Financial Mathematics

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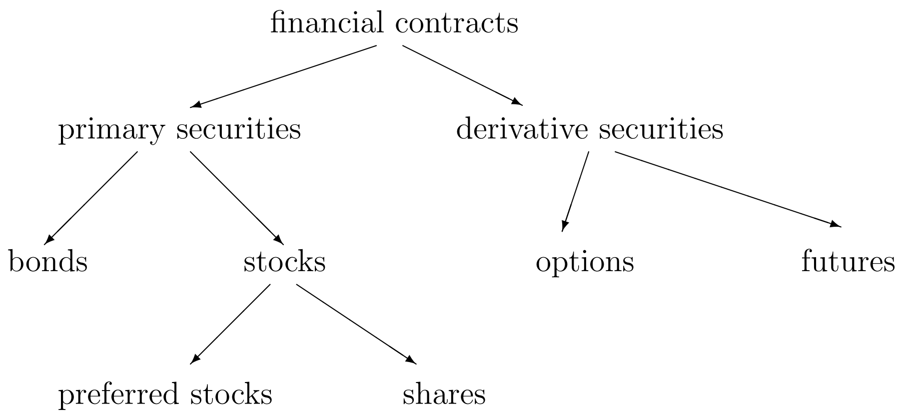
1. Introduction

1.1 Financial markets

Financial markets are places where individuals and corporations can buy or sell financial securities and products. These markets are not only a possibility to purchase assets, they also are used for risk transfer.
Already centuries ago financial contracts have been made. It is known that in the Antique Greece olives were sold by the farmers using forward contracts (i.e. quality and price was aggreed upon in advance).
In the beginning of the 17th century the first official stock exchange was opened in Amsterdam. Especially tulips, originally from Turkey, became extremely popular among rich merchants. Traders purchased bulbs at higher and higher prices planning to re-sell them for profit. Due to the nature of (growing) tulips which only can be moved in a certain time of the year the concept of futures contracts was developped. But suddenly the interest in tulips decreased and prices fell rapidly ('tulip mania', 'speculative bubble').

In the United States the Chicago Board of Trade (CBOT) was created in 1848 as an exchange market for futures and options.

1.2 Types of financial contracts



a security is a piece of paper representing a promise

bonds are certificates issued by a government or a public company promising to repay a fixed interest rate at a specified time

a share (or stock) is a security representing partial ownership of a company and/or makes dividend payments according to the profits. Shares are traded on a *stock exchange*

preferred stocks are entitled to a fixed dividend

an asset (in Finance) is anything owned, whether in possession or by right to take possession, by a person or a company, the value of which can be expressed in monetary terms

$$\left.\begin{array}{l}\text{stock}\\ \text{cash}\end{array}\right\}\text{ current assets}$$

land
building
machinery } fixed assets

goodwill
copyrights
education
... } intangible assets

a forward contract is an agreement between two parties to buy or sell an asset (which can be of any kind) at a pre-agreed future point in time.

a futures contract is a forward contract the has been standardized:

- amount to be traded: for example a fixed number of barrels of oil

  currency (US dollar often)
- quality
- delivery month
- last trading date

Futures contracts are traded on a *futures exchange*

an option gives the holder of the option the right to buy (or sell) a security (shares) at a predefined time (or timeperiod) in the future and for a pre-determined amount.

Types of options:
- stock options
- foreign exchange options
- interest rate options (=largest derivatives market in the world)
- warrants
- options on bonds
- swaptions
- ...

