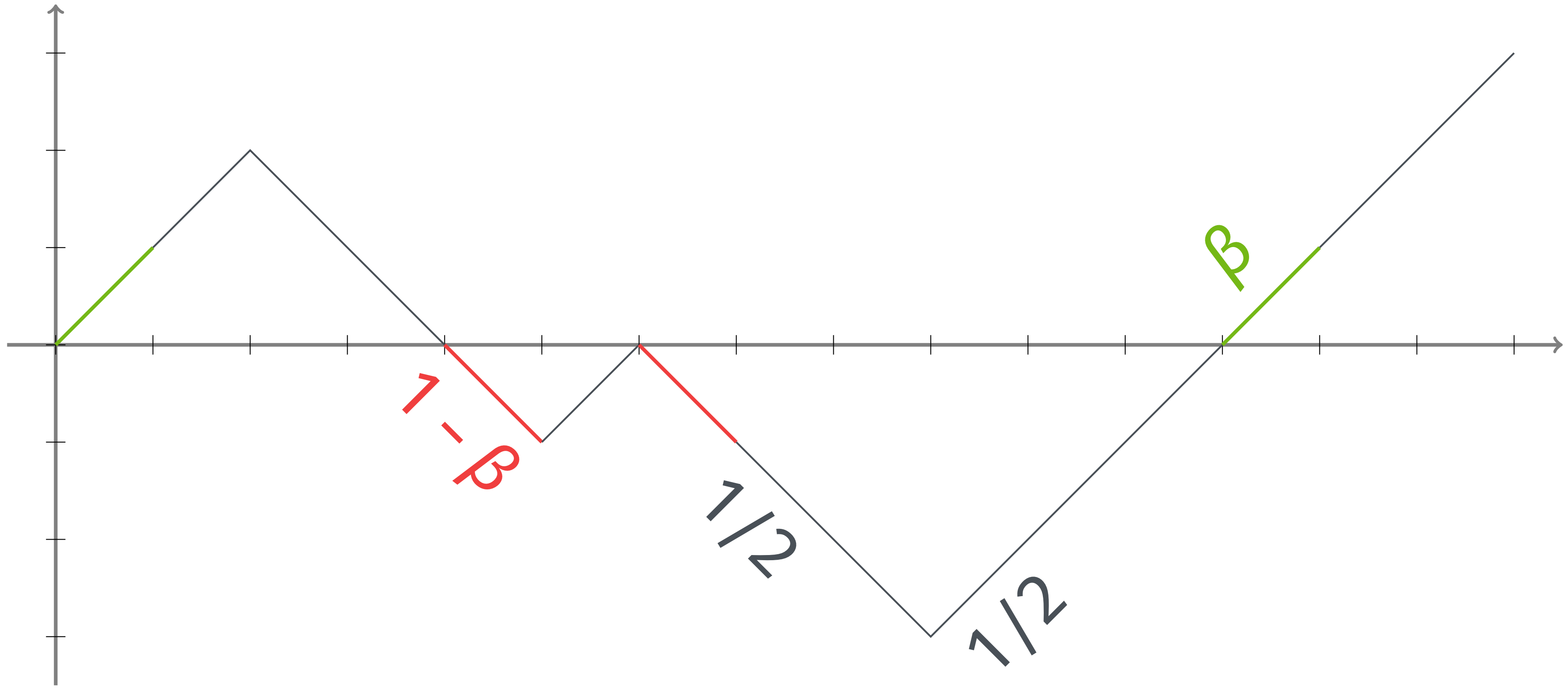
SRW = Random walk on $\mathbb{Z}$ with the dynamic

$$\mathbb{P}_\beta[S_{n+1} = x + 1 \mid S_n = x] = \begin{cases} \beta & \text{if } x = 0 \\ 1/2 & \text{if } x \neq 0 \end{cases}$$

$$\mathbb{P}_\beta[S_{n+1} = x - 1 \mid S_n = x] = \begin{cases} 1 - \beta & \text{if } x = 0 \\ 1/2 & \text{if } x \neq 0 \end{cases}$$

How to estimate $\beta \in (0, 1)$ from one path with N samples ?

Skew Random Walk (SRW)



Skew Random Walk

- The likelihood with N steps is

$$\Lambda_N(\beta) = \beta^{N_+}(1-\beta)^{N_-} \times \frac{1}{2}^{N-N_+-N_-},$$

with N_+ = # upward transitions from 0,
N_- = # downward transitions from 0.

- Only the pairs (S_k, S_{k+1}) with $S_k = 0$ contains information about β.
- The maximum likelihood estimator (MLE) is

$$\beta_N = \frac{N_+}{N_+ + N_-} = \frac{N_+}{A} \text{ with } A = \#\{k \le N \mid S_k = 0\}.$$

- β_N = # positive excursions / # excursions

MLE for Skew Random Walk

Theorem (AL, 2018)

(i) β_N is a **consistent estimator** of β, that is β_N converges in probability to β under $\mathbb{P}_\beta$.

(ii) $N^{1/4}(\beta_N - \beta)$ converges in distribution to $\sqrt{\beta(1-\beta)}H$ with

$$H \overset{\text{law}}{=} G/\sqrt{L_1} \text{ mixed normal distribution,}$$
$$G \sim N(0,1),$$
$$L_1 \text{ Brownian motion's } \textbf{local time}.$$

Why mixed normal limit ? Why $N^{1/4}$ and not $N^{1/2}$?

