Overcoming the curse of dimensionality: from nonlinear Monte Carlo to the training of deep neural networks

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ANNs: For every $d \in \mathbb{N}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$\mathcal{A}(x) = (\max\{x_1, 0\}, \ldots, \max\{x_d, 0\}),$$

let

$$\mathbf{N} = \cup_{L \in \mathbb{N}} \cup_{\ell_0, \ldots, \ell_L \in \mathbb{N}} \left(\times_{n=1}^{L} (\mathbb{R}^{\ell_n \times \ell_{n-1}} \times \mathbb{R}^{\ell_n}) \right),$$

and $\forall\, L \in \mathbb{N}$, $\ell_0, \ldots, \ell_L \in \mathbb{N}$, $U = ((W_1, B_1), \ldots, (W_L, B_L)) \in \times_{n=1}^{L} (\mathbb{R}^{\ell_n \times \ell_{n-1}} \times \mathbb{R}^{\ell_n})$ let $\mathcal{R}_U \colon \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_L}$ and $\mathcal{P}_U \in \mathbb{N}$ satisfy for all $x_0 \in \mathbb{R}^{\ell_0}, \ldots, x_L \in \mathbb{R}^{\ell_L}$ with $\forall\, n \in \{1, \ldots, L\} \colon x_n = \mathcal{A}(W_n x_{n-1} + B_n)$ that

$$\mathcal{R}_U(x_0) = W_L x_{L-1} + B_L, \qquad \mathcal{P}_U = \textstyle\sum_{n=1}^{L} (\ell_n \ell_{n-1} + \ell_n).$$

Theorem (Hutzent.-J-Kruse-Nguyen 2020 *PDEA*; Ackerm.-Kruse-Kuck.-J-Padgett 2024)

Let $T, \kappa > 0$, *let* $f \colon \mathbb{R} \to \mathbb{R}$ *be Lipschitz,* $\forall\, d \in \mathbb{N}$ *let* $g_d \in C^1(\mathbb{R}^d, \mathbb{R})$ *and* $u_d \colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *be an at most poly. grow. solution of*

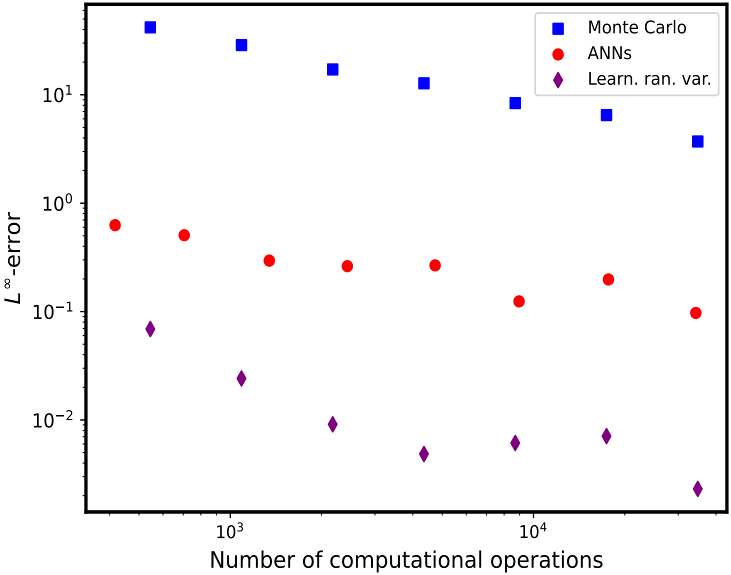
$$\tfrac{\partial u_d}{\partial t} = \Delta_x u_d + f(u_d), \qquad u_d(0, \cdot) = g_d$$

with $|g_d(x)| + \|(\nabla g_d)(x)\| \leq \kappa d^\kappa (1 + \|x\|^\kappa)$, *and assume* $\forall\, d \in \mathbb{N}, \varepsilon \in (0, 1] \colon \exists\, G \in \mathbf{N} \colon \forall\, x \in \mathbb{R}^d \colon |g_d(x) - \mathcal{R}_G(x)| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa)$ *and* $\mathcal{P}_G \leq \kappa d^\kappa \varepsilon^{-\kappa}$.
Then $\exists\, c > 0 \colon \forall\, d \in \mathbb{N}, \varepsilon \in (0, 1] \colon \exists\, U \in \mathbf{N}$:

$$\int_{[0,T] \times [0,1]^d} |u_d(y) - \mathcal{R}_U(y)|\, dy \leq \varepsilon \qquad \text{and} \qquad \mathcal{P}_U \leq c\, d^c \varepsilon^{-c}.$$

