From correlated to white transport noise in fluid models.

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The Navier-Stokes equations on a domain $\mathcal{O} \in \mathbb{R}^d$, $d = 2, 3$:

$$\begin{cases} \partial_t v = \nu \Delta v + (v \cdot \nabla) v + \nabla p, \\ \mathrm{div}\ v = 0, \end{cases}$$

with suitable boundary conditions and initial condition.

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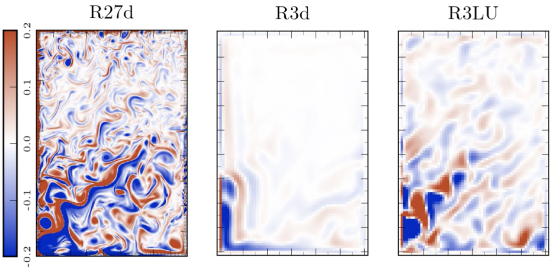
- The starting point of some models trying to represent the small scales is to split the velocity into a large scale component u and the small scales modelled by a white noise in time: $v = u + \dot{\xi}$.
- A white noise in time is delta correlated in time:

.
$$\mathbb{E}(\dot{\xi}(t,x)\dot{\xi}(s,y)) = c(x,y)\delta_{t-s}$$

- This is an idealization of a process which has a small correlation length. It can be approximated by $\frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x)$.
- This assumes a strong (infinite) separation of scales.



- Idealised configuration of the North Atlantic ocean thanks to the primitive equations.
- The figure on the left (resp. center) is done with a fine (resp. coarse) grid and a deterministic equation.
- The figure on the right introduces stochasticity in the coarse simulation through the LU form of the primitive equations. (Li-Mémin)

Classical derivation of Navier-Stokes equations

$$\partial_t X_t = v(t, X_t) \tag{1}$$

A conserved quantities q satisifies:

$$\int_{V_t} q(t,y)dy = \int_{V_0} q(0,y)dy,$$

where V_t is the image of V_0 by the flow of (2).
Using classical arguments, we find that a conserved quantities q satisifies:

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and for an incompressible fluid:

$$\text{div}\, v(t,x) = 0$$

.

Classical derivation of Navier-Stokes equations

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From Newton's law, we obtain the equation for the momentum $q = \rho v$ (take $\rho = 1$):

$$\partial_t v + v \cdot \nabla v = \nu \Delta v + \nabla p.$$

LU model

- For the derivation of LU model (Mikulevicius-Rozovsky, Mémin), we choose $v = u + \dot{\xi}$ with:

$$\dot{\xi}(t,x)dt = \int_{\mathcal{O}} \sigma(x,y)dW(t,y) = d\widetilde{W}(t,x)$$

  a correlated noise in space, $\frac{dW}{dt}$ is a space time white noise.

- Write

$$dX_t = u(t,X_t)dt + d\widetilde{W}(t,X_t)$$

  and compute a stochastic transport using Ito calculus and Ito-Wentzel formula.

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- We obtain a stochastic transport theorem and the stochastic LU Navier-Stokes equation with transport noise (Mémin):

$$du + (u - u_S)\cdot\nabla u\,dt + d\tilde{W}\circ\nabla u = \nu\Delta u\,dt + \nu\Delta d\tilde{W} + \nabla p\,dt,$$

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- $\circ$ is the Stratonovich product:

$$d\tilde{W} o \nabla u = d\tilde{W} \cdot \nabla u + \frac{1}{2}\mathrm{div}(a \cdot \nabla u),$$

with $a_{ij}(x) = \int_{\mathcal{O}} \sigma_{ik}(x, y)\sigma_{kj}(y, x) dy$.

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- The Stratonovich form of the stochastic lagrangian equation for the particles is

$$\begin{aligned} dX_t &= u(t,X_t)dt+d\widetilde{W}(t,X_t) \\ &= u(t,X_t)dt-u_S dt+d^o\widetilde{W}(t,X_t), \end{aligned}$$

with $u_S=\frac{1}{2}\mathrm{div}a$.

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- Thanks to the Stratonovich product, the energy equality holds (formally):

$$\frac{1}{2} d\|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 dt = \nu(u, \Delta d\tilde{W})_{L^2} + \frac{1}{2} C_\sigma dt.$$

Correlated noise

- Replace the time white noise by a correlated noise, *i.e.* do not assume complete separation of scales, and replace $d\widetilde{W}(t, X_t)$ by $\frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, X_t)$:

$$\partial_t X_t = \tilde{u}^\epsilon(t, X_t) + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, X_t),$$

  m is a centered, stationary and ergodic process.