- The MLE depends from a random number of samples A (the occupation time).
- The occupation time A at 0 is of order $\sqrt{N}$.
- $A/\sqrt{N}$ converges in distribution to the local time L_1 at point 0,

$$L_1 := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^1 1_{B_s \in [-\epsilon, \epsilon]} \,\mathrm{d}s, \ \text{ a.s.}$$

Rem With A^+ := time spent above 0, $\mathbb{E}[A^+/N] = \beta$.
Yet A^+/N converges in distr. to a variant of the arc-sine law
$\Longrightarrow$ A^+/N is an *useless* estimator of β.

Skew Brownian motion

Skew Brownian motion (SBM) of parameter $\theta \in (-1, 1)$:

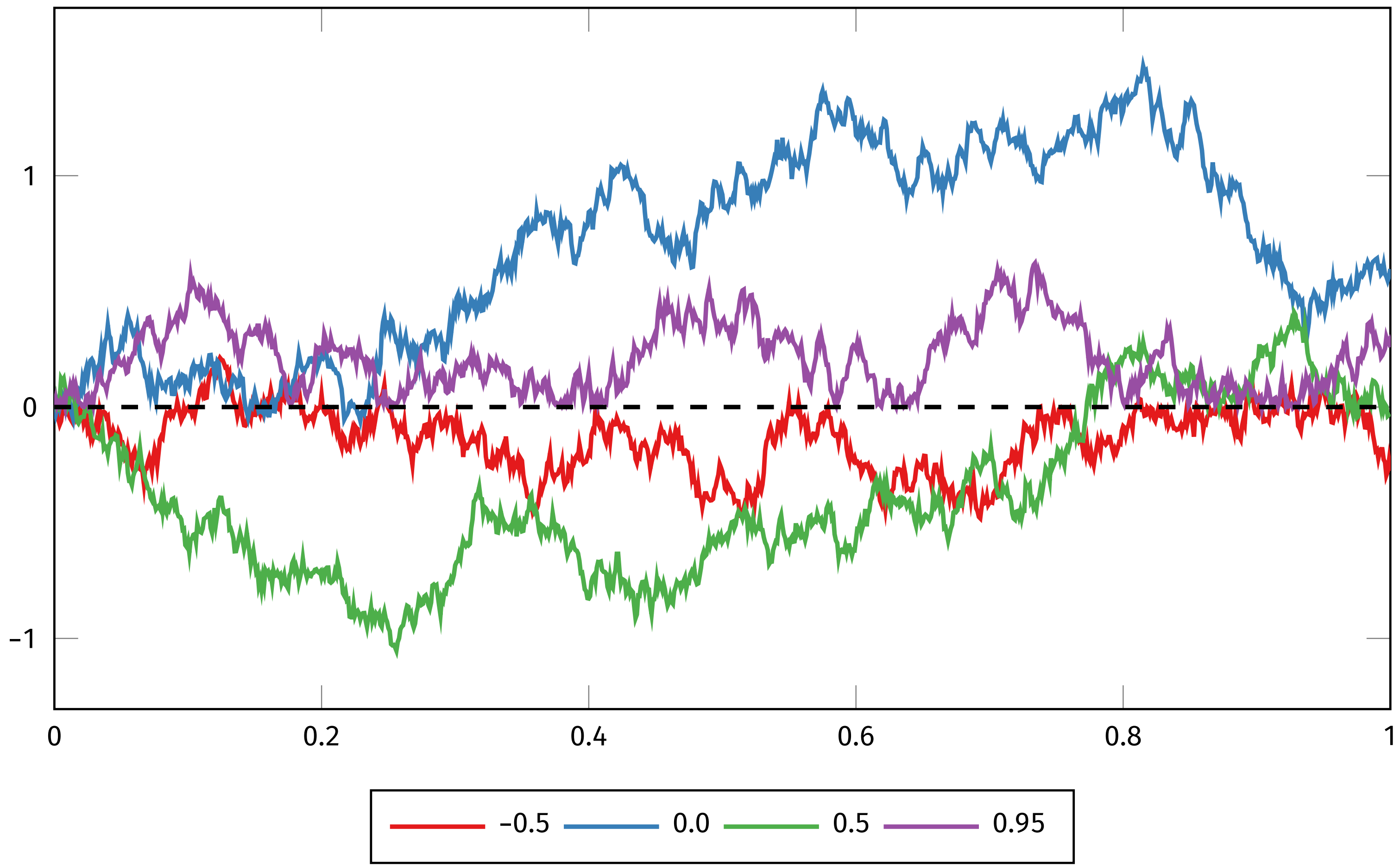
$$X_t = B_t + \theta L_t(X), \quad t \geq 0$$

with B Brownian motion

$L_t(X)$ local time at point 0 of X (not B)

- The SBM is the limit (Donsker) of the Skew Random walk with $\beta = (1 + \theta)/2$.
- There are 10+ possible constructions of the SBM.
- Away from 0, the SBM behaves like a BM.
  The local time acts only when the process reaches 0.
- Its distribution is singular with the one of the BM.

