# Stochastic Processes 2
## Exercises 4

Monday, 19th April, 2010
MaD 245, 10-12

For questions 1 and 2:
Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n=0}^\infty)$ be a stochastic basis, $\mathcal{F}_\infty := \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$, and $Z \in L_1$. What can we say about the almost sure and $L_1$-convergence of

1. $\mathbb{E}(Z|\mathcal{F}_n) \to_n \mathbb{E}(Z|\mathcal{F}_\infty)$,

2. $\mathbb{E}(Z|\mathcal{F}_n) \to_n Z$?

   (Proofs/counterexamples)

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For questions 3-7:
Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function, i.e. $|f(x) - f(y)| \leq L|x - y|$.
Let

$$\xi_n(t) := \sum_{k=1}^{2^n} \frac{k-1}{2^n} \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(t),$$

$\Omega := [0, 1)$, $\mathcal{F}_n := \sigma(\xi_n)$, and

$$M_n(t) := \frac{f(\xi_n(t) + 2^{-n}) - f(\xi_n(t))}{2^{-n}}.$$

3. Prove that $(\mathcal{F}_n)_{n=0}^\infty$ is a filtration and that $\mathcal{B}([0,1)) = \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$.

4. Prove that $(M_n)_{n=0}^\infty$ is a martingale with $|M_n(t)| \leq L$.

5. Prove that there is an integrable function $g : [0,1) \to \mathbb{R}$ such that $M_n = \mathbb{E}(g|\mathcal{F}_n)$ a.s.

6. Prove that $f(\frac{k}{2^n}) = f(0) + \int_0^{\frac{k}{2^n}} g(t)dt$ for $k = 0, ..., .2^n - 1$.

7. Prove that $f(x) = f(0) + \int_0^x g(t)dt$ for $x \in [0,1]$, i.e. $g$ is the generalized derivative of $f$.