Probabilistic Conformal Field Theory

Antti Kupiainen

International Seminar on SDEs and Related Topics 11/2025

Conformal Field Theory (CFT)

Statistical mechanics systems exhibit conformal symmetry in large spatial scales at the critical temperature of 2nd order phase transition

- Large scale behaviour is **independent** of small scale details
- **Universality classes** are described by **Conformal Field Theories** (CFT)
- Quantum Field Theories become CFTs in small scales
- CFTs have special symmetries allowing for precise conjectures

CFT has also a rich mathematical structure

- Representation theory of infinite dimensional Lie algebras (Virasoro, Kac-Moody, W-algebras)
- Geometry of moduli spaces, knot theory,...

However, a rigorous constructive foundation for CFT is still lacking.

In this talk I'll discuss a **probabilistic** approach to 2d CFTs .

Conformal invariance

A probabilistic CFT is characterised by a family of **random fields**

$$\{V_\alpha(x)\,|\,x\in\mathbb{R}^d,\alpha\in I\}$$

and their **correlation functions**

$$\langle\prod_{i=1}^{n}V_{\alpha_i}(x_i)\rangle.$$

Correlation functions are invariant under rotations and translations of $\mathbb{R}^d$ and under scaling

$$\langle\prod_i V_{\alpha_i}(\lambda x_i)\rangle=\prod_i\lambda^{-2\Delta_{\alpha_i}}\langle\prod_i V_{\alpha_i}(x_i)\rangle\quad(*)$$

Δ_α is called scaling dimension or **conformal weight**.

Conformal invariance: $(*)$ extends to conformal maps $x\to\Lambda(x)$,

In $d=2$: $\mathbb{R}^2\simeq\mathbb{C}$ and Λ is Möbius map and $\lambda^{-2\Delta_\alpha}\to|\Lambda'(z)|^{-2\Delta_\alpha}$.

Structure Constants

Natural setup for 2d CFT is the **Riemann sphere**: $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (and, more generally, a **Riemann surface**).

Using conformal map to send three points $z_1, z_2, z_3 \in \mathbb{C}$ to $\{0, 1, \infty\}$

$\implies$ 3-point functions are determined up to constants

$$\langle \prod_{k=1}^{3} V_{\alpha_k}(z_k) \rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_{\alpha_1\alpha_2\alpha_3}$$

with $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc.

$$C_{\alpha_1\alpha_2\alpha_3} = \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle$$

are called the **structure constants** of the CFT.

Bootstrap hypothesis

Operator Product Expansion (OPE) hypothesis:

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3)\dots\rangle = \sum_{\alpha} C_{\alpha_1\alpha_2\alpha}\mathcal{D}_{x_2}\langle V_{\alpha}(x_2)V_{\alpha_3}(x_3)\dots\rangle$$

- Differential operator $\mathcal{D}_{x_2}$ **determined** by conformal symmetry
- Iterating OPE: all correlation functions are determined by $C_{\alpha_1\alpha_2\alpha_3}$

How to find CFTs?

Consistency of OPE can be used to guess possible solutions Belavin, Polyakov, Zamolodchicov, 1984, Rychkov et al 2008

How to **justify** bootstrap?

Derive it from a **probabilistic path integral**.

One Dimensional Free Field: Wiener measure

Let $X : \mathbb{R} \to \mathbb{R}$ be two-sided Brownian motion and $c \in \mathbb{R}$. The 1d free field

$$\phi(t) = c + X(t)$$

is defined by the measure μ on continuous paths $\phi : \mathbb{R} \to \mathbb{R}$ with

$$\langle F \rangle = \int F(\phi) d\mu(\phi) = \int_{\mathbb{R}} \mathbb{E}[F(c + X(\cdot))] dc$$

for suitable $F : C(\mathbb{R}) \to \mathbb{R}$.

