Stochastic Processes 2
Exercises 6

Monday, 3rd May, 2010
MaD 245, 10-12
(Choose 5 questions)

1. Let $X : \Omega \to \mathbb{R}$ be a random variable with $\mathbb{E}|X| < \infty$. Check the fact that we have used many times during the course:

$$\lim_{c\to\infty} \int_{\{|X|\geq c\}} |X| d\mathbb{P} = 0.$$

Does it bring to your mind some new concept introduced at this course?

2. Let $\Omega := [0,1)$ and let $(h_n)_{n=0}^{\infty}$, be the sequence of Haar-functions defined by $h_0 \equiv 1$ and

$$h_{2^{n-1}+k}(t) := \begin{cases} -1 & : \quad t \in \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \\ 1 & : \quad t \in \left[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right) \\ 0 & : \quad \text{else} \end{cases}$$

for $k = 0, ..., 2^{n-1} - 1$ if $n \geq 1$. Define $\mathcal{F}_n := \sigma(h_0, ..., h_n)$ for $n = 0, 1, ...$

   (a) Draw pictures of some of the first functions.

   (b) Describe $\mathcal{F}_n$ as easy as possible.

   (c) Is $(M_n)_{n=0}^{\infty}$ with $M_n := h_0 + \cdots + h_n$ a martingale?

3. Let $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ iid Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. Let $S_n := \varepsilon_1 + \cdots + \varepsilon_n$. Prove that

$$\mathbb{E}(\varepsilon_1 | \sigma(S_n)) = \frac{S_n}{n} \quad a.s.$$

4. Assume that $\varepsilon_1, ..., \varepsilon_n : \Omega \to \mathbb{R}$ are independent random variables such that $\mathbb{P}(\varepsilon_i = 1) = p$ and $\mathbb{P}(\varepsilon_i = -1) = q$ for some $p, q \in (0,1)$ with $p + q = 1$. Define the stochastic process $X_k := e^{a(\varepsilon_1 + ... + \varepsilon_k) + bk}$ for $k = 1, ..., n$ and $X_0 := 1$ with $a > 0$ and $b \in \mathbb{R}$ and the filtration $(\mathcal{F}_k)_{k=0}^{n}$ with $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_k := \sigma(\varepsilon_1, ..., \varepsilon_k)$.

   (a) Assume that $-a + b < 0 < a + b$. Find $p, q \in (0,1)$ such that the process $(X_k)_{k=0}^{n}$ is a martingale.

   (b) Assume that $-a + b > 0$. Why there cannot exist random variables $\varepsilon_1, ..., \varepsilon_n : \Omega \to \{-1, 1\}$ such that $(X_k)_{k=0}^{n}$ is a martingale?

5. Let $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ be independent random variables with $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. Define $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_n := \sigma(\varepsilon_1, ..., \varepsilon_n)$ if $n \geq 1$.

   (a) Let $M_0(\omega) := 1$ and

$$M_n(\omega) := e^{3\left(\varepsilon_1(\omega) + \cdots + \varepsilon_n(\omega)\right) - bn}$$

   if $n \geq 1$. For which $b \in \mathbb{R}$ the process $M = (M_n)_{n=0}^{\infty}$ is a martingale?

   (b) Let $N_n := \varepsilon_1 + \cdots + \varepsilon_n$ if $n \geq 1$ and $N_0 := 0$. Is $N = (N_n)_{n=0}^{\infty}$ uniformly integrable?

6. Let $\mathbb{P}_k$ and μ_k be measures on $\Omega_k := \{-1, 1\}$ with $\mathcal{F}_k := 2^{\Omega_k}$ such that $\mathbb{P}_k(\{-1\}) = \mathbb{P}_k(\{1\}) = 1/2$, and $\mu_k(\{-1\}) = p_k$ and $\mu_k(\{1\}) = q_k$ with $p_k + q_k = 1$, where

$$p_k := \frac{1}{2} + \sqrt{\frac{1}{k} - \frac{1}{k^2}} \quad \text{for} \quad k \geq 4$$

and $p_k, q_k \in (0,1)$ for $k = 1, 2, 3, 4$. Decide whether $\times_{n=1}^{\infty} \mu_k$ is absolutely continuous with respect to $\times_{k=1}^{\infty} \mathbb{P}_k$.

Hint: Check that $((1 - \frac{1}{k})^2 - \frac{1}{2})^2 = p_k q_k$ for $k \geq 4$ and use a discrete version of Proposition 3.9.7.

7. Let $\Pi = \{(x, y) : x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and $A_n : \Pi \to \Pi$ a rotation by $\frac{2\pi}{2^n}$ degree, $n = 0, 1, 2, ...$ Let $\mathcal{F}_n$ be the set of all Borel sets $B \subseteq \Pi$ such that $A_n(B) = B$.

   (a) Show that $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \cdots$.

   (b) Given $f \in L_1(\Pi, \lambda)$, compute $\mathbb{E}(f|\mathcal{F}_n)$, where λ is the normalized Lebesgue measure on Π.

8. What have we learnt about the (symmetric) random walk $S = (S_n)_{n=0}^{\infty}$,

$$S_n := \varepsilon_1 + \cdots + \varepsilon_n,$$

where $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ are independent Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$?
(No proofs; recall questions and possible answers. This question we will consider more at the last lecture.)