- Then standard calculus can be used to derived a stochastic transport theorem with corraled noise. A conserved quantities q satisifies:

$$\mathbb{D}_t^\epsilon q = \partial_t q(t, x) + \text{div}\ ((\tilde{u}^\epsilon(t, x) + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x))q(t, x)) = 0.$$

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- We have to be careful, the limit $\epsilon \to 0$ will be a Stratonovich equation.

- In
$$\partial_t X_t = \tilde{u}^\epsilon(t, X_t) + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, X_t),$$

- The noise is stationary (in time) and has a zero average:
$$\int_E m(t,x) d\mu(m) = 0.$$
But, this does not imply, that $m(\frac{t}{\epsilon^2}, X_t)$ has zero averaged or that it is decorrelated to $\tilde{u}^\epsilon(t, X_t)$. Contrary to a Ito noise.

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- Assume for simplicity that m has correlation length 1:
$$\frac{1}{\epsilon}\mathbb{E}\left(m(\frac{t}{\epsilon^2}, X_t)\right) = \frac{1}{\epsilon}\mathbb{E}\left(m(\frac{t}{\epsilon^2}, X_t) - m(\frac{t}{\epsilon^2}, X_{t-\epsilon^2})\right)$$

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$$= \frac{1}{\epsilon}\mathbb{E}\left(\int_{t-\epsilon^2}^t \nabla m(\frac{t}{\epsilon^2}, X_s) \cdot \left(\tilde{u}^\epsilon(s, X_s) + \frac{1}{\epsilon} m(\frac{s}{\epsilon^2}, X_s)\right) ds\right)$$

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&\sim \frac{1}{\epsilon^2}\mathbb{E}\left(\int_{t-\epsilon^2}^t \nabla m(\frac{t}{\epsilon^2}, X_t) \cdot m(\frac{s}{\epsilon^2}, X_t) ds\right) \\
&\sim \mathbb{E}\left(\nabla m(\frac{t}{\epsilon^2}, X_t) \cdot L(m)(\frac{t}{\epsilon^2})(X_t)\right),
\end{aligned}$$
where $L(m)(t) = \int_{-\infty}^0 \mathbb{E}(m(t+s))ds$.

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$$\partial_t X_t = \tilde{u}^\epsilon(t, X_t) + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, X_t),$$

The noise has not a zero average (which should be the case for a Ito noise). Assume for simplicity that m has correlation length 1:

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where $L(m)(t) = \int_{-\infty}^{0} \mathbb{E}(m(t+s))ds$. This is the inverse of the generator of m applied to the function $I(m) = m$.
- Finally, using stationarity of m and the fast decorrelation:
$$\frac{1}{\epsilon}\mathbb{E}\left(m(\frac{t}{\epsilon^2}, X_t)\right) \sim \mathbb{E}\left(\int_E \nabla n(X_t) \cdot L(n)(X_t) d\nu(n)\right).$$
- This is precisely the Ito-Stokes drift u_S of the expected limit equation.
- The large scale velocity is in fact $u^\epsilon = \tilde{u}^\epsilon + u_S$.

Stochastic transport theorem with decorraled noise

$$\partial_t X_t = \tilde{u}^\epsilon(t, X_t) + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, X_t).$$

Using classical arguments, we find that a conserved quantities q satisifies:

$$\mathbb{D}_t^\epsilon q = \partial_t q + (\tilde{u}^\epsilon(t, x) + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x)) \cdot \nabla q = 0.$$

From Newton's law, we obtain the equation for the momentum of the large scales (take $\rho = 1$):

$$\begin{aligned}\partial_t u^\epsilon + (u^\epsilon - u_S^\epsilon + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x)) \cdot \nabla u^\epsilon \\ = \nu \Delta [u^\epsilon - u_S^\epsilon + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x)] + \nabla p^\epsilon.\end{aligned}$$

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At the limit $\epsilon \to 0$, we obtain (AD-Hug-Mémin):

$$du + (u - u_S) \cdot \nabla v \, dt + d\tilde{W} o \nabla u = \nu \Delta (u - u_S) dt + \nu \Delta d\tilde{W} + \nabla p \, dt,$$

with a noise $\tilde{W}$ whose correlation operator σ is explicit in terms of m.

Idea of the limit.