This gives a rigorous sense to the formal **path integral**

$$\langle F \rangle = \int_{\phi:\mathbb{R}\to\mathbb{R}} F(\phi) e^{-S(\phi)} D\phi$$

where the **action functional** is given by

$$S(\phi) = \tfrac{1}{2} \int_{\mathbb{R}} (\frac{d\phi(t)}{dt})^2 dt$$

Path integrals

This extends to other **action functionals**:

- **Feynman-Kac**. Perturb by a function $V : \mathbb{R} \to \mathbb{R}$:

$$\langle F \rangle = \int_{\phi:\mathbb{R}\to\mathbb{R}} F(\phi) e^{-\frac{1}{2}\int_{\mathbb{R}} (\dot{\phi}(t)^2 + V(\phi(t)) dt} D\phi := \int_{\mathbb{R}} \mathbb{E}[F(c+X) e^{-\frac{1}{2}\int_{\mathbb{R}} V(c+X(t) dt}] dc$$

- **Sigma model**. Let $\phi : \mathbb{R} \to M$, M a manifold with Riemannian metric m

$$\langle F \rangle = \int_{\phi:\mathbb{R}\to M} F(\phi) e^{-\frac{1}{2}\int_{\mathbb{R}} |\dot{\phi}(t)|_m^2 dt} D\phi$$

  where in local coordinates of M, $|\dot{\phi}|_m^2 = \sum_{ij} \dot{\phi}^i \dot{\phi}^j m_{ij}(\phi)$. We can define this in terms of the Brownian motion on M.

The random fields are given then by

$$V_\alpha(t) = W_\alpha(\phi(t))$$

for some functions $W_\alpha : M \to \mathbb{R}$ and their **correlation functions** are

$$\langle \prod_{i=1}^{N} V_{\alpha_i}(t_i) \rangle = \int_{\phi:\mathbb{R}\to\mathbb{R}} \prod_{i=1}^{N} W_{\alpha_i}(\phi(t_i)) e^{-S(\phi)} D\phi$$

Path integral for d>1

Setup:

- Field $\phi : \mathbb{R}^d \to M$
- **Local** action functional

$$S(\phi) = \int_{\mathbb{R}^d} L(\phi(x), \nabla\phi(x))dx$$

Formal expression for correlation functions

$$\langle \prod_{i=1}^{n} V_i(x_i) \rangle = \int_{\phi:\mathbb{R}^d \to M} \prod_{i=1}^{n} W_i(\phi(x_i)) e^{-S(\phi)} D\phi$$

- How to find candidates for conformally invariant S?
- How to construct the path integral?
- How to derive bootstrap from path integral?

Classical 2d Conformal Field Theories

In 2d it is natural to replace $\mathbb{R}^2$ by a **Riemann surface**:

- Field $\phi : \Sigma \to M$. M manifold.
- (Σ, h) 2d surface, h Riemannian metric
- Action functional $S(\phi, h)$

$S(\phi, h)$ is **conformally invariant** if

- $S(\phi, h) = S(\phi \circ \psi, \psi^* h)$ for $\psi : \Sigma \to \Sigma$ diffeomorphism.
- S is **Weyl** invariant:

$$S(\phi, e^\sigma h) = S(\phi, h), \quad \sigma \in C^\infty(\Sigma)$$

- Thus S depends only on the complex structure determined by h

Free field

Let $\phi : \Sigma \to \mathbb{R}$. Action functional is the Dirichlet energy

$$S(\phi, h) = \frac{1}{4\pi} \int_\Sigma |d\phi(x)|_h^2 \, dv_h(x)$$

where dv_h is Riemannian volume and $|\cdot|_h$ in metric in $T^*\Sigma$.

In complex coordinates z on Σ

$$S(\phi, h) = \frac{1}{\pi} \int_\Sigma \partial_z \phi(z) \partial_{\bar{z}} \phi(z) \, d^2 z$$

Weyl invariant. Extrema of S are harmonic functions.

Sigma model

Let (M, m) be a Riemannian manifold and $\phi : \Sigma \to M$.

$$S(\phi, h) = \frac{1}{4\pi} \int_\Sigma |d\phi(x)|^2_{h,m} \, dv_h(x)$$

where $|\cdot|_{h,m}$ is in metric on $d\phi(x) \in T_x^*\Sigma \otimes T_{\phi(x)}M$.

In complex coordinate z on Σ and local coordinates ϕ^i on M

$$S(\phi, h) = \frac{1}{\pi} \int_\Sigma \sum_{ij} \partial_z \phi^i(z) \partial_{\bar{z}} \phi^j(z) m_{ij}(\phi(z)) d^2 z$$

with $m = \sum_{ij} m(\phi)_{ij} d\phi^i \otimes d\phi^j$.

Weyl invariant. Extrema harmonic maps $\Sigma \to M$.