Hutzenthaler-J-Kruse-Nguyen-von Wurstemberger 2020 *PRSA*,
Gonon-Graeber-J 2023, Becker-J-Müller-von Wurstemberger 2023 *Math. Finance*

$x \in [90, 110], \quad T \in [\frac{1}{100}, 1], \quad r \in [-\frac{1}{10}, \frac{1}{10}], \quad \sigma \in [\frac{1}{100}, \frac{1}{2}], \quad K \in [90, 110]$



(addition, substraction, multiplication, division, exponential, GeLU)

Theorem (Hannibal, J, & Thang 2024; J & Riekert 2024 SIAM JUQ (to appear))

Let $d \in \mathbb{N}$, $a \in \mathbb{R}$, $b > a$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be prob. sp., for every* $m, n \in \mathbb{N}_0$ *let* $X_n^m \colon \Omega \to [a, b]^d$ *and* $Y_n^m \colon \Omega \to \mathbb{R}$ *be RVs, assume for all* $i \in \mathbb{N}$, $j \in \mathbb{N} \backslash \{i\}$ *that* $\mathbb{P}(X_0^i = X_0^j) = 0$, *for every* $k \in \mathbb{N}_0$ *let* $\mathfrak{d}_k, L_k \in \mathbb{N} \backslash \{1\}$, $\mathbf{l}_k = (\mathbf{l}_k^0, \ldots, \mathbf{l}_k^{L_k}) \in \mathbb{N}^{L_k+1}$ *satisfy* $\mathbf{l}_k^0 = d$, $\mathbf{l}_k^{L_k} = 1$, *and* $\mathfrak{d}_k = \sum_{i=1}^{L} \mathbf{l}_k^i (\mathbf{l}_k^{i-1} + 1)$, *for every* $k, n \in \mathbb{N}_0$, $\theta \in \mathbb{R}^{\mathfrak{d}_k}$ *let* $\mathcal{N}_k^{v,\theta} = (\mathcal{N}_{k,1}^{v,\theta}, \ldots, \mathcal{N}_{k,\mathbf{l}_k^v}^{v,\theta}) \colon \mathbb{R}^d \to \mathbb{R}^{\mathbf{l}_k^v}$, $v \in \{0, 1, \ldots, L_k\}$, *satisfy for all* $x \in \mathbb{R}^d$, $i \in \{1, \ldots, \mathbf{l}_k^{v+1}\}$ *that*

$$\mathcal{N}_{k,i}^{v+1,\theta}(x) = \theta_{\mathbf{l}_k^{v+1}\mathbf{l}_k^v + i + \sum_{h=1}^{v} \mathbf{l}_k^h(\mathbf{l}_k^{h-1}+1)} + \sum_{j=1}^{\mathbf{l}_k^v} \theta_{(i-1)\mathbf{l}_k^v + j + \sum_{h=1}^{v} \mathbf{l}_k^h(\mathbf{l}_k^{h-1}+1)} \big(x_j \mathbb{1}_{\{0\}}(v) + \max\{\mathcal{N}_{k,j}^{v,\theta}(x), 0\} \mathbb{1}_{\mathbb{N}}(v)\big),$$

for every $k, n \in \mathbb{N}_0$ *let* $M_n^k \in \mathbb{N}$, $\gamma_n^k \in \mathbb{R}$, *let* $\mathcal{R}_n^k \colon \mathbb{R}^{\mathfrak{d}_k} \times \Omega \to \mathbb{R}$ *satisfy*

$$\mathcal{R}_n^k(\theta) = \tfrac{1}{M_n^k} \Big[\sum_{m=1}^{M_n^k} |\mathcal{N}_k^{L_k,\theta}(X_n^m) - Y_n^m|^2\Big],$$