Rewrite the equation as:

$$\partial_t u^\epsilon = \mathcal{E}(u^\epsilon, p^\epsilon) - \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x) \cdot \nabla u^\epsilon + \frac{1}{\epsilon} \nu \Delta m(\frac{t}{\epsilon^2}, x)$$

and compute

$$\partial_t \left(u^\epsilon(t,x) - \epsilon \int_{t/\epsilon^2}^{\infty} m(s,x) ds \cdot \nabla u^\epsilon(t,x) \right)$$

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$$\partial_t \left(u^\epsilon(t,x) - \epsilon \int_{t/\epsilon^2}^{\infty} m(s,x) ds \cdot \nabla u^\epsilon(t,x) \right)$$
$$= \mathcal{E}(u^\epsilon, p^\epsilon) + \frac{1}{\epsilon} \nu \Delta m(\frac{t}{\epsilon^2}, x)$$
$$+ \epsilon \int_{t/\epsilon^2}^{\infty} m(s,x) ds \cdot \nabla \mathcal{E}(u^\epsilon, p^\epsilon)$$
$$+ \int_{t/\epsilon^2}^{\infty} m(s,x) ds \cdot \nabla \left((m(\frac{t}{\epsilon^2}, x) \cdot \nabla u^\epsilon + \nu \Delta m(\frac{t}{\epsilon^2}, x) \right)$$

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$$- \int_E L(n)(x) \nabla (n(x) \cdot \nabla u^\epsilon + \nu \Delta n) d\nu(n)$$
$$+ \int_E L(n)(x) \nabla (n(x) \cdot \nabla u^\epsilon + \nu \Delta n) d\nu(n)$$

Remarks

- The proof uses the perturbed test function method based on Markov generators, correctors, martingale problems ....
- The limit in

$$\partial_t u^\epsilon + (u^\epsilon - u_S^\epsilon + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x)) \cdot \nabla u^\epsilon = \nu \Delta [u^\epsilon - u_S^\epsilon + \frac{1}{\epsilon} m(\frac{t}{\epsilon^2}, x)] + \nabla p^\epsilon.$$

  can also be done using rough path theory (Hofmanova-Leahy-Nilsen) but we need to assume more on the small scales: they should converge to a white noise in the sense of rougth paths.
- The method is very robust and can be applied to many fluid models.

The perturbed test function method

Let us illustrate this method on a model inspired by works by A. Majda, P. Kramer, I. Timorfeyev, E. Van den Eijden, F. Flandoli proposed to study a multiscale fluid equation:

$$\begin{cases} \partial_t u = \nu \Delta u + (u+v)\cdot \nabla u + \nabla p \\ dv = (\nu \Delta v + \frac{1}{\varepsilon} C v) dt + (u+v)\cdot \nabla v dt + \nabla q + \frac{1}{\varepsilon}\phi dW, \end{cases}$$

- With the notations

  $$H = \{u \in (L^2(\mathcal{O}))^d,\ \mathrm{div} u = 0,\ u \cdot n = 0 \text{ on } \partial\mathcal{O}\},$$

  P the Leray projector on H, the Stokes operator

  $$A = \nu P \Delta \text{ on } D(A) = (H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}))^d \cap H$$

  and $b(u,v) = P(u \cdot \nabla v)$.
- Rewrite the equation as:

  $$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = (Av + \frac{1}{\varepsilon} C v) dt + b(u+v,v) dt + \frac{1}{\varepsilon}\phi dW, \end{cases}$$

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- The term $\frac{1}{\varepsilon}Cv$ mimics the separation of scales.
- Formally, when $\varepsilon \to 0$, we get $v = (-C)^{-1}\phi dW$ and the Navier-Stokes equation with transport noise:

$$du = Au + b(u,u) + b^o((-C)^{-1}\phi dW, u).$$

This misses a term.

The perturbed test function method

Consider first the simpler case:

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = \frac{1}{\varepsilon} Cvdt + \frac{1}{\varepsilon}\phi dW, \end{cases}$$

- $v(t) = e^{\frac{C}{\varepsilon}t}v_0 + \frac{1}{\varepsilon}\int_0^t e^{\frac{C}{\varepsilon}(t-s)}\phi dW(s).$
- $$\begin{aligned} \mathbb{E}(|v(t)|_H^2) &= |e^{\frac{C}{\varepsilon}t}v_0|_H^2 + \frac{1}{\varepsilon^2}\int_0^t \|e^{\frac{C}{\varepsilon}(t-s)}\phi\|_{\mathcal{L}_2(H)}^2 ds \\ &= |e^{\frac{C}{\varepsilon}t}v_0|_H^2 + \frac{1}{\varepsilon}\int_0^{t/\varepsilon} \|e^{Ct}\phi\|_{\mathcal{L}_2(H)}^2 ds. \end{aligned}$$