Skew Brownian motion

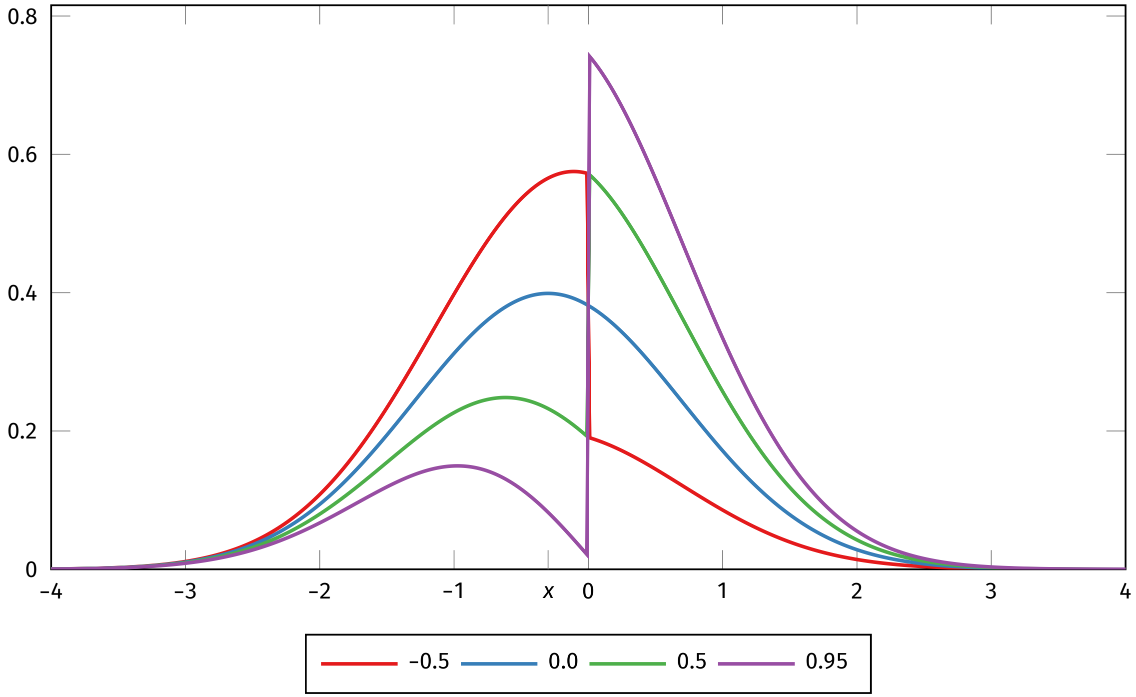


Density transition function for the SBM

The SBM has an explicit formula for the density transition function (Walsh, 1978)

$$p_\theta(t, x, y) = g(t, y - x) + \theta \operatorname{sgn}(y) \cdot g(t, |x| + |y|)$$

$$g(t, \cdot) \text{ Gaussian density of } N(0, t)$$



MLE for the SBM

Data: $\{X_{i\Delta t}\}_{i=0,\dots,n}$ with $\Delta t := T/n$

$\rightsquigarrow$ high-frequency, up to time T.

$$\Lambda_n(\theta) := \prod_{i=0}^{n-1} p_\theta(\Delta t, X_{i\Delta t}, X_{(i+1)\Delta t}) \qquad \text{likelihood}$$

$$L_n(\theta) := \log \Lambda_n(\theta) \qquad \text{log-likelihood (concave)}$$

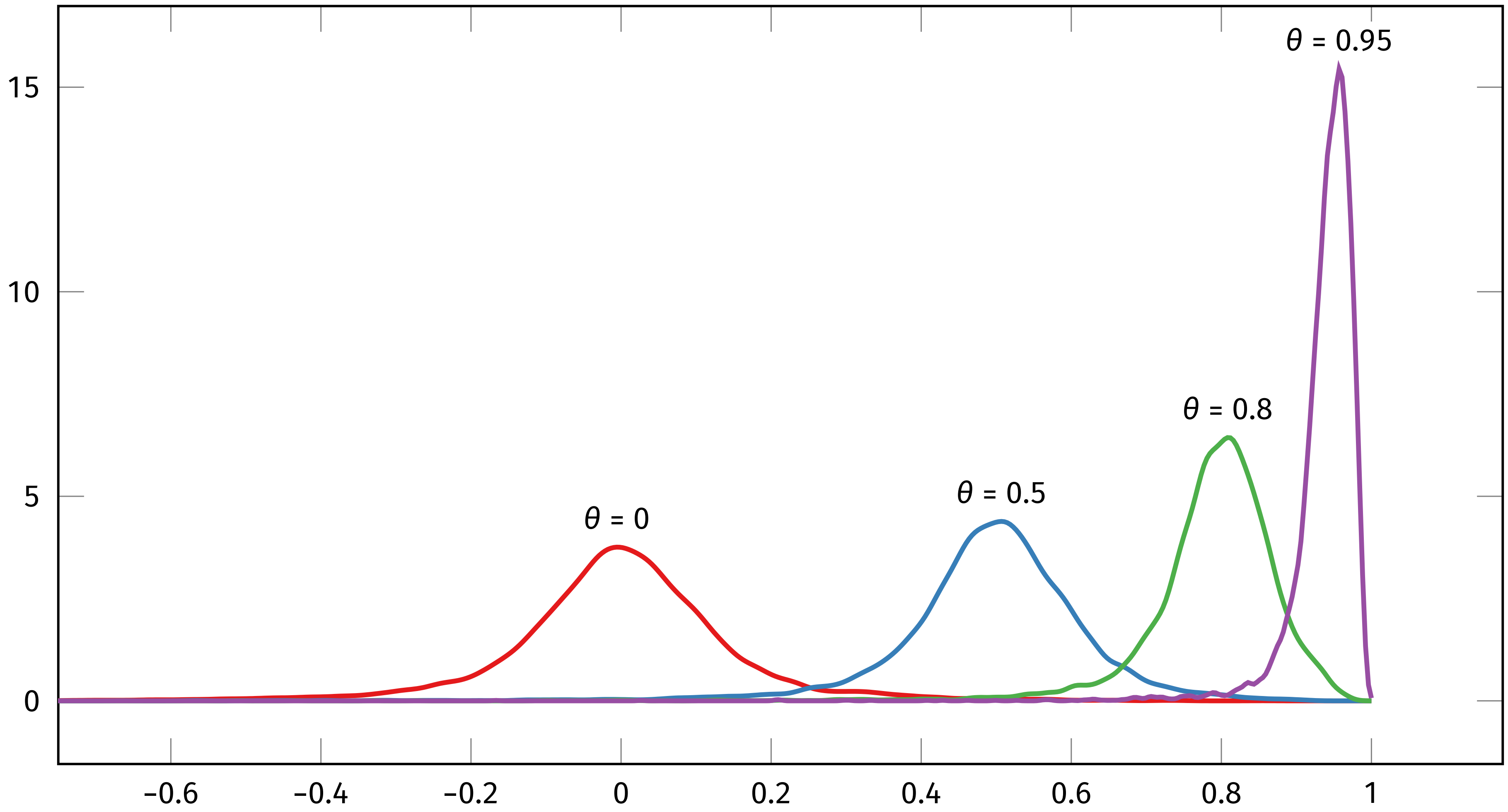
$$S_n(\theta) := \partial_\theta L_n(\theta) \qquad \text{score}$$

