Sizes of Up-to-n Halting Testers

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1 Motivation

- Standard proof of **undecidability of halting testing** is
  - hard for my students: looks like a magician's trick to them
  - common target of objections in net and other discussions (well, nothing changes the minds of some people . . .)

$\Rightarrow$ I have wanted a proof that would raise less objections, would "feel better"

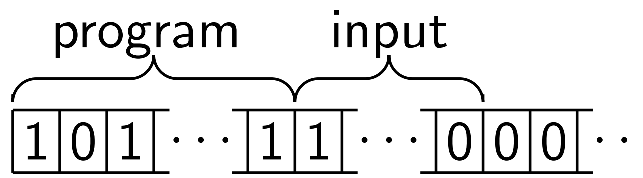
$\Rightarrow$ Got interested in the "Busy Beaver" proof (discussed later)

- Triggered by yet another crazy net discussion, this June I tried to write the Busy Beaver proof in a very clear and convincing form
- Accidentally got an easy but interesting result that I had not known before
- No one else seemed to have seen it either

$\Rightarrow$ So here it comes
  - here formulated using Turing machines
  - programming language version (not here) intended for software people
  - the theorems of the two versions have surprising differences!

2 Up-to-n (3-way) Deciders 1/2

- **Decision problem** $\varphi := \{0,1\}^* \to \{\text{“yes”}, \text{“no”}\}$
- **Decider for φ** := universal Turing machine program that computes φ



- **Up-to-n ($n \in \mathbb{N}$) decider for φ** := universal Turing machine program that
  - if $|\text{input}| \leq n$, then replies $\varphi(\text{input})$
  - if $|\text{input}| > n$, then may do anything: reply wrong, fail to terminate, ...
- **Up-to-n 3-way decider for φ** := universal Turing machine program that
  - if $|\text{input}| \leq n$, then replies $\varphi(\text{input})$
  - if $|\text{input}| > n$, then replies “too big”
- We study families $(D_n)_{n \in \mathbb{N}}$ of up-to-n (3-way) deciders
- For every φ, such a family exists
  - look up the answer in a pre-determined array of $2^0 + 2^1 + \ldots + 2^n$ bits

  $\Rightarrow$ its size is $2^{n+1} + O(n)$ (Section 3 explains $O(n)$)
- φ is decidable if and only if it has an up-to-n decider of size $O(1)$

2 Up-to-n (3-way) Deciders 2/2

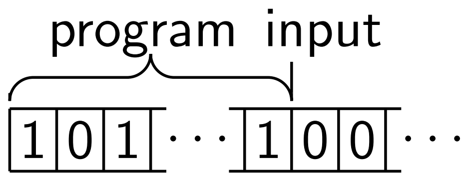
Grey part not in the paper

- Let μ_n be the above-mentioned bit string of answers for $|\text{input}| \leq n$
  - $|\mu_n| = 2^{n+1} - 1$
- |a smallest up-to-n 3-way decider| $\cong$ the s.d.-Kolmogorov complexity of μ_n
  - s.d. $:=$ self-delimiting (Section 3 discusses)

  $\geq$: run D_n for all strings ε, 0, 1, 00, ... until it starts saying "too big"

  $\leq$: construct μ_n, check "too big" against $|\mu_n|$, pick "yes" / "no" from μ_n

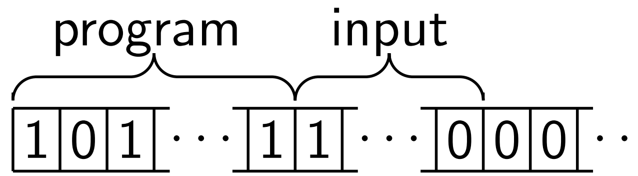
- Let
  - e-i $:=$ empty input $= 0^\omega$
  - lg $:=$ base-2 logarithm



- previous slide: $O(1) \leq |D_n| \leq 2^{n+1} + O(n)$
- **Our main result:** if φ is e-i halting testing, then
  - the size of the smallest up-to-n decider is between $n - \lg n - 2 \lg\lg n - O(1)$ and $n + O(1)$
  - the size of the smallest up-to-n 3-way decider is between $n \pm O(1)$

