Stochastic Processes 2
Exercises 1

Monday, 22nd March, 2010
MaD 245, 10-12

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $X, Y : \Omega \to \{0,1\}$ independent random variables. How many elements does the sub-σ-algebra

$$\mathcal{G} := \sigma(X, Y)$$

contain?

For questions 2 and 3:
Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration and $\sigma, \tau : \Omega \to \{0, 1, 2, ...\} \cup \{\infty\}$ stopping times.

2. Show that $\sigma + \tau$ is a stopping time.

3. Show that $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

For questions 4 and 5:
Let $\epsilon_i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots$ independent random variables such that $\mathbb{P}(\epsilon_i = -1) = \mathbb{P}(\epsilon_i = 1) = \frac{1}{2}$ for all $i = 1, 2, \ldots$ Define $M_0 := 0$ and $M_n := \epsilon_1 + \ldots + \epsilon_n$, $n \geq 1$. Let $\mathcal{F}_0$ be the trivial σ-algebra and $\mathcal{F}_n := \sigma(\epsilon_1, \ldots, \epsilon_n)$. Are the following functions stopping times ($\inf \emptyset := \infty$)?

4. (a) $\sigma(\omega) := \inf \{n \geq 0 : M_n = 0\}$
   (b) $\sigma(\omega) := \inf \{n \geq 0 : M_n \in \,]10, 12[\}$
   (c) $\sigma(\omega) := \inf \{n \geq 0 : M_n \in \,]10, 12[\} - 1$
   (d) $\sigma(\omega) := \inf \{n \geq 0 : M_n \in \,]10, 12[\} + 1$

5. (a) $\sigma(\omega) := \inf \{n \geq 0 : M_{n+1} \in \,]10, 12[\}$
   (b) $\sigma(\omega) := \inf \{n \geq 0 : M_{n+1} \in \,]10, 11[\}$
   (c) $\sigma(\omega) := \inf \{n \geq 0 : M_{n-1} = 10\}$
   (d) $\sigma(\omega) := \inf \{n \geq 0 : M_{n-1} = 10\} - 1$

6. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n=0}^{\infty})$ be a stochastic basis and $M = (M_n)_{n=0}^{\infty}$ be a martingale such that $M_0 = 0$ a.s. and $\mathbb{E}M_n^2 < \infty$ for all $n = 0, 1, 2, ...$ Define $\langle M \rangle_0 := 0$,

$$\langle M \rangle_n := \sum_{k=1}^{n} \mathbb{E}((M_k - M_{k-1})^2 | \mathcal{F}_{k-1}) \in [0, \infty).$$

The process $\langle M \rangle := (\langle M \rangle_n)_{n=0}^{\infty}$ is called *(predictable) bracket process.* Show that

   (a) $(M_n^2 - \langle M \rangle_n)_{n=0}^{\infty}$ is a martingale,
   (b) $\mathbb{E}M_n^2 = \mathbb{E}\langle M \rangle_n$ for all $n = 0, 1, ...$