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- v is of order $\varepsilon^{-1/2} \rightsquigarrow w = \varepsilon^{1/2} v$ and

$$\begin{cases} \partial_t u = Au + b(u,u) + \frac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \frac{1}{\varepsilon} Cwdt + \frac{1}{\varepsilon^{1/2}}\phi dW, \end{cases}$$

Generators

- v is of order $\varepsilon^{-1/2} \rightsquigarrow w = \varepsilon^{1/2} v$ and

$$\begin{cases} \partial_t u = Au + b(u,u) + \dfrac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \dfrac{1}{\varepsilon} Cwdt + \dfrac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

- Then $U_\varepsilon(x,y,t) = \mathbb{E}(\varphi(u(x,y,t), w(x,y,t)))$ satisfies

$$\frac{dU_\varepsilon}{dt} = \mathcal{L}_\varepsilon U_\varepsilon, \quad U_\varepsilon(0) = \varphi.$$

with the generator:

$$\mathcal{L}_\varepsilon \varphi(x,y) = \langle Ax + b(x,x), D_x\varphi(x,y)\rangle + \frac{1}{\varepsilon^{1/2}} \langle b(y,x), D_x\varphi(x,y)\rangle$$

$$+ \frac{1}{\varepsilon} \langle Cy, D_y\varphi(x,y)\rangle + \frac{1}{2\varepsilon} Tr(\phi^2 D_{yy}^2 \varphi(x,y)).$$

Perturbed test function method

$$\begin{cases} \partial_t u = Au + b(u,u) + \frac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \frac{1}{\varepsilon} Cwdt + \frac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

$$\begin{aligned} \mathcal{L}_\varepsilon \varphi(u,w) &= \langle Au + b(u,u), D_u\varphi(u,w)\rangle + \frac{1}{\varepsilon^{1/2}} \langle b(w,u), D_u\varphi(u,w)\rangle \\ &+ \frac{1}{\varepsilon}\mathcal{L}_w\varphi(u,w) \end{aligned}$$

with $\mathcal{L}_w\varphi(u,w) = \langle Cw, D_w\varphi(u,w)\rangle + \frac{1}{2} Tr(\phi^2 D^2_{ww}\varphi(u,w))$.

- We want to find the limit equation for u and take a test function $\varphi(u)$. The goal is to get rid of the singular terms and of the dependence in w.
- Use correctors: $\varphi_\varepsilon(u,w) = \varphi(u) + \varepsilon^{1/2}\varphi_1(u,w) + \varepsilon\varphi_2(u,w)$

Perturbed test function method

$$\mathcal{L}_\varepsilon \varphi(u,w) = \langle Au + b(u,u), D_u\varphi(u,w)\rangle + \frac{1}{\varepsilon^{1/2}}\langle b(w,u), D_u\varphi(u,w)\rangle + \frac{1}{\varepsilon}\mathcal{L}_w\varphi(u,w)$$

with $\mathcal{L}_w\varphi(u,w) = \langle Cw, D_w\varphi(u,w)\rangle + \frac{1}{2}Tr(\phi^2 D^2_{ww}\varphi(u,w))$.

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-

$$\begin{aligned}\mathcal{L}_\varepsilon \varphi_\varepsilon(u,w) &= \langle Au + b(u,u), D_u\varphi(u)\rangle + \frac{1}{\varepsilon^{1/2}}\langle b(w,u), D_u\varphi(u)\rangle \\ &+\varepsilon^{1/2}\langle Au + b(u,u), D_u\varphi_1(u,w)\rangle + \langle b(w,u), D_u\varphi_1(u,w)\rangle \\ &+\frac{1}{\varepsilon^{1/2}}\mathcal{L}_w\varphi_1(u,w) \\ &+\varepsilon\langle Au + b(u,u), D_u\varphi_2(u,w)\rangle + \varepsilon^{1/2}\langle b(w,u), D_u\varphi_2(u,w)\rangle \\ &+\mathcal{L}_w\varphi_2(u,w)\end{aligned}$$

Perturbed test function method

$$\begin{cases} \partial_t u = Au + b(u,u) + \frac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \frac{1}{\varepsilon} Cw\,dt + \frac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

$$\begin{aligned} \mathcal{L}_\varepsilon \varphi(u,w) &= \langle Au + b(u,u), D_u\varphi(u,w)\rangle + \frac{1}{\varepsilon^{1/2}} \langle b(w,u), D_u\varphi(u,w)\rangle \\ &+ \frac{1}{\varepsilon} \mathcal{L}_w \varphi(u,w) \end{aligned}$$

with $\mathcal{L}_w \varphi(u,w) = \langle Cw, D_w\varphi(u,w)\rangle + \frac{1}{2} Tr(\phi^2 D^2_{ww}\varphi(u,w))$.