  $\Rightarrow$ for $\mu_n =$ e-i halting testing answers, s.d.-Kolmogorov$(\mu_n) = \lg|\mu_n| \pm O(1)$

3 Self-Delimiting Representations 1/2



- The program must know where it ends and the input begins
  $\Rightarrow$ it is assumed that programs are **self-delimiting**
  - no proper prefix of a program is a program

- We will need self-delimiting representations for arbitrary $\beta \in \{0,1\}^*$

- $\ell(\beta)$:= the size of the representation of β

- **Theorem** No self-delimiting representation system for all finite bit strings satisfies $\ell(\beta) = |\beta| + \lg|\beta| + O(1)$.
  - if such a system exists, then there is a c such that when $|\beta| \geq 1$, then $\ell(\beta) \leq |\beta| + \lg|\beta| + c$
  - by Kraft's inequality, for any $m \in \mathbb{N}$,

$$1 \geq \sum_{1 \leq |\beta| \leq m} 2^{-\ell(\beta)} = \sum_{n=1}^{m} \sum_{|\beta|=n} 2^{-\ell(\beta)} \geq \sum_{n=1}^{m} \sum_{|\beta|=n} 2^{-(n+\lg n+c)}$$

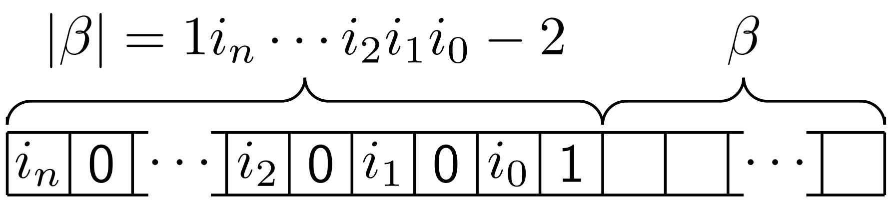
$$= \sum_{n=1}^{m} 2^n 2^{-(n+\lg n+c)} = 2^{-c} \sum_{n=1}^{m} \frac{1}{n}$$

3 Self-Delimiting Representations 2/2

- **Theorem** No self-delimiting representation system for all non-negative integers satisfies $\ell(n) = \lg n + \lg\lg n + O(1)$.
  - when $\beta \in \{0,1\}^*$ and $n > 0$, let $1\beta \leftrightarrow n$, then previous theorem
- Actually, for any $k \in \mathbb{N}$, no representation system satisfies $\ell(n) = \lg n + \lg\lg n + \ldots + (\lg)^k n + O(1)$

$$\sum_{i=2^{n-1}+1}^{2^n} \frac{1}{i \lg i \cdots (\lg)^{k+1} i} \geq \frac{2^{n-1}}{2^n \lg 2^n \cdots (\lg)^{k+1} 2^n} = \frac{1}{2} \frac{1}{n \lg n \cdots (\lg)^k n}$$

- So a non-negative integer n needs more than that many bits!
- A self-delimiting representation system with $\ell(\beta) = |\beta| + 2\lfloor \lg(|\beta| + 2) \rfloor$



- Practical programming languages have $\ell(\beta) = c|\beta| + O(1)$ for some $c > 1$

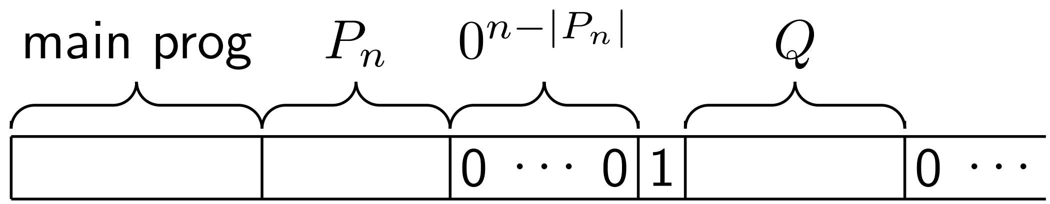
4 Upper Bound

- If n is so small that no Q with $|Q| \leq n$ e-i halts, then H_n is trivial to design
  - **if** $|Q| \leq n$ **then** reply "no" **else** reply "too big"

