MATS352 Stochastic analysis
Excercise session 6

Monday, 5.5.2014
MaD 380 at 10.15

Notation: $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is a stochastic basis satisfying the usual conditions, and $B = (B_t)_{t\geq 0}$ is a standard $(\mathcal{F}_t)_{t\geq 0}$- Brownian motion.

Complete the proof of Proposition 5.9: Assume $L \in \mathcal{L}_2$.

1. **Uniqueness** (of the limit; independence from the approximating sequence): If there are two sequences $K^{(n)}$ and $L^{(n)}$ with
$$d(K^{(n)}, L) \to 0 \text{ and } d(L^{(n)}, L) \to 0$$
as $n \to \infty$, and for each $t \geq 0$,
$$I_t(K^{(n)}) \underset{L_2}{\to} X_t, \;\; I_t(L^{(n)}) \underset{L_2}{\to} Y_t,$$
then $\mathbb{P}\left(X_t = Y_t\right) = 1$ for all $t \geq 0$.

Hint: Compute $\|X_t - Y_t\|_{L_2}$.

2. **Linearity**: for $a, b \in \mathbb{R}$ and $K, L \in \mathcal{L}_2$,
$$\mathbb{P}\left(J_t(aK + bL) = aJ_t(K) + bJ_t(L) \text{ for all } t \geq 0\right) = 1.$$

3. **Itô isometry**: for $L \in \mathcal{L}_2$,
$$\|J_t(L)\|_{L_2} = \| \left(\int_0^t L_u^2 du \right)^{\frac{1}{2}} \|_{L_2} \text{ for all } t \geq 0.$$

4. **Continuity** (of the mapping $J\colon \mathcal{L}_2 \to \mathcal{M}_2^{c,0}$): if $L^{(n)}, L \in \mathcal{L}_2$ with $d(L^{(n)}, L) \to 0$, then
$$\mathbb{E} \sup_{t\in[0,T]} [J_t(L) - J_t(L^{(n)})]^2 \to 0 \text{ for all } T \geq 0.$$

5. **Uniqueness** (of the mapping $J\colon \mathcal{L}_2 \to \mathcal{M}_2^{c,0}$): if $J'\colon \mathcal{L}_2 \to \mathcal{M}_2^{c,0}$ is another mapping extending $I\colon \mathcal{L}_0 \to \mathcal{M}_2^{c,0}$ to $\mathcal{L}_2$ and satisfying the above conditions (1.-4.), then
$$\mathbb{P}\left(J_t(L) = J'_t(L) \text{ for all } t \geq 0\right) = 1 \text{ for all } L \in \mathcal{L}_2.$$

[From last week:]

6. Let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be Gaussian random variables. Assume that $X_n \underset{L_2}{\to} X$ as $n \to \infty$ for some random variable $X\colon \Omega \to \mathbb{R}$. Show that X is Gaussian.

7. Let $f\colon [0, \infty[\to \mathbb{R}$ be a continuous function. For $t \geq 0$, define
$$X_t := \int_0^t f(u) dB_u.$$
Show that $X = (X_t)_{t\geq 0}$ is a Gaussian process with mean zero and covariance
$$\mathbb{E}(X_s X_t) := \int_0^{\min(s,t)} [f(u)]^2 du.$$

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8. Check that the geometric Brownian motion S defined by
$$S_t := e^{B_t - \frac{t}{2}} \text{ for all } t \geq 0$$
satisfies the following *stochastic differential equation*:
$$dS_t = S_t dB_t,$$
that is,
$$S_t - S_0 = \int_0^t S_u dB_u$$
for all $t \geq 0$.

Hint: Itô's formula

9*. Prove **Lemma 6.6**: Let $Y_n, Z_n, Z\colon \Omega \to \mathbb{R}$ be random variables such that $Y_n \underset{a.s.}{\to} 0$ and $Z_n \underset{\mathbb{P}}{\to} Z$ as $n \to \infty$. Show that $Y_n Z_n \underset{\mathbb{P}}{\to} 0$.

10*. Collect the steps to define the Itô integral.