Liouville theory

In general $\int_\Sigma V(\phi(x))dv_h(x)$ is **not** Weyl invariant.

However, the **Liouville action**

$$S(\phi,h)=\frac{1}{4\pi}\int_\Sigma (|d\phi|_h^2+QR_h\phi+\mu e^{2b\phi})\,dv_h$$

with R_h scalar curvature is covariant if $Q=\frac{1}{b}$:

$$S(\phi,e^\sigma h)=S(\phi+\sigma,h)-6Q^2A(\sigma,h)$$

where $A(\sigma,h)$ is the **Weyl anomaly**

$$A(\sigma,h)=\frac{1}{96\pi}\int_\Sigma (|d\sigma|_h^2+R_h\sigma)dv_h$$

Extrema ϕ_0 are metrics $e^{2b\phi_0}h$ with constant curvature (Picard, Poincare).

Path integral

How to define the formal path integral for observables $F(\phi)$

$$\langle F \rangle = \int_{\phi:\Sigma\to M} F(\phi) e^{-S(\phi)} D\phi \ \ ?$$

Free field: let

$$\phi = c + X_h$$

where $c \in \mathbb{R}$ and define

$$\langle F \rangle := Z(h) \int_{\mathbb{R}} \mathbb{E} F(c + X_h) dc$$

- X_h is the **Gaussian free field (GFF)**
- $X_h \in H^{-s}(\Sigma)$ is a Gaussian random distribution with covariance

  $$\mathbb{E}[X_h(x) X_h(x')] = G_h(x, x')$$

  G_h is the Green function of the Laplace Beltrami operator Δ_h.
- $Z(h) = (\det{}'(-\Delta_h))^{-\frac{1}{2}}$ is the partition function of the GFF

Liouville theory

Non-linear terms in action require **renormalisation**.

For **Liouville theory** we need to define $e^{2b\phi}$ for a distribution ϕ:

- **Mollify** the free field $\phi \to \rho_\epsilon * \phi$
- **Renormalise** $e^{2b\phi}$ by

$$e^{2b\phi(z)} dv_h(z) := \lim_{\epsilon \to 0} \epsilon^{2b^2} e^{2b\phi_\epsilon(z)} dv_h(z)$$

= **Gaussian Multiplicative Chaos (GMC)** measure.

Then define

$$\langle F \rangle := Z(h) \int_{\mathbb{R}} \mathbb{E} F(c + X_h) e^{-\frac{1}{4\pi} \int_\Sigma (QR_h\phi + \mu e^{2b\phi})\, dv_h} dc$$

Primary fields are **vertex operators**: For $\alpha \in \mathbb{C}$

$$V_\alpha(z) := \lim_{\epsilon \to 0} \epsilon^{\frac{\alpha^2}{2}} e^{\alpha\phi_\epsilon}$$

Bootstrap holds for LCFT: C.Guillarmou, A.K., R.Rhodes, V. Vargas, (Annals 2020, Acta 2024, Annals 2026)

Sigma models

Action functional is **not** a perturbation of GFF.

Mollify $\phi \to \phi_\epsilon$, for instance let $\phi_\epsilon : \epsilon\mathbb{Z}^2 \to M$

Renormalise $S \to S_\epsilon$

Typically **no** conformal invariance as $\epsilon \to 0$.

Example: **Heisenberg model** $M = S^N$, round metric and $N \geq 2$:

- Have to renormalise as $S_\epsilon = \frac{1}{T_\epsilon} S$
- Have to take $T_\epsilon \to 0$ as $\epsilon \to 0$ (asymptotic freedom, Polyakov 1981)
- Limit is believed to be massive (not conformal).

The same holds for all compact symmetric spaces M.

Wess-Zumino-Witten model

Let $B = \sum B_{ij} d\phi^i \wedge d\phi^j$ be a 2-form on M then

$$S_{top} = \int_\Sigma \phi^* B = \int_\Sigma B_{ij}(\phi(z)) d\phi^i(z) \wedge d\phi^j(z)$$

is independent of metric i.e. **topological**.

Take $M = G$ a compact Lie Group, (e.g. $SU(N)$). $\mathcal{G}$ Lie algebra.