$\Rightarrow$ From now on we assume that some Q with $|Q| \leq n$ e-i halts

- Let P_n be a slowest e-i halting program of size at most n bits
  - exists, because there are $< 2^{n+1}$ programs of size at most n bits

- H_n and its input Q use the tape like this



  - P_n is self-delimiting, because it is a program

  $\Rightarrow$ main program gets n by finding the end of P_n and then a 1

  $\Rightarrow$ can check if Q is too big

  - then main program e-i executes Q and P_n one step at a time
  - Q terminates first: "yes", P_n terminates first: "no"
  - main program does not depend on n

$\Rightarrow$ An $n + O(1)$ family of up-to-n 3-way e-i halting testers exists

5 Earlier Upper Bound Results

- Knowing n first bits of G. Chaitin's Ω facilitates up-to-n e-i halting testing
  - $\Omega := \sum_{P \text{ e-i halts}} 2^{-|P|}$
  - also his programs are self-delimiting, so $0 < \Omega < 1$
  - simulate all programs (also those $> n$ bits) maintaining a lower approximation of Ω until it matches $\Omega_{1:n}$
- $\Omega_{1:n}$ is not self-delimiting
  $\Rightarrow$ only the bound $n + \lg n + 2 \lg \lg n + O(1)$ obtained
- It is widely known that knowing (the running time of) a slowest e-i halting program of size $\leq n$ facilitates up-to-n e-i halting testing
  - from this, proof of $n + O(1)$ for up-to-n e-i halting testers is immediate
  - I do not know if anyone has carefully (self-delimiting!) formulated it
- Extending the proof to 3-way testers seems new
  - 3-way is important for the lower bounds, this observation seems new
  - 3-way result yields self-delimiting Kolmogorov complexity result

6 Lower Bounds

- Let H_n be any family of up-to-n 3-way e-i halting testers

```
print 1
for β := ε, 0, 1, 00, 01, 10, 11, 000, ... do
  if β is a program then
    r := H_n(β)
    if r = "too big" then halt
    if r = "yes" then e-i simulate β and print its result
```

- The above program
  - is of size $O(1) + |H_n|$
  - e-i halts
  - catenates 1 and the outputs of all e-i halting programs $\leq n$ bits

  $\Rightarrow$ must be of size $> n$

$\Rightarrow$ $|H_n| > n - O(1)$

- If H_n is not 3-way, the program is modified to test $|\beta| > n$ and halt then
  - the program needs a representation of n

  $\Rightarrow$ $\lg n + 2 \lg \lg n + O(1)$ additional bits

$\Rightarrow$ $|H_n| > n - \lg n - 2 \lg \lg n - O(1)$

7 Earlier Lower Bound Results

- For non-3-way testers, the proof is a variant of the Busy Beaver proof of non-existence of halting testers
  - Busy Beaver $:=$ n-state TM that prints as many 1's as possible and halts
  - an $O(\log n)$ bit program prints something that a $< n$ bit program cannot ↯
- G. Chaitin proved in a similar way that to produce $\Omega_{1:n}$, a program of size $n - O(1)$ is needed
  - however, $\Omega_{1:n}$ is affected by e-i halting programs $> n$ bits

  $\Rightarrow$ it is not obvious how H_n would yield $\Omega_{1:n}$

  $\Rightarrow$ no obvious lower bound for $|H_n|$

$\Rightarrow$ A gap in my (and others'?) knowledge on the self-delimiting Kolmogorov complexity of $\Omega_{1:n}$
  - between $n - O(1)$ and $n + \lg n + 2 \lg \lg n + O(1)$

- Although the lower bound results in this talk are simple, it seems that they have not been formulated precisely before

Thank you for attention!