let $\mathfrak{G}_n^k \colon \mathbb{R}^{\mathfrak{d}_k} \times \Omega \to \mathbb{R}^{\mathfrak{d}_k}$ *be generalized gradient of* $\mathcal{R}_n^k$, *and let* $\Theta_n^k \colon \Omega \to \mathbb{R}^{\mathfrak{d}_k}$ *be RV, assume* $\forall\, k, n \in \mathbb{N} \colon \Theta_n^k = \Theta_{n-1}^k - \gamma_n^k \mathfrak{G}_n^k(\Theta_{n-1}^k)$, $\liminf_{k\to\infty} \mathbf{l}_k^1 = \infty$, *and* $\liminf_{k\to\infty} \mathbb{P}\big(\inf_{\theta \in \mathbb{R}^{\mathfrak{d}_k}} \mathcal{R}_0^k(\theta) > 0\big) = 1$, *let* $(c_k)_{k\in\mathbb{N}} \subseteq (0, \infty)$ *satisfy for all* $k \in \mathbb{N}$ *that* $c_k \Theta_0^k$ *is standard normal. Then*

$$\liminf_{k\to\infty} \mathbb{P}\Big(\inf_{n\in\mathbb{N}_0} \mathcal{R}_0^k(\Theta_n^k) > \inf_{\theta\in\mathbb{R}^{\mathfrak{d}_k}} \mathcal{R}_0^k(\theta)\Big) = 1.$$

Theorem (Dereich & J 2024: Convergence rates for Adam; Dereich, Graeber, & J 2024)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be prob. sp., let* $d \in \mathbb{N}$, $a \in \mathbb{R}$, $b > a$, *let* $X_{n,m} \colon \Omega \to [a, b]^d$, $(n, m) \in \mathbb{N}^2$, *be i.i.d. RVs, let* $\mathfrak{l} \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *satisfy for all* $\theta, x \in \mathbb{R}^d$ *that*

$$\mathfrak{l}(\theta, x) = \|\theta - x\|^2,$$

let $\xi \in \mathbb{R}^d$, $p, \varepsilon, \alpha \in (0, \infty)$, $\beta \in (\alpha^2, 1)$, $(\gamma_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ *satisfy* $\limsup_{n\to\infty}(\gamma_n + \gamma_n^{-2}|\gamma_n - \gamma_{n+1}|) = 0$, *for every* $M \in \mathbb{N}$ *let* $\mathcal{M}^M \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^d$, $\mathbb{M}^M \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^d$, *and* $\Theta^M \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^d$ *be stoch. proc. satisfying for all* $n \in \mathbb{N}$, $i \in \{1, \dots, d\}$ *that*

$$\mathcal{M}_0^M = 0, \qquad \mathcal{M}_n^M = \alpha \mathcal{M}_{n-1}^M + \frac{(1-\alpha)}{M}\left[\sum_{m=1}^M (\nabla_\theta \mathfrak{l})(\Theta_{n-1}^M, X_{n,m})\right],$$

$$\mathbb{M}_0^{M,i} = 0, \qquad \mathbb{M}_n^{M,i} = \beta \mathbb{M}_{n-1}^{M,i} + \frac{(1-\beta)}{M^2}\left[\sum_{m=1}^M \left(\tfrac{\partial}{\partial \theta_i}\mathfrak{l}\right)(\Theta_{n-1}^M, X_{n,m})\right]^2,$$

$$\Theta_0^M = \xi, \qquad \Theta_n^{M,i} = \Theta_{n-1}^{M,i} - \gamma_n\big[\varepsilon + [(1-\beta^n)^{-1}\mathbb{M}_n^{M,i}]^{1/2}\big]^{-1}\mathcal{M}_n^{M,i},$$

Then there exist $c > 0$, $(\vartheta_M)_{M\in\mathbb{N}} \subseteq \mathbb{R}^d$ *such that for all* $n \in \mathbb{N}$, $M \in \mathbb{N} \cap [c, \infty)$:

- $\mathbb{P}\big(\limsup_{N\to\infty} \|\Theta_N^M - \vartheta_M\| = 0\big) = 1$,
- $\big(\mathbb{E}[\|\Theta_n^M - \vartheta_M\|^p]\big)^{1/p} \le c\sqrt{\gamma_n}$, *and*
- $\|\vartheta_M - \mathbb{E}[X_{1,1}]\| \le c\, M^{-1}$.

Many thanks for your attention!

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