Stochastic Processes 2
Exercises 3

Monday, 12th April, 2010
MaD 245, 10-12

1. Prove that the process M defined in Example 3.8.5 is a martingale.
2. Prove for the process M defined in Example 3.8.5 that

$$\lim_N \int_0^1 \sup_{n=1,\ldots,N} M_n(t)dt = \infty \quad \text{but} \quad \int_0^1 M_N(t)dt = 1.$$

   What is the connection to Doob's maximal inequalities?
3. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1), \mathcal{B}([0,1)), \lambda)$ and

$$f_n(t) := \chi_{[0,1/n)}(t) n^{\frac{1}{p}}.$$

   For what $0 < p < \infty$ is the family $(f_n)_{n=1}^\infty$ uniformly integrable?

For questions 4 - 6:
Let $\Omega := [0,1)$ and

$$M_n(t) := h_0(t) + \cdots + h_n(t),$$

where

$$h_n(t) := 2^n \chi_{[0,1/2^{n+1})} - 2^n \chi_{[1/2^{n+1},1/2^n)}.$$

Let $\mathcal{F}_n := \sigma(h_0, ..., h_n)$.

4. Show that $M = (M_n)_{n\geq 0}$ is a martingale.
5. Is there a constant $c > 0$ such that for all $N = 1, 2, ...$ one has

$$\int_0^1 \sup_{n=1,\ldots,N} |M_n(t)|dt \leq c \int_0^1 |M_N(t)|dt?$$

6. Is there a random variable $M_\infty : [0,1) \to \mathbb{R} \in L_1$ such that

$$M_n = \mathbb{E}(M_\infty | \mathcal{F}_n) \quad a.s.?$$