long position someone agrees to buy the asset

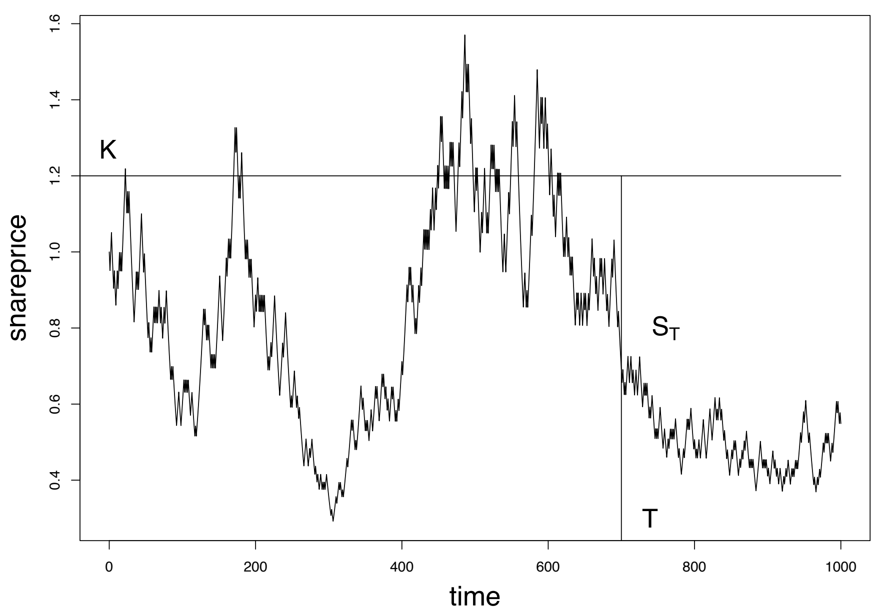
short position someone agrees to sell the asset

One purpose of derivatives is as a form of insurance to move risk from someone who cannot afford a major loss to someone who could absorb the loss, or is able to hedge against the risk by buying some other derivatives.

The central topic of Financial Mathematics is the fair valuation of derivatives. One key equation used to value derivatives is the Black-Scholes-Equation (published 1973). Fischer Black and Myron Scholes received the Nobel Prize in Economics for this. After 1973 trading with options increased rapidly.

1.3 Example: the European call option

Someone buys at time 0 a "European call option". Then he can (but does not have to) buy a given number of shares (1 share here) for a fixed price K (= the so called "strike price") at a fixed time T.



If $S_T > K$ then he will buy the shares for the price K and if he sells them immediately his
gain $= S_T - K -$ price of the option

If $S_T \leq K$ he will not buy and his
loss $=$ price of the option

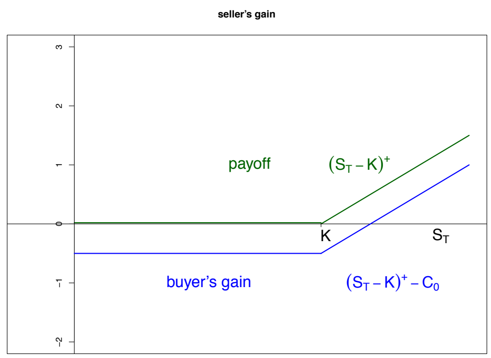
Question: How to determine a "fair" price for an option?

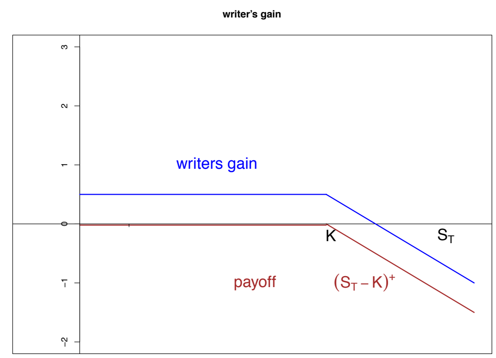
1. If the price would be 0: then the *option holder* (= the one who bought the option) could make a riskless profit: this is against the "rules of the market"

2. if the option price is too high and if there is no sign that the share price S_T will be much higher than the strike price K, nobody will buy this option.

