MATS352 Stochastic analysis
Excercise session 5

Monday, 28.4.2014

MaD 380 at 10.15

Notation: $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is a stochastic basis satisfying the usual conditions, and $B = (B_t)_{t\geq 0}$ is a standard $(\mathcal{F}_t)_{t\geq 0}$- Brownian motion.

1. Show that for simple processes, stochastic integral with respect to B is linear (Proposition 5.5).

2. For $t \geq 0$, let $X_t := \int_0^t B_s^2 ds$.
   (a) Show that X_t is a random variable.
   (b) Compute $\mathbb{E}|X_t|$.
   (c) Compute $\mathbb{E}X_t^2$.

3. Let $B^{(1)}$ and $B^{(2)}$ be independent Brownian motions, and define the 2-dimensional process X with $X_t = (B_t^{(1)}, B_t^{(2)})$ for all $t \geq 0$. Show that $\langle y, X\rangle = (\langle y, X_t\rangle)_{t\geq 0}$ is a standard 1-dimensional Brownian motion for any $y \in \mathbb{R}^2$ with $\|y\| = 1$.

4. Let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be Gaussian random variables. Assume that $X_n \underset{L_2}{\to} X$ as $n \to \infty$ for some random variable $X \colon \Omega \to \mathbb{R}$. Show that X is Gaussian.

5. Let $f \colon [0, \infty[\to \mathbb{R}$ be a continuous function. For $t \geq 0$, define
$$X_t := \int_0^t f(u) dB_u.$$
Show that $X = (X_t)_{t\geq 0}$ is a Gaussian process with mean zero and covariance
$$\mathbb{E}(X_s X_t) := \int_0^{\min(s,t)} [f(u)]^2 du.$$

6. For $n \in \mathbb{N}$, let $t_k = t_{k,n} = \frac{k}{n}$, $k = 0, 1 \ldots, n$. Show that for $p > 0$,
$$n^{\frac{p}{2}-1} \sum_{k=1}^{n} |B_{t_k} - B_{t_{k-1}}|^p \underset{\mathbb{P}}{\to} c_p$$
as $n \to \infty$ for some constant c_p depending only on p.

Hint: Recall the weak law of large numbers.

7*. Show that
$$n^{\frac{p-1}{2}} \left(\sum_{k=1}^{n} |B_{t_k} - B_{t_{k-1}}|^p - n^{1-\frac{p}{2}} c_p \right) \underset{d}{\to} Z$$
for some Gaussian random variable Z.