- $\varphi_\varepsilon(u,w) = \varphi(u) + \varepsilon^{1/2}\varphi_1(u,w) + \varepsilon\varphi_2(u,w)$
  $\rightsquigarrow (b(w,u), D_u\varphi(u)) + \mathcal{L}_w\varphi_1(u,w) = 0.$
- This is a Poisson equation. We know that $Im\mathcal{L}_w = \{\psi : H \to \mathbb{R},\ \int_H \psi(v)\nu(dv) = 0\}$, where ν is the invariant measure of $dv = Cv\,dt + \phi dW$.

Perturbed test function method

$$\begin{cases} \partial_t u = Au + b(u,u) + \frac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \frac{1}{\varepsilon} Cw\, dt + \frac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

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with $\mathcal{L}_w \varphi(u,w) = \langle Cw, D_w\varphi(u,w)\rangle + \frac{1}{2} Tr(\phi^2 D^2_{ww}\varphi(u,w))$.

- $\varphi_\varepsilon(u,w) = \varphi(u) + \varepsilon^{1/2}\varphi_1(u,w) + \varepsilon\varphi_2(u,w)$ $\rightsquigarrow (b(w,u), D_u\varphi(u)) + \mathcal{L}_w\varphi_1(u,w) = 0.$
- This is a Poisson equation. We know that $Im\mathcal{L}_w = \{\psi : H \to \mathbb{R}, \int_H \psi(v)\nu(dv) = 0\}$, where ν is the invariant measure of $dv = Cv\, dt + \phi dW$. $\rightsquigarrow \varphi_1(u,w) = (b((-C)^{-1}w, u), D_u\varphi(u))$.

Perturbed test function method

$$\begin{aligned}\mathcal{L}_\varepsilon\varphi(u,w) &= \langle Au+b(u,u), D_u\varphi(u,w)\rangle + \frac{1}{\varepsilon^{1/2}}\langle b(w,u), D_u\varphi(u,w)\rangle \\ &+\frac{1}{\varepsilon}\mathcal{L}_w\varphi(u,w)\end{aligned}$$

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$$\begin{aligned}\mathcal{L}_\varepsilon\varphi_\varepsilon(u,w) &= \langle Au+b(u,u), D_u\varphi(u)\rangle \\ &+\varepsilon^{1/2}\langle Au+b(u,u), D_u\varphi_1(u,w)\rangle + \langle b(w,u), D_u\varphi_1(u,w)\rangle \\ &+\varepsilon\langle Au+b(u,u), D_u\varphi_2(u,w)\rangle + \varepsilon^{1/2}\langle b(w,u), D_u\varphi_2(u,w)\rangle \\ &+\mathcal{L}_w\varphi_2(u,w).\end{aligned}$$

$$\rightsquigarrow \langle b(w,u), D_u\varphi_1(u,w)\rangle + \mathcal{L}_w\varphi_2(u,w) = \int_H \langle b(w,u), D_u\varphi_1(u,w)\rangle\,\nu(dw)$$

.

Perturbed test function method

$$\begin{cases} \partial_t u = Au + b(u,u) + \dfrac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \dfrac{1}{\varepsilon} Cwdt + \dfrac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

- $\varphi_\varepsilon(u,w) = \varphi(u) + \varepsilon^{1/2}\varphi_1(u,w) + \varepsilon\varphi_2(u,w)$

$$\begin{aligned} \mathcal{L}_\varepsilon \varphi_\varepsilon(u,w) = \; & \langle Au + b(u,u), D_u\varphi_\varepsilon(u,w)\rangle \\ & + \int_H \left\langle D_u\varphi(u), b((-C)^{-1}y, b(y,u))\right\rangle d\nu(y) \\ & \quad + \int_H \left\langle D^2_{uu}\varphi(u)\cdot b(y,u), b((-C)^{-1}y,u)\right\rangle d\nu(y) \\ & \qquad + O(\varepsilon^{1/2}) \\ & = \mathcal{L}_0\varphi(u) + O(\varepsilon^{1/2}). \end{aligned}$$

$$\rightsquigarrow du = Au + b(u,u) + b^{o}((-C)^{-1}\phi dW, u).$$

The full problem

- Split $v = r + \varepsilon^{1/2} w$:

$$\begin{cases} \partial_t u = Au + b(u + \varepsilon^{-1/2} w + r, u), \\ dw = \varepsilon^{-1}(\varepsilon A w + C w) dt + \varepsilon^{-1/2} \phi dW, \\ \partial_t r = \varepsilon^{-1}(\varepsilon A r + C r) dt + b(u + \varepsilon^{-1/2} w + r, \varepsilon^{-1/2} w + r). \end{cases}$$