The MLE θ_n is

$$\theta_n := \arg\max_\theta L_n(\theta) \text{ or equivalently } S_n(\theta_n) = 0.$$

θ_n is easy to compute numerically.

MLE for the SBM



MLE for the SBM

Theorem (AL-EM-ST, 2019; AL-SM, 2023)

(i) θ_n is a **consistent estimator** of θ under $\mathbb{P}_\theta$.

(ii) Asymptotic mixed-normality:

$$n^{1/4}(\theta_n - \theta) \xrightarrow[n\to\infty]{dist} s(\theta)\frac{W(L_T)}{L_T}$$

with

W Brownian motion indep. from X

L SBM's local time (its law does not depend on θ)

$$s(\theta) \approx \frac{\sqrt{1-\theta^2}}{\sqrt{1.3 + 0.23\theta^2 + 0.07\theta^4}} \text{ (not exact yet accurate)}$$

Key result on convergence: LLN

For f such that $\int x^2 |f(x)|\,\mathrm{d}x < +\infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(\sqrt{n} X_{iT/n}, \sqrt{n} X_{(i+1)T/n}) \xrightarrow[n\to\infty]{} c(F) L_T$$

$$c(F) := (1+\theta) \int_0^{+\infty} F(x)\,\mathrm{d}x + (1-\theta) \int_{-\infty}^{0} F(x)\,\mathrm{d}x$$

$$F(x) := \int_{-\infty}^{+\infty} p_\theta(1, x, y) f(x, y)\,\mathrm{d}y.$$

- (J. Jacod, 1998) for the BM and SDE
- (AL, EM & ST, 2019) for the SBM

Key result on convergence

This LLN mixes several behaviors:

- transformation into a martingale $\rightsquigarrow$ $F(x) := \mathbb{E}_x[f(x, X_1)]$
- averaging over the invariant measure of the SBM $((1 \pm \theta)\mathbb{1}_{\pm x \geq 0}\,\mathrm{d}x)$ $\rightsquigarrow$ expression of $c(F)$
- concentration around 0 $\rightsquigarrow$ local time

A CLT may also be proved, with more technicality:

- (J. Jacod, 1998) for the BM and SDE
- (S. Mazzonetto, 2019) for SBM, see also (C.Y. Robert, 2022)

Expansion of the MLE

Recall that θ_n solves $S_n(\theta_n) = 0$. Search for θ_n in the form

$$\theta_n = \theta + \sum_{k\geq 1} \frac{E_{k,n}(\theta)}{n^{k/4}}.$$

Expand $S_n(\theta_n)$ using the Taylor formula

$$S_n(\theta_n) = \sum_{k\geq 0} \frac{1}{k!}\partial_\theta^k S_n(\theta)(\theta_n - \theta)^k$$

and seek $E_{k,n}(\theta)$ to vanish the expansion.

The $E_{k,n}(\theta)$'s depend on the $\partial_\theta^k S_n(\theta)$. The expansion is not unique. We select $E_{k,n}(\theta)$ so that they converge as $n \to \infty$.