Sigma model action for $g : \Sigma \to G$

$$S_G(g) = -\frac{1}{2\pi} \int_\Sigma \mathrm{Tr}(g^{-1}\partial_z g)(g^{-1}\partial_{\bar{z}} g) d^2 z$$

Renormalisation as in Heisenberg model $S_G \to T_\epsilon^{-1} S_G \implies$ **not** a CFT

Witten (1984): add a topological term with

$$dB = \frac{1}{12\pi i} \mathrm{Tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg)$$

locally in G. Then, for $k \in \mathbb{Z}$, $e^{-kS_{top}(g)}$ is globally defined.

Conformal Field Theory

The formal path integral

$$\langle F \rangle_k = \int_{g:\Sigma\to G} F(g) e^{-k(S_G + S_{top})} Dg$$

is expected to give rise to a CFT for all $k \in \mathbb{N}$:

- WZW model has a **Kac-Moody symmetry** extending the conformal symmetry
- Most rational CFT's (e.g. 2d Ising model) can be obtained algebraically from the WZW model

Rigorous construction of the WZW path integral is still open.

However, if we replace G by **positive** elements in $G^{\mathbb{C}}$ a probabilistic construction in terms of GMC can be given and leads to an interesting CFT.

$G^{\mathbb{C}}/G$ WZW model

Consider WZW action on **positive** elements $\mathcal{P} = \{gg^* | g \in G^{\mathbb{C}}\}$ of $G^{\mathbb{C}}$:

$$S_{WZW}(gg^*), \quad g \in G^{\mathbb{C}}$$

$\mathcal{P} \cong$ homogenous space $G^{\mathbb{C}}/G$ since $gU(gU)^* = gg^*$ if $U \in G$.
Define formally $G^{\mathbb{C}}/G$ model by the path integral

$$\langle F \rangle_{G^{\mathbb{C}}/G} = \int F(gg^*) e^{kS_{WZW}(gg^*)} D(gg^*)$$

with $D(gg^*)$ the $G^{\mathbb{C}}$ invariant measure on $G^{\mathbb{C}}/G$.

- $\mathrm{Tr}|(gg^*)^{-1} d(gg^*)]^3$ is **exact** and the parameter $k \in \mathbb{R}_+$ is **not** quantised.
- **Duality** between G WZW and $G^{\mathbb{C}}/G$ model (Gawedzki '89, Witten '91)
- $G^{\mathbb{C}}/G$ model admits a probabilistic formulation in terms of GMC!

σ model on hyperbolic space

Let $G = SU(2)$. $g \in SL(2,\mathbb{C})$ can uniquely be written $g = bU$, $U \in SU(2)$ and

$$b = \begin{pmatrix} e^{\phi/2} & e^{-\phi/2}\gamma \\ 0 & e^{-\phi/2} \end{pmatrix} \quad \phi \in \mathbb{R}, \gamma \in \mathbb{C}.$$

The sigma model action becomes

$$S_G(bb^*) = -\frac{1}{\pi}\int_{\mathbb{C}} (2\partial_z\phi\partial_{\bar{z}}\phi + e^{-2\phi}(|\partial_z\gamma|^2 + |\partial_{\bar{z}}\gamma|^2)d^2z.$$

i.e. a σ- model on **3d hyperbolic space H_3**:

$$(\phi,\gamma_1,\gamma_2) \in \mathbb{R}^3, \;\; \text{metric} \;\; |d\phi||^2 + e^{2\phi}(|d\gamma_1|^2 + |d\gamma_2|^2)$$

S_{top} becomes

$$S_{top}(bb^*) = -\frac{1}{\pi}\int_{\mathbb{C}} (e^{-2\phi}(|\partial_{\bar{z}}\gamma|^2 - |\partial_z\gamma|^2)d^2z.$$

and altogether

$$S_{WZW}(bb^*) = -\frac{1}{\pi}\int_{\mathbb{C}} (\partial_{\bar{z}}\phi\partial_{\bar{z}}\phi + e^{-2\phi}|\partial_{\bar{z}}\gamma|^2)d^2z.$$

H_3 CFT

The formal H_3 model path integral is

$$\langle F \rangle = \int F(\phi, \gamma) e^{-\frac{k}{\pi} \int_\Sigma (|\partial_{\bar{z}} \phi|^2 + e^{-2\phi} |\partial_{\bar{z}} \gamma|^2) d^2 z} \prod_x d\mu(\phi(x), \gamma(x))$$

with $SL(2, \mathbb{C})$ invariant measure $d\mu(\phi, \gamma) = e^{2\phi} d\phi d^2\gamma$ on H_3.