Summary: European call-option; C_0 :=option price

The "gain" (outcome) of the	
option holder (= buyer)	writer (=seller of the option)
$\begin{cases} S_T - K - C_0 \text{ if } S_T > K \\ -C_0 \text{ if } S_T \leq K \end{cases}$	$\begin{cases} K - S_T + C_0 \text{ if } S_T > K \\ C_0 \text{ if } S_T \leq K \end{cases}$





The purpose of a European call option:

1. The writer reduces the risk in case S_t will go down: he gets C_0.
2. The buyer hopes that $S_T > K + C_0$ and takes the risk that S_t will go down. In this case he loses the price C_0 of the option.

purpose: it is a form of insurance (for the writer)

A fair price of an European call option $f(S_T) = (S_T - K)^+$ (Example)

A fair price of $f(S_T)$ would be a price where both the writer and the buyer could not make riskless profit. We consider the following example:

Assume 2 trading dates: 0 and T.

$$\begin{array}{ll} \textit{at time 0} & \text{share price } S_0 = 20 \text{ \$} \\ \textit{at time } T & S_T = \begin{cases} 20 \text{ \$ with probability } p \\ 7.5 \text{ \$ with probability } 1-p \ (0 < p < 1). \end{cases} \end{array}$$

Let the strike price be $K = 15$ (dollar).

$$\Rightarrow \text{the option writer has to pay } \begin{cases} 5 \text{ \$ if } S_T & = & 20 \\ \text{nothing if } S_T & = & 7.5 \end{cases}$$

We can do the following: "hedging" (=counterbalancing action to protect oneself from losing). Let us assume here for simplicity that the interest rate r= 0. That means one can borrow from the bank without paying interest.
The writer sells the option, so he gets C_0. He borrows (-φ_0) dollar from the bank and can buy

$$\varphi_1 = \frac{C_0 - \varphi_0}{10} \qquad (S_0 = 10)$$

shares at time 0.
The portfolio (φ_0,φ_1) is correctly chosen if

$$\left.\begin{array}{rcl} \varphi_0 1 + \varphi_1 20 & = & 5 \\ \varphi_0 1 + \varphi_1 7.5 & = & 0 \end{array}\right\}$$

We get

$$\begin{array}{lcl} \varphi_0 & = & -7.5\varphi_1 \\ 12.5\varphi_1 & = & 5 \\ \varphi_1 & = & \frac{5}{12.5} = 0.4 \end{array}$$

and

$$\varphi_0 = -7.5 \times 0.4 = -3.$$

Then

$$C_0 = 10\varphi_1 + \varphi_0 = 4 - 3 = 1$$

is the fair price for the option.
Hence at the time 0 the writer gets 1 dollar for the option and borrows 3 dollars from the bank. With these 4 dollars he can buy 0.4 shares.

Case 1: $S_T = 20$. The option is exercised at a cost of 5\$. The writer repays the loan (cost 3\$) and sells the shares (gain $0.4 \times 20 = 8$). Balance of trade: 8-5-3 = 0.

Case 2: $S_T = 7.5$ The option is not exercised (cost = 0). The writer repays the loan (cost 3\$) and sells the shares (gain $0.4 \times 7.5 = 3$) Balance of trade: 3-3= 0.

If $C_0 > 1$, then the writer can make (by hedging like above) the riskless profit $C_0 - 1$.

If $C_0 < 1$ the option holder can make a riskless profit by the following procedure: buy the option (cost $-C_0$), sell 0.4 shares (gain: 4) and put $4 - C_0$ to the bank account. Then, at time T

$$\begin{cases} 4 - C_0 + 5 - 0.4 \times 20 = 1 - C_0 & \text{for} \quad S_T = 20 \\ 4 - C_0 - 0.4 \times 7.5 = 1 - C_0 & \text{for} \quad S_T = 7.5 \end{cases}$$

Where the 5 is the payoff of the option and $1 - C_0$ is the riskless profit.

Summary

In this example we did

1. find the hedging portfolio (φ_0, φ_1) by solving the equation

$$\varphi_0 + \varphi S_T = f(S_T) \qquad (\text{ here } f(S_T) = (S_t - K)^+).$$

2. We calculated the "fair price" namely how much money a trader would need at time 0 to have the amount $f(S_T)$ at time T:

$$\text{"fair price"} = \varphi_0 + \varphi_1 S_0.$$

Remark

One can compute the fair price of an option also by using a 'martingale-measure'. For this we introduce probability theory.