Expansion of the MLE

$$\theta_n = \theta + \sum_{k\geq 1} \frac{E_{k,n}(\theta)}{n^{k/4}}$$

Since

$$0 = S_n(\theta_n) = S_n(\theta) + \partial_\theta S_n(\theta)\frac{E_{1,n}(\theta)}{n^{1/4}} + \cdots,$$

we identify

$$E_{1,n}(\theta) = n^{1/4}\frac{-S_n(\theta)}{\partial_\theta S_n(\theta)} \xrightarrow[n\to\infty]{\text{dist}} s(\theta)\frac{W(L_T)}{L_T} \text{ under } \mathbb{P}_\theta$$

because $S_n(\theta_n)/n^{1/4}$ and $\partial_\theta S_n(\theta)/n^{1/2}$ converge.

Expansion of the MLE

As $\theta \mapsto p_\theta(t,x,y)$ is analytic, define

$$[S]_{k,n}(\theta) := \frac{1}{\sqrt{n}} \partial_\theta^k S_n(\theta).$$

Thanks to the LLN or CLT

$$[S]_{k,n}(\theta) \xrightarrow[n\to\infty]{\text{proba}} \xi_k(\theta) L_T$$

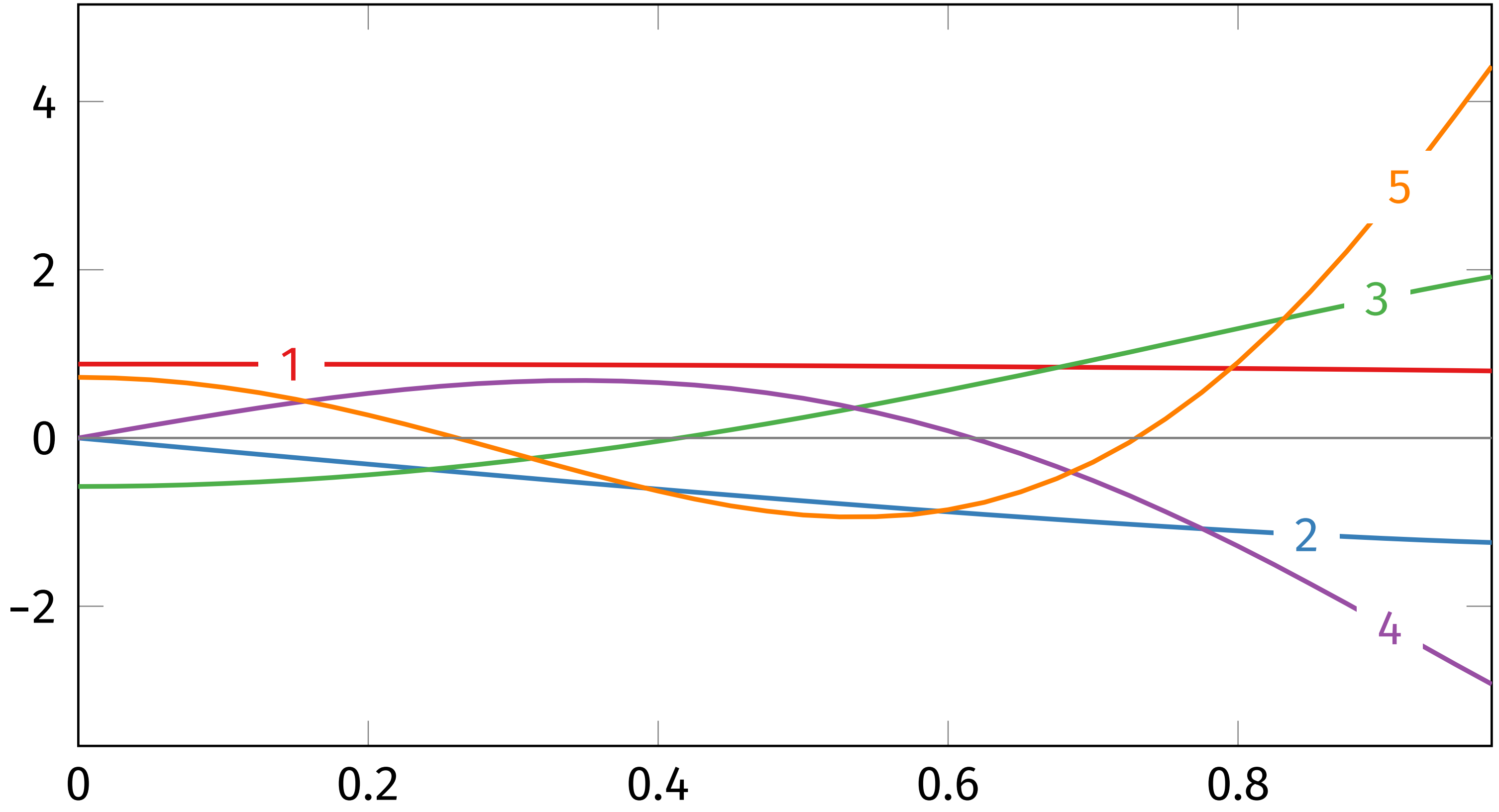
$$n^{1/4}[S]_{k,n}(\theta) \xrightarrow[n\to\infty]{\text{stable}} \Xi_k(\theta)\sqrt{L_T} G \text{ with } G \sim N(0,1) \text{ when } \xi_k(\theta) = 0.$$

Expansion of the MLE

$$d_{k,n}(\theta) := \frac{-1}{k!} \frac{[S]_k(\theta)}{[S]_1(\theta)} \xrightarrow[n\to\infty]{\text{proba}} d_k(\theta) := \frac{-\xi_k(\theta)}{k!\,\xi_1(\theta)}.$$

- $\xi_1(\theta) \neq 0$ (linked to Fisher information)
- If $\theta = 0$, then $\xi_{2k}(0) = 0$, $k \geq 0$.
- Joint convergence holds for $([S]_{2k,n}(0))_{k\leq m}$ toward a Gaussian vector.

