Stochastic Processes 2
Exercises 5
Monday, 26th April, 2010
MaD 245, 10-12

Theorem 0.1 *Let $M = (M_n)_{n=0}^{\infty}$ be a martingale with respect to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$, where $\mathcal{F}_n := \sigma(M_0, ..., M_n)$. Assume a stopping time $\tau \geq 1$ with $\mathbb{E}\tau < \infty$ and that*

$$\chi_{\{\tau \geq n\}} \mathbb{E}(|M_{n+1} - M_n| | \mathcal{F}_n) \leq c \text{ a.s.}$$

for some $c > 0$ and all $n = 0, 1, 2, ...$ Then

$$\mathbb{E}|M_\tau| < \infty \quad \text{and} \quad \mathbb{E}M_\tau = \mathbb{E}M_0.$$

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1. WALD's **Identity**: Assume iid random variables $\xi_1, \xi_2, ... : \Omega \to \mathbb{R}$ with $\mathbb{E}|\xi_1| < \infty$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_n := \sigma(\xi_1, ..., \xi_n)$ for $n \geq 1$, and let $\tau \geq 1$ be a stopping time with $\mathbb{E}\tau < \infty$. Prove that

$$\mathbb{E}(\xi_1 + \cdots + \xi_\tau) = \mathbb{E}\xi_1 \mathbb{E}\tau. \tag{1}$$

   Hint: Use $M_n := \xi_1 + \cdots + \xi_n - n\mathbb{E}\xi_1$ and Theorem 0.1.

2. Prove the WALD identity (1) if τ and $\xi_1, \xi_2, ...$ are independent.

   Hint: See the proof of Proposition 3.2.8.

3. Let $M_n := \varepsilon_1 + \cdots + \varepsilon_n$ and $M_0 := 0$, where $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ are iid with $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$. Let

$$\tau(\omega) := \inf\{n \geq 0 : M_n(\omega) = -10\}.$$

   Prove that $\mathbb{E}\tau = \infty$.

   Hint: Assume that $\mathbb{E}\tau < \infty$ and use WALD's identity.

For questions 4-7:
Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_k)_{k=0}^n)$ with $\Omega = \{\omega_1, ..., \omega_N\}$, $\mathbb{P}(\{\omega_i\}) > 0$, and a process $(Z_k)_{k=0}^n$ such that Z_k is $\mathcal{F}_k$-measurable. Define

$$U_n := Z_n$$

and, backwards,

$$U_k := \max\{Z_k, \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}$$

for $k = 0, ..., n-1$.

4. Show that $(U_k)_{k=0}^n$ is a super-martingale.

5. Show that $(U_k)_{k=0}^n$ is the smallest super-martingale which dominates $(Z_k)_{k=0}^n$: if $(V_k)_{k=0}^n$ is a super-martingale with $Z_k \leq V_k$, then $U_k \leq V_k$ a.s.

6. Show that $\tau(\omega) := \inf\{k = 0, ..., n : Z_k(\omega) = U_k(\omega)\}$ (with $\inf \emptyset := n$) is a stopping time.

   The process $(U_k)_{k=0}^n$ is called SNELL-envelop of $(Z_k)_{k=0}^n$.