2. Basics of Probability theory

1 Finite probability spaces

Definition 2.1. Let $\Omega = \{\omega_1, \cdots, \omega_N\}$ be a finite set. Assume

$$p_i > 0, i = 1, \cdots, N \text{ such that}$$

$$\sum_{i=1}^{N} p_i = 1.$$

Then $\mathbb{P}$ is a *probability measure*: For $A \subseteq \Omega$ we set $\mathbb{P}(A) := \sum_{\omega_i \in A} \mathbb{P}(\{\omega_i\})$.

Example 2.2. Rolling a die

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\mathbb{P}(\{\omega\}) = \frac{1}{6},\ \omega \in \Omega.$$

$A :=$ 'rolling an odd number'
$\mathbb{P}(A) =?$

It follows from the definition that

$$\mathbb{P}(\Omega) = \sum_{i=1}^{N} \mathbb{P}(\{\omega_i\}) = \sum_{i=1}^{N} p_i = 1.$$

$$\mathbb{P}(\emptyset) = \sum_{\omega_i \in \emptyset} \mathbb{P}(\{\omega_i\}) = 0.$$

$$\begin{aligned} \text{We define } F \;&:=\; 2^{\Omega} \text{ be the } \textit{power set} \text{ of } \Omega \\ &=\; \text{the set of } \textit{all} \text{ subsets of } \Omega. \end{aligned}$$

Example 2.3. For $\Omega = \{1, 2\}$ we have $2^\Omega = \{\{1, 2\}, \{1\}, \{2\}, \emptyset\}$. The power set 2^Ω has $2^{\#\Omega}$ elements.

Definition 2.4. [σ **- field,** σ **- algebra**]
Let Ω be a non-empty set. A system $\mathcal{F}$ of subsets $A \subseteq \Omega$ is a σ-field or σ-algebra on Ω if

1. $\emptyset, \Omega \in \mathcal{F}$,
2. $A \in \mathcal{F} \Rightarrow A^C := \Omega \backslash A \in \mathcal{F}$,
3. $A_1, A_2, ... \in F \Rightarrow \bigcup_{i=1}^\infty A_i \in \mathcal{F}$.

Remark 2.5. If Ω is finite, it is enough to check in (3) that $A_1, A_2 \in \mathcal{F}$ implies $A_1 \cup A_2 \in \mathcal{F}$.

Examples

1. 2^Ω is a σ-field.
2. Let Ω be a set and assume $A_1, ..., A_M$ is a *finite partition* of Ω i.e. $A_1, ..., A_M$ are mutually disjoint:
$$A_i \cap A_j = \emptyset, \forall i \neq j$$
and
$$\bigcup_{i=1}^M A_i = \Omega.$$