The coefficients $\xi_k(\theta)$



$$\theta \mapsto \frac{(-1)^k(1-\theta^2)^k}{k!}\xi_k(\theta)$$

Expansion for the MLE

Theorem (AL-EM-ST, 2014 for $\theta = 0$; AL-SM, 2023)

$$\theta_n = \theta + \sum_{k\geq 1} D_{k,n}(\theta) d_{0,n}(\theta)^k \qquad \text{[Expansion in } n^{-1/4}\text{]}$$

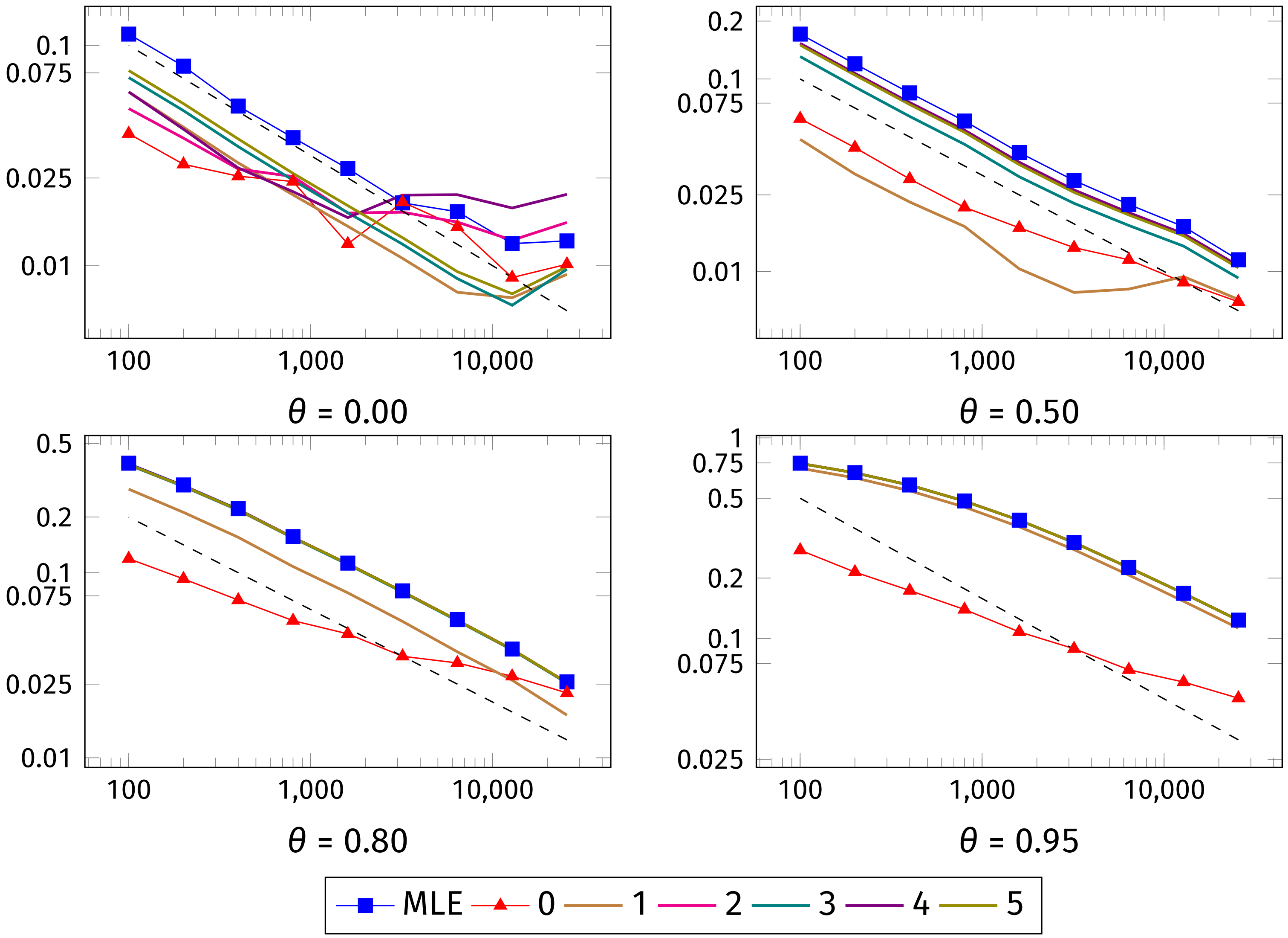
where $D_{1,n}(\theta) := 1$ and

$$D_{k,n}(\theta) := \sum_{m=2}^{k} d_{m,n}(\theta) \sum_{k_1+\cdots+k_m=k} D_{k_1,n}(\theta) \cdots D_{k_m,n}(\theta).$$

