MATS352 Stochastic analysis
Excercise session 2

Monday, 31.3.2014

MaD 380 at 10.15

1. (From last week:) Let all paths of X be LCRL. For a fixed $t_0 > 0$, define

$$\begin{aligned} A_{t_0} &:= \{\omega \in \Omega \colon t \mapsto X_t(\omega) \text{ is continuous at each } t \in (0, t_0] \\ &\qquad \text{and right-continuous at } 0\}. \end{aligned}$$

   Assuming that X is adapted to a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$, show that $A_{t_0} \in \mathcal{F}_{t_0}$.

2. Let $\mathcal{A} := \sigma(A_1 \times A_2 \times \cdots \times A_n \times \mathbb{R} \times \mathbb{R} \times \cdots)$, where $n \in \mathbb{N}$ and $A_k \in \mathcal{B}(\mathbb{R})$ for all $k = 1, \ldots, n$. Moreover, let $X = (X_t)_{t \geq 0}$ be a stochastic process. Show that $C \in \sigma(X_s \colon s \in [0, t])$ if and only if there exist $t_1, t_2, \ldots \in [0, t]$ and $A \in \mathcal{A}$ with

$$C = \{\omega \in \Omega \colon (X_{t_1}(\omega), X_{t_2}(\omega), \ldots) \in A\}.$$

3. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis. Define

$$\mathcal{F}_t^+ := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}.$$

   Show that $(\mathcal{F}_t^+)_{t \geq 0}$ is a filtration and that $\mathcal{F}_t \subset \mathcal{F}_t^+$ for all $t \geq 0$.

4. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis, and let X be an adapted process with continuous paths. For $t_0 > 0$, define

$$A_{t_0} := \{\omega \in \Omega \colon \sup_{|t - t_0| < \varepsilon_\omega} X_t(\omega) \leq X_{t_0}(\omega) \text{ for some } \varepsilon_\omega > 0\}$$

   Do we have

$$A_{t_0} \in \bigcap_{\varepsilon > 0} \mathcal{F}_{t_0 + \varepsilon} \ ?$$

5. Let $\sigma > 0$, $m \in \mathbb{R}$ and $X \colon \Omega \to \mathbb{R}$ a random variable with

$$\mathbb{P}(X \in A) = \frac{1}{\sqrt{2\pi}\sigma} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}} dx,$$

   where $A \in \mathcal{B}(\mathbb{R})$. Show that $\mathbb{E}X = m$ and $\mathbb{E}[(X - m)^2] = \sigma^2$.

6. Let $W = (W_t)_{t \geq 0}$ be a Gaussian process with

$$\mathbb{E}W_t = 0 \text{ and } \mathbb{E}W_s W_t = \min(s, t).$$

   Show that $\mathbb{E}[(W_d - W_c)(W_b - W_a)] = 0$ for all $0 \leq a \leq b \leq c \leq d$. Are the increments $(W_d - W_c)$ and $(W_b - W_a)$ independent?