This should give rise to a CFT for $k > 2$ with central charge $c = \frac{3k}{k-2}$.

- Euclidean AdS_3 CFT
- Admits a map to **Liouville CFT**
- Possible setup for "quantum" Langlands correspondece

Probabilistic formulation of H_3 CFT

Action $\int_\Sigma (\partial_z \phi \partial_{\bar z} \phi + e^{-2\phi} \partial_z \bar\gamma \partial_{\bar z} \gamma)$ motivates the following:

- Let $\phi = c + X_h$ be the free field
- $\int_\Sigma e^{-2\phi} |\partial_{\bar z} \gamma|^2 d^2 z = (\gamma, \mathcal{D}_\phi \gamma)_{L^2(dv_h)}$ where

  $$\mathcal{D}_\phi = -\partial_{\bar z}^* e^{-2\phi} \partial_{\bar z}$$

  is the **Witten Laplacean**.
- Thus **conditionally on ϕ**, γ is Gaussian with covariance

  $$\mathbb{E}_\phi \overline{\gamma(u)} \gamma(v) = \mathcal{D}_\phi^{-1}(u,v) = \frac{1}{k} \int \overline{\partial_{\bar z}^{-1}(u,z)} \partial_{\bar z}^{-1}(v,z) e^{2\phi(z)} d^2 z.$$

- Total mass of this Gaussian is Ray-Singer determinant (on genus 0)

  $$\det{}' \mathcal{D}_\phi^{-1} = e^{\frac{2}{\pi} \int_{\mathbb{C}} |\partial_{\bar z} \phi|^2 d^2 z} e^{-\frac{1}{4\pi} \int \phi R_h dv_h} \det{}' \mathcal{D}_0^{-1}$$

- Hence k is **shifted** to $k-2$.

Renormalisation

Rescaling $\phi \to b\phi$ with $b = \frac{1}{\sqrt{k-2}}$ we make the

Definition The H_3 path integral for $\Sigma = \hat{\mathbb{C}}$ is defined by

$$\langle F(\phi,\gamma)\rangle_{H_3} := \det{}'(\Delta)^{-\frac{3}{2}} \int_{\mathbb{R}\times\mathbb{C}} \mathbb{E}(\mathbb{E}_{X_h} e^{-\frac{b}{4\pi}\int \phi R_h dv_h} F(b\phi,\gamma))\, dc\, d^2\gamma_0$$

where $\phi = c + X_h$ and $\gamma = \gamma_0 + \Gamma$ and conditionally on X_h

$$\mathbb{E}_{X_h}\overline{\Gamma(u)}\Gamma(v) = \frac{1}{\pi^2}\int \frac{1}{u-z}\frac{1}{\bar{v}-\bar{z}} e^{2b\phi} d^2 z.$$

where $e^{2b\phi} d^2 z$ denotes the **GMC measure**.

Theorem (C.Guillarmou &A.K& R. Rhodes 2025) $\langle - \rangle_{H_3}$ defines a CFT with central charge $c = \frac{3k}{k-2}$ and primary fields

$$W_{j,\mu}(z) = e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(z)} e^{2(j+1)\phi(z)}, \quad j,\mu \in \mathbb{C}$$

H_3 - Liouville correspondence

Theorem (GKR2024) The correlation functions of H_3 CFT can be expressed in terms of those of Liouville theory as

$$\langle \prod_{i=1}^{N} W_{j_i,\mu_i}(z_i) \rangle_{H_3} = F(\mathbf{z},\mathbf{y}) \langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{N-2} V_{-\frac{1}{b}}(y_j) \rangle_L$$

where $F(\mathbf{z},\mathbf{y})$ is explicit and $\alpha_i = 2b(j_i + 1) + \frac{1}{b}$.

- Similar formuli hold for all $\mathrm{genus}(\Sigma) > 0$ and also for bb^* twisted by an arbitrary rank two holomorphic vector bundle on Σ ($=$ gauge field)
- These formuli generalise those conjectured by Ribault, Teschner, Hikida, Schomerus.