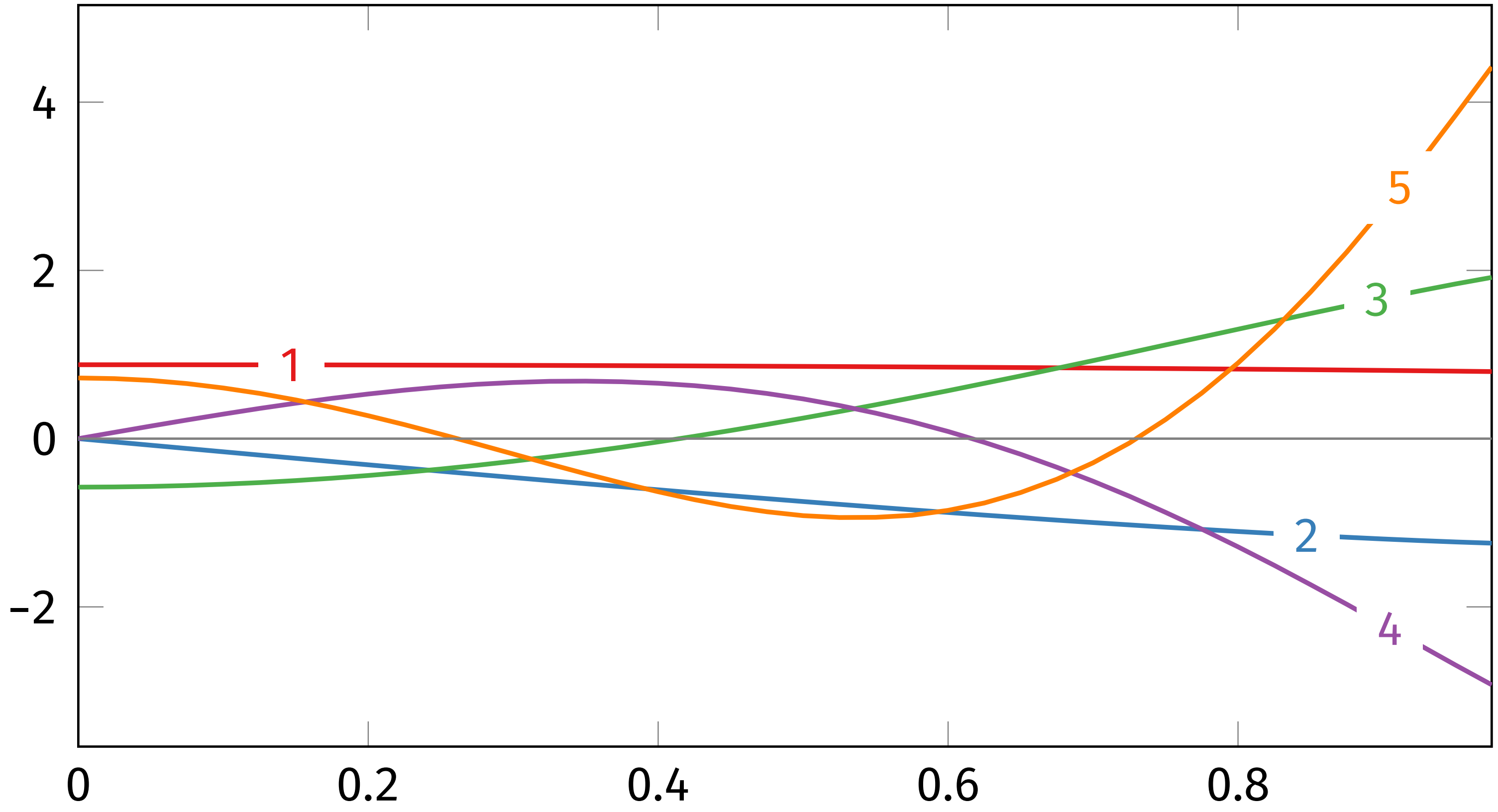
Besides,

$$n^{1/4} d_{0,n}(\theta) \xrightarrow[n\to\infty]{\text{stable}} s(\theta)\frac{W(L_T)}{L_T} \overset{\text{law}}{=} s(\theta)\frac{G}{\sqrt{L_T}},\ G \sim N(0,1).$$

KS distance between $D_{k,n}(\theta)$ and $D_k(\theta)$, $1 \le k \le 5$, θ_n and θ

