MATS352 Stochastic analysis
Excercise session 4

Tuesday, 22.4.2014
MaD 381 at 10.15

Notice the unusual time and place!

Notation: $\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is a stochastic basis satisfying the usual conditions, and $B = (B_t)_{t\geq 0}$ is a standard $(\mathcal{F}_t)_{t\geq 0}$- Brownian motion.

1. (a) Show that $B \in \mathcal{M}_2^{c,0}$.
   (b) Define $S_t := e^{B_t - \frac{t}{2}}$ ("geometric Brownian motion") and show that $S = (S_t)_{t\geq 0} \in \mathcal{M}_2^c$.
2. $X : \Omega \to \mathbb{R}$ a random variable with $\mathbb{E}X^2 < \infty$. Show that $M = (M_t)_{t\geq 0}$ defined by $M_t := \mathbb{E}(X|\mathcal{F}_t)$ is a square-integrable martingale (i.e $M \in \mathcal{M}_2$).
3. Let $a > 0$ and define $X_t := \sqrt{a} B_{\frac{t}{a}}$. Show that $X = (X_t)_{t\geq 0}$ is a Brownian motion. What about filtration?
4. Show that $\langle B \rangle_t = t$ (a.s.).
5. Show that B has infinite 1-variation on any interval $[0, t]$ (a.s.).
6. Collect the steps to prove the existence of Brownian motion.

7*. Simulate and draw N paths of Brownian motion on $[0, 1]$ (choose $N = 1, 5, 10, \ldots$). Draw the functions $t \mapsto \pm\sqrt{2t \log \log t}$ in the same picture.

Hint: Choose $n \in \mathbb{N}$ for accuracy and simulate n independent $X_k \sim N(0, \frac{1}{n})$.

Increase your time interval to $[0, T]$, $T = 10, 100, \ldots$ Can you observe the "law of the iterated logarithm"?