- An averaging phenomenon appears for r, we expect that it converges to

$$\bar{r} = (-C)^{-1} \int_H b(w, w) d\nu(w).$$

- This is the Ito-Stokes drift.
- The perturbed test function method is easy to adapt.

Assumptions

- $Tr(-C)^{-1}\phi^2 \ < \infty$.
- There exists $\Gamma \geq \gamma > 1/4$ such that for $s \in \mathbb{R},\ \beta > 0$:

$$|x\|^2_{H^{s+\beta\gamma}} \lesssim \|(-C)^{\beta/2}x\|^2_{H^s} \lesssim \|x\|^2_{H^{s+\beta\Gamma}}.$$

- $\nu = \mathcal{N}(0, \frac{1}{2}(-C)^{-1}\phi^2)$. It is supported by H^{s_0} for some s_0 depending on d, Γ:

$$\int_H \|w\|^2_{H^{s_0}}\nu(dw) < \infty.$$

- C and ϕ commute.

Theorem [D.,Pappalettera]
Let $u_0,\ v_0 \in H$ be given. For $\varepsilon > 0$ there exists a weak solution to:

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = \varepsilon^{-1}(\varepsilon Av + Cv)dt + b(u+v,v)dt + \varepsilon^{-1}\phi dW. \end{cases}$$

with initial data $u_0,\ v_0$ which is uniformly bounded in

$$\left(L^\infty(\Omega, C([0,T],H) \cap L^2([0,T],H^1))\right) \times \left(L^2(\Omega, C([0,T],H) \cap L^2([0,T],H^1))\right)$$

The laws of $(u_\varepsilon)_{\varepsilon>0}$ are tight in $L^2(0,T,H) \cap C([0,T],H^{-\beta})$ for $\beta > 0$ and every limit point is a weak solution of

$$du = Au + b(u+\bar{r},u) + b^o((-C)^{-1}\phi dW, u).$$

For $d = 2$, the solutions are probabilistically strong and convergence holds in probability.
Moreover on the torus, if $u_0 \in (H^1(\mathbb{T}^2))^2$, we can take $C = I$,
Remark: The same result can be proved using rough path theory. (D., Hofmanova).

Stochastic variational principle

- Another way to obtain Euler equations is to use a variational principle and write that the velocity and density should be a critical point of:

$$\mathcal{E}(v,\rho,\lambda,p) = \int_0^t \left(\frac{1}{2}\rho|v|^2 + \langle \lambda, \partial_t \rho + \nabla \cdot (\rho v)\rangle - \langle p, \rho - 1\rangle\right) dt$$

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- SALT (Holm): write that v splits into $u + \frac{d\tilde{W}}{dt}$.
- Replace $\frac{1}{2}\rho|v|^2$ by $\frac{1}{2}\rho|u|^2$.
- Consider a Stratonovich product in the term $\nabla \cdot (\rho v)$
- We obtain

$$du = u\nabla u dt + \nabla p dt + d\tilde{W} \circ \nabla u + \nabla d\tilde{W} \,\bar{\circ}\, u.$$

Where $(\nabla v \,\bar{\circ}\, w)_k = \sum_\ell (\partial_{x_k} v_\ell) w_\ell$.

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- We want to work with a correlated noise and smooth the white noise. Define $\xi^\epsilon(t,x) = \sum_i \int_{t-\epsilon}^{t+\epsilon} h((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s)$, with $\tilde{W}(t,x) = \sum_i \xi_i(x,t)\beta_i(t)$. Write $v = u + \xi^\epsilon$.
- If one chooses $\frac{1}{2}\rho|u|^2$ in the energy, a regularized SALT Euler stochastic equation is obtained.

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- With the true energy, we obtained the Euler equation for v:

$$\partial_t(u + \xi^\epsilon) + (u + \xi^\epsilon) \cdot \nabla(u + \xi^\epsilon) = -\nabla p.$$

The limit $\epsilon \to 0$ is problematic ...

