MATS352 Stochastic analysis
Excercise session 3

Monday, 7.4.2014

MaD 380 at 10.15

1. Let X and Y be two Gaussian processes. Show that they have the same finite-dimensional distributions if and only if they have the same mean and covariance functions.

2. Let $W = (W_t)_{t \geq 0}$ be a Gaussian process with
$$\mathbb{E}W_t = 0 \text{ and } \mathbb{E}W_s W_t = \min(s, t).$$
Show that $W_b - W_a \underset{d}{=} W_{b-a}$ for all $0 \leq a \leq b$. What is this distribution?

3. Let $W = (W_t)_{t \geq 0}$ be a Gaussian process with
$$\mathbb{E}W_t = 0 \text{ and } \mathbb{E}W_s W_t = \min(s, t).$$
Define $Y_0 := 0$ and $Y_t = tW_{\frac{1}{t}}$ for $t > 0$. Show that $Y = (Y_t)_{t \geq 0}$ is a Gaussian process and compute $\mathbb{E}Y_s Y_t$. What do you notice?

4. (Brownian bridge) Let $W = (W_t)_{t \geq 0}$ be a Gaussian process with
$$\mathbb{E}W_t = 0 \text{ and } \mathbb{E}W_s W_t = \min(s, t).$$
For $t \in [0, 1]$, define $X_t := W_t - tW_1$. Show that $X = (X_t)_{t \in [0,1]}$ is a Gaussian process with $\mathbb{E}X_t = 0$ and $\mathbb{E}X_t X_s = \Gamma(t, s)$, where
$$\Gamma(t, s) = \begin{cases} s(1-t) \text{ if } 0 \leq s \leq t \leq 1, \\ t(1-s) \text{ if } 0 \leq t \leq s \leq 1. \end{cases}$$
Is there a modification of X that has continuous paths?

5. Check that the family of measures $(\mu_{(t_1,\ldots,t_n)})_{(t_1,\ldots,t_n) \in \Delta}$ defined in the proof of Proposition 3.7 is consistent.

6. Let $X = (X_1, X_2) \colon \Omega \to \mathbb{R}^2$ be a Gaussian random vector with mean vector $m = (m_1, m_2)$ and covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1}^2$ for some m_1, m_2 and $\sigma_{ij} \in \mathbb{R}$, i.e.
$$\mathbb{E}X_1 = m_1, \ \mathbb{E}X_2 = m_2 \text{ and } \mathbb{E}[(X_i - m_i)(X_j - m_j)] = \sigma_{ij}.$$
Assume that we have a matrix $A = (a_{ij})_{i,j=1}^2$ satisfying
   (a) $a_{ij} = a_{ji}$ for all $i, j = 1, 2$, and
   (b) $AA = \Sigma$.

   Show that
$$\mathbb{P}(X \in B) = \frac{1}{2\pi} \int_{A^{-1}(B)} e^{-\frac{|x - A^{-1}m|^2}{2}} dx$$
for $B \in \mathcal{B}(\mathbb{R}^2)$ if A^{-1} exists. What does it mean that A^{-1} does not exist? What kind of formula do we get then?