(Semi) Classical limit

Path integrals can be studied by **large deviation** methods as parameters tend to 0 or ∞. In the Liouville theory this means $b \to 0$ or $b \to \infty$. This leads to interesting geometry.

Classical limit: $k \to \infty \Leftrightarrow b \to 0$

Extremiser of the classical Liouville action is a metric $e^{2b\phi}|dz|^2$ on the surface Σ with constant (negative) curvature (Picard, Poincare). Vertex operators create conical singularities in the metric.

Extremiser of classical H_3 action is a hermitean metric on a rank 2 holomorphic vector bundle whose unitary connection is flat (Donaldson). The vertex operators create parabolic structures at z_i (Taktajan).

Semiclassical limit $=$ critical limit: $k \to 2 \Leftrightarrow b \to \infty$

Semiclassical limit of the H_3-Liouville correspondence is argued (Gaiotto, Teschner) to give rise to the analytic Langlands correspondence of Etingof, Frenkel and Kazhdan

Hence H_3-Liouville correspondence has been called "quantum analytic Langlands correspondence" (Teschner)

Outlook

GFF and GMC seem to be basic ingredients of probabilistic 2d CFT

- Toda CFT
- **Imaginary** Liouville theory can be defined in terms of imaginary GMC and should describe Potts models with $c < 1$.
- Sine-Gordon-Liouville theory and 2d black hole CFT
- Supersymmetric Liouville theory

Challenge: $d > 2$!

Thank You!

Map to Liouville theory

Need to compute

$$\mathbb{E}\left(\mathbb{E}_X[e^{-\frac{b}{4\pi}\int \phi R_h dv_h}\prod_{i=1}^{N} e^{\mu_i\gamma(z_i)-\bar{\mu}_i\overline{\gamma(z_i)}}e^{\alpha_i\phi(z_i)})]\right)$$

The conditional γ-expectation is explicit:

$$\mathbb{E}_X\prod_{i=1}^{N} e^{\mu\gamma(z_i)-\bar{\mu}\overline{\gamma(z_i)}} = e^{\gamma_0\sum_i \mu_i-\bar{\gamma}_0\sum_i\bar{\mu}_i)}e^{-\sum_{ij}\bar{\mu}_i\mu_j\mathbb{E}_X\overline{\Gamma_h(z_i)}\Gamma_h(z_j)}$$

The γ_0 integral gives $\delta(\sum_i \mu_i)$ and then using

$$\sum_{ij}\bar{\mu}_i\mu_j\mathbb{E}_X\overline{\Gamma_h(z_i)}\Gamma_h(z_j) = \frac{b}{\pi^2}\sum_{ij}\bar{\mu}_i\mu_j\int\frac{1}{\bar{z}_i-\bar{w}}\frac{1}{z_j-w}e^{2b\phi(w)}d^2w$$

we get

$$\int_{\mathbb{C}}\mathbb{E}_X\prod_{i=1}^{N} e^{\mu\gamma(z_i)-\bar{\mu}\overline{\gamma(z_i)}}d^2\gamma_0 = e^{-\int|\sum_i\frac{\mu_i}{z_i-w})|^2e^{2b\phi(w)}d^2w}$$

Map to Liouville theory

The function

$$B(w) = \sum_{i=1}^{N} \frac{\mu_i}{z_i - w} = \frac{\sum_i \mu_i \prod_{j \neq i}(z_j - w)}{\prod_i (z_i - w)}$$

is meromorphic with N poles at z_i and $N-2$ zeros at some points y_i (since $\sum_i \mu_i = 0$). Hence

$$|B(w)|^2 = C \frac{\prod_{i=1}^{N-2} |w - y_i|^2}{\prod_{i=1}^{N} |w - z_i|^2} = C e^{\sum_{i=1}^{N-2} 2 \log |w - y_i| - \sum_{i=1}^{N} 2 \log |w - z_i|}$$

The Green function $G_h(w, z) = -\log |w - z|$ plus metric dependent terms. Hence

$$|B(w)|^2 \propto e^{-\sum_{i=1}^{N-2} 2G_h(w - y_i) + \sum_{j=1}^{N} 2G_h(w - z_j)}$$

A (Girsanov) shift in the ϕ integral creates vertex operator insertions $V_{-\frac{1}{b}}(y_i)$ and a shift $\alpha_i \to \alpha_i + \frac{1}{b}$ in the $V_{\alpha_i}(z_j)$