Stochastic variational principle

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$$\mathcal{E}(v,\rho,\lambda,p)=\int_0^t\left(\frac{1}{2}\rho|v|^2+\langle\lambda,\partial_t\rho+\nabla\cdot(\rho v)\rangle-\langle p,\rho-1\rangle\right)dt$$

$$v=u+\xi^\epsilon,\quad \xi^\epsilon(t,x)=\sum_i\int_{t-\epsilon}^{t+\epsilon}h((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s).$$

- $$\partial_t u+\partial_t\xi^\epsilon+(u+\xi^\epsilon)\cdot\nabla u+u\cdot\nabla\xi^\epsilon+\xi^\epsilon\cdot\nabla\xi^\epsilon=-\nabla p.$$

  Getting rid of the colored terms we obtain the LU stochastic equations (without the Ito-Stokes drift):

$$du+u\cdot\nabla u\,dt+dWo\nabla u=-\nabla q\,dt.$$

Stochastic variational principle

- We now want to use a variational principle to get information on the noise, more precisely on the ξ_i's:

$$\tilde{W}(t,x) = \sum_i \xi_i(x,t)\beta_i(t)$$

.

- Since they influence only the law, it is natural to work with averaged quantities. Take the functional:

$$\mathcal{S}(u,\rho,(\xi)_i,\lambda,p)$$
$$= \mathbb{E}\left(\int_0^t \left(\frac{1}{2}\rho|u+\xi^\epsilon|^2 + \langle \lambda, \partial_t\rho + \nabla\cdot(\rho(u+\xi^\epsilon))\rangle - \langle p, \rho - 1\rangle\right) dt\right)$$

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with $\xi^\epsilon(t,x) = \sum_i \int_{t-\epsilon}^{t+\epsilon} h((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s)$.

- Because of the correlation of the noise and of the other variables, this functional is complicated and it is difficult to write equations for critical points.

Stochastic variational principle

- Take the functional:

$$\mathcal{S}(u,\rho,(\xi)_i,\lambda,p)$$
$$= \mathbb{E}\left(\int_0^t \left(\frac{1}{2}\rho|u+\xi^\epsilon|^2 + \langle\lambda, \partial_t\rho + \nabla\cdot(\rho(u+\xi^\epsilon))\rangle - \langle p, \rho-1\rangle\right)dt\right)$$

with $\xi^\epsilon(t,x) = \sum_i \int_{t-\epsilon}^{t+\epsilon} h((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s)$.

- We want to write an equivalent expression with uncorrelated quantities.
- We have seen that it is natural to replace the energy term by

$$\frac{1}{2}|u-u_S+\tilde{\xi}^\epsilon|^2$$

where $\tilde{\xi}^\epsilon$ and $u-u_S$ are uncorrelated. We choose:

$$\tilde{\xi}^\epsilon(t,x) = \sum_i \int_t^{t+\epsilon} \tilde{h}((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s).$$

Stochastic variational principle

- Take the functional:

$$\mathcal{S}(u,\rho,(\xi)_i,\lambda,p) = \mathbb{E}\left(\int_0^t \left(\frac{1}{2}\rho|u+\xi^\epsilon|^2 + \langle\lambda, \partial_t\rho + \nabla\cdot(\rho(u+\xi^\epsilon))\rangle - \langle p, \rho - 1\rangle\right) dt\right)$$

with $\xi^\epsilon(t,x) = \sum_i \int_{t-\epsilon}^{t+\epsilon} h((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s)$.

- We want to write an equivalent expression with uncorrelated quantities. Assume

$$d\xi_i = \mu_i dt + \sum_j \int_t^{t+\epsilon} \tilde{h}_\epsilon(t-s)\Lambda^i_j(t)d\beta^j_s.$$

notice that up to negligible terms:

$$\partial\rho + \nabla\cdot(\rho(u+\xi^\epsilon)$$
$$\sim \partial_t\rho + \nabla\cdot(\rho u) + \frac{1}{2}\sum_i \nabla\cdot(\Lambda^i_i\rho) - \frac{1}{2}\nabla\cdot(\xi_i\nabla\cdot(\xi_i\rho))$$
$$+ \sum_i \nabla\cdot(\rho\xi_i(t))\int_t^{t+\epsilon}\tilde{h}_\epsilon(t-s)d\beta^i_s.$$

Stochastic variational principle

- Take the functional:

$$\mathcal{S}(u,\rho,(\xi)_i,\lambda,p)$$
$$= \mathbb{E}\left(\int_0^t \left(\frac{1}{2}\rho|u+\xi^\epsilon|^2 + \langle \lambda, \partial_t \rho + \nabla\cdot(\rho(u+\xi^\epsilon))\rangle - \langle p, \rho - 1\rangle\right) dt\right)$$

with $\xi^\epsilon(t,x) = \sum_i \int_{t-\epsilon}^{t+\epsilon} h((t-s)/\epsilon)\xi_i(x,t)d\beta_i(s)$.