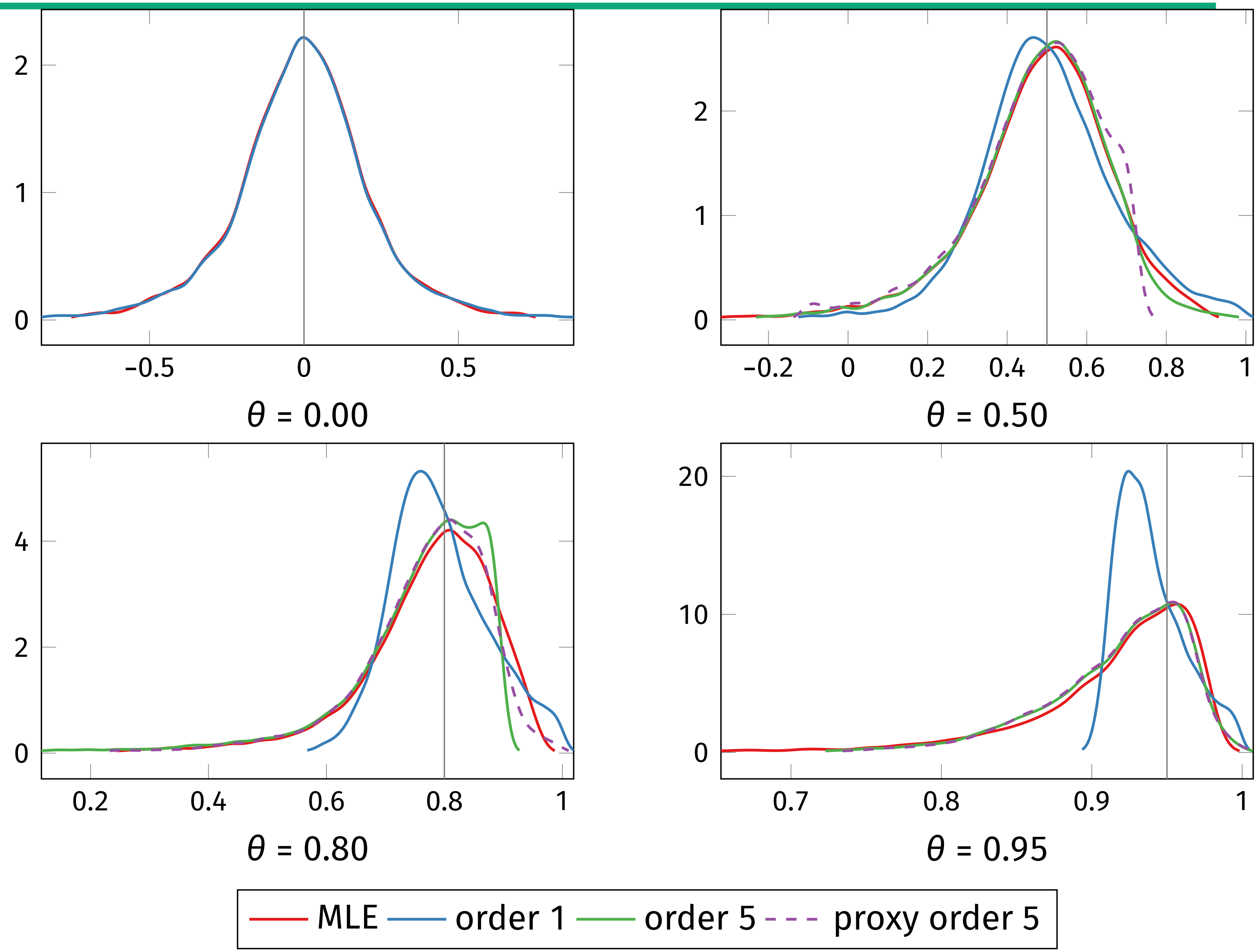


The coefficients $D_k(\theta)$



$$\theta \mapsto D_k(\theta)s(\theta)^k(1-\theta^2)^{k/2-1}$$

Using $D_{k,n}(\theta)$ and $D_k(\theta)$ (“proxy”) to approximate θ_n



More on the expansion

- The terms $d_{k,n}(\theta)$ and $D_{k,n}(\theta)$ are related by

$$f(g(z)) = -z$$

with $f(z) = \sum_{k\geq 1} d_{k,n}(\theta) z^k$ and $g(z) = \sum_{k\geq 1} D_{k,n}(\theta) z^k$.

- The $D_{k,n}(\theta)$ are random but $D_k(\theta)$ are deterministic.
- For θ close to ± 1, there is a boundary layer: the truncated expansion becomes inacurate as

$$|D_k(\theta) s(\theta)^k| = O\left(\frac{1}{(1-\theta^2)^{k/2-1}}\right).$$

Phase transition

- When $\theta = 0$ (the SBM is a BM), $D_{2,n}(0)$ vanishes asymptotically.
- We then replace the LNN by the CLT when possible $\implies$ Some of the $n^{-1/2}$ are replaced by $n^{-1/4}$.
- When $\theta = 0$,

$$\theta_n = d_{0,n}(0) + \sum_{\substack{k \geq 3 \\ k \text{ odd}}} \frac{a_{k,n}}{n^{k/4}}$$

where the $a_{k,n}$'s are given by a recursive relation, and

$$a_{k,n}(\theta) \xrightarrow[n \to \infty]{} \text{polynomial}(\sqrt{L_T}, G_1, \dots, G_n).$$

What we have done?

- Consistency and asymptotic property of the MLE θ_n: asymptotic mixed normality, rate of convergence 1/4.
- Convergence of the score of its derivatives.
- Study of the behavior of the coefficients with θ.

$\Leftarrow$ Key methods: extensions of the results of (J. Jacod, 1998).

- Expansion of the MLE of power of $1/n^{1/4}$.

$\Leftarrow$ Key method: "Asymptotic inverse function theorem" (AL-SM, 2022)

- Numerical studies: simulation of the SBM (AL-G. Pichot, 2012)

Still to do...

- The asymptotic expansion aims at quantifying the semi-asymptotic behavior of the MLE. We obtained

$$\theta_n = F_n(P_n)$$

$$P_n \text{ (asymptotically pivotal) } \xrightarrow[n\to\infty]{\text{law}} G/\sqrt{L_T}$$

$$F_n \text{ (random) } \xrightarrow[n\to\infty]{} F \text{ (deterministic)}$$

- F_n is well approximated by a polynomial of low order (empirically around 5).
- How to control the distance between P_n and $G/\sqrt{L_T}$?
- How to build confidence intervals beyond Wald's one? Hypotheses tests?

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