- Assume that all quantities decompose into decorrelated large and small scales. For instance:

$$\rho = \bar{\rho} + \sum_i \rho_i \int_t^{t+\epsilon} \tilde{h}_\epsilon(t-s)d\beta_s^i,$$

similar for λ, p.

Stochastic variational principle

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$$\mathcal{S}(u, \rho, (\xi)_i, \lambda, p) \\ = \mathbb{E}\left(\int_0^t \left(\frac{1}{2}\rho|u+\xi^\epsilon|^2 + \langle \lambda, \partial_t \rho + \nabla\cdot(\rho(u+\xi^\epsilon))\rangle - \langle p, \rho - 1\rangle\right) dt\right)$$

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similar for $\lambda,\ p$.

- Everything can be nicely expressed and after (lengthy) computations we obtain (with $u_S = 0$ for simplicity):

$$\partial_t \xi_i + (u\cdot\nabla)\xi_i + (\xi_i\cdot\nabla)u = -\nabla p_i,\ \nabla\cdot\xi_i = 0.$$

Stochastic variational principle

- To sum up we obtain the following large scale Euler equation, with an explicit expression of the small-scale component evolution (AD-Mémin):

$$
\begin{aligned}
&\partial_t u + \big((u + \xi^\epsilon)\cdot\nabla\big)u = -\nabla p,\\
&\nabla\cdot u = 0,\\
&\xi^\epsilon = \sum_i \int_{t-\epsilon}^{t+\epsilon} h_\epsilon(t-s)\xi_i(x,t)\,d\beta_s^i, \text{ with}\\
&\partial_t \xi_i + (u\cdot\nabla)\xi_i + (\xi_i\cdot\nabla)u = -\nabla p_i,\\
&\nabla\cdot\xi_i = 0.
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$$
\begin{aligned}
&\partial_t u + \big((u + \xi^\epsilon) \cdot \nabla\big) u = -\nabla p, \\
&\nabla \cdot u = 0, \\
&\xi^\epsilon = \sum_i \int_{t-\epsilon}^{t+\epsilon} h_\epsilon(t-s) \xi_i(x,t) d\beta_s^i, \text{ with} \\
&\partial_t \xi_i + (u \cdot \nabla)\xi_i + (\xi_i \cdot \nabla) u = -\nabla p_i, \\
&\nabla \cdot \xi_i = 0.
\end{aligned}
$$

Letting $\epsilon \to 0$ we then obtain the stochastic LU system:

$$
\begin{aligned}
&d_t u + \Big(u dt + \sum_i \xi_i \circ d\beta^i\Big) \cdot \nabla u = -\nabla p dt, \\
&\nabla \cdot u = \nabla \cdot \xi_i = 0, \\
&\partial_t \xi_i + (u \cdot \nabla)\xi_i + (\xi_i \cdot \nabla) u = -\nabla q_i.
\end{aligned}
$$

(PhD Moskowitz).

The Ornstein-Uhlenbeck case

- Let us consider the particular smoothing of the noise:

$$\xi^\epsilon = \sum_i \xi^{\epsilon,i} = \sum_i \xi_i(t) Z_t^{\epsilon,i} = \sum_i \xi_i(t) \int_t^{t+\epsilon} e^{\frac{1}{\epsilon}(t-s)} d\beta_s^i.$$

then

$$d_t \xi^{\epsilon,i} = -\left(u \cdot \nabla \xi^{\epsilon,i} + \xi^{\epsilon,i} \cdot \nabla u + \nabla \tilde{q}_i\right) dt - \frac{1}{\epsilon} \xi^{\epsilon,i} dt + \frac{1}{\epsilon} \xi_i d\beta_i,$$

and

$$\begin{aligned} d_t \xi^\epsilon = & -\left(u \cdot \nabla \xi^\epsilon + \xi^\epsilon \cdot \nabla u + \nabla p^\epsilon\right) dt \\ & - \frac{1}{\epsilon} \xi^\epsilon + \frac{1}{\epsilon} \sum_i \xi_i d\beta_t^i. \end{aligned}$$

- This is reminiscent of ideas proposed heuristically by Majda, Timofeyev and Van den Eijden and before by Hasselmann.

Thanks for your attention