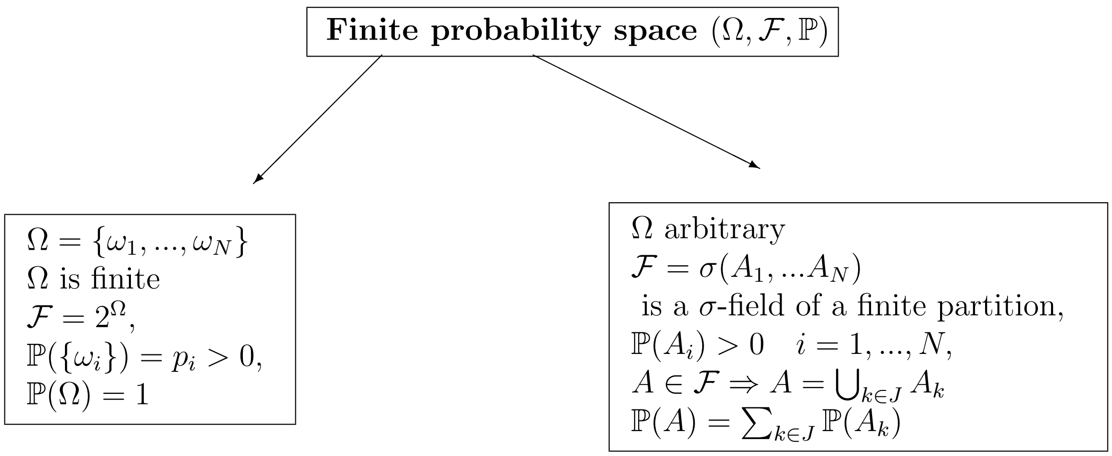
Then

$$\begin{aligned} \mathcal{F} &= \left\{ \bigcup_{i=J} A_i : J \subseteq \{1, ..., M\} \right\} \\ &= \{\emptyset, A_1, ..., A_M, A_1 \cup A_2, A_1 \cup A_3, ..., \Omega\} \end{aligned}$$

is a σ-field. We say $\mathcal{F}$ is *generated* by $A_1, ..., A_N$ and use the notation

$$\mathcal{F} := \sigma(A_1, ...A_N).$$

A finite probability space can be thought of in two ways:


Finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$
$\Omega = \{\omega_1, ..., \omega_N\}$
Ω is finite
$\mathcal{F} = 2^{\Omega}$,
$\mathbb{P}(\{\omega_i\}) = p_i > 0$,
$\mathbb{P}(\Omega) = 1$
Ω arbitrary
$\mathcal{F} = \sigma(A_1, ...A_N)$
is a σ-field of a finite partition,
$\mathbb{P}(A_i) > 0 \quad i = 1, ..., N$,
$A \in \mathcal{F} \Rightarrow A = \bigcup_{k \in J} A_k$
$\mathbb{P}(A) = \sum_{k \in J} \mathbb{P}(A_k)$

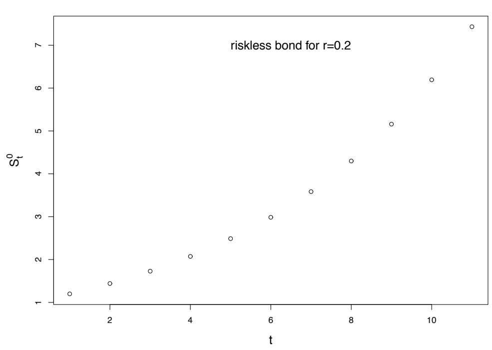
3. The Cox-Ross-Rubinstein model (CRR-model, binomial tree model)

We want to model the time-development of shares and bonds with a simple model:

Assume $\mathbb{T} = \{0, 1, ..., T\}$ are *trading dates* (T = *trading horizon*).

$S^0 = (S^0_0, S^0_1, ..., S^0_T)$ is a *riskless bond* (or *bank account*).
$S^1 := S = (S_1, ..., S_T)$ is a risky (i.e. random) *stock.*

We assume a constant interest rate $r > 0$, i.e. if $S^0_0 = 1$, then $S^0_1 = 1 + r$, $S^0_k = (1 + r)^k, k = 0, 1, ..., T$.



The random behavior of the stock S will be modeled as follows:$0 < p < 1$ fixed.

$$S_{n+1} = \begin{cases} S_n(1+a) \text{ with probability } 1-p \\ S_n(1+b) \text{ with probability } p \end{cases}$$

$$-1 < a < b$$

If we choose

$$\Omega := \{\omega = (\epsilon_1, ..., \epsilon_T) : \epsilon_i \in \{1+a, 1+b\}\}$$

then

$$S_t(\omega) = S_0\epsilon_1, \epsilon_2, ..., \epsilon_t, \quad t \in \mathbb{T}$$

Hence each $\omega \in \Omega$ corresponds to one "possible case" of a stock development. We can also compute the probability of each case:

$$\mathbb{P}\big(\{(\epsilon_1, ..., \epsilon_T)\}\big) = p^k(1-p)^{T-k}$$

where

$$k := \#\{i : \epsilon_i = 1+b\}.$$

is the binomial distribution. The defined $\mathbb{P}$ is clearly a probability measure on Ω:

$$\mathbb{P}(\{\omega\}) > 0 \quad \forall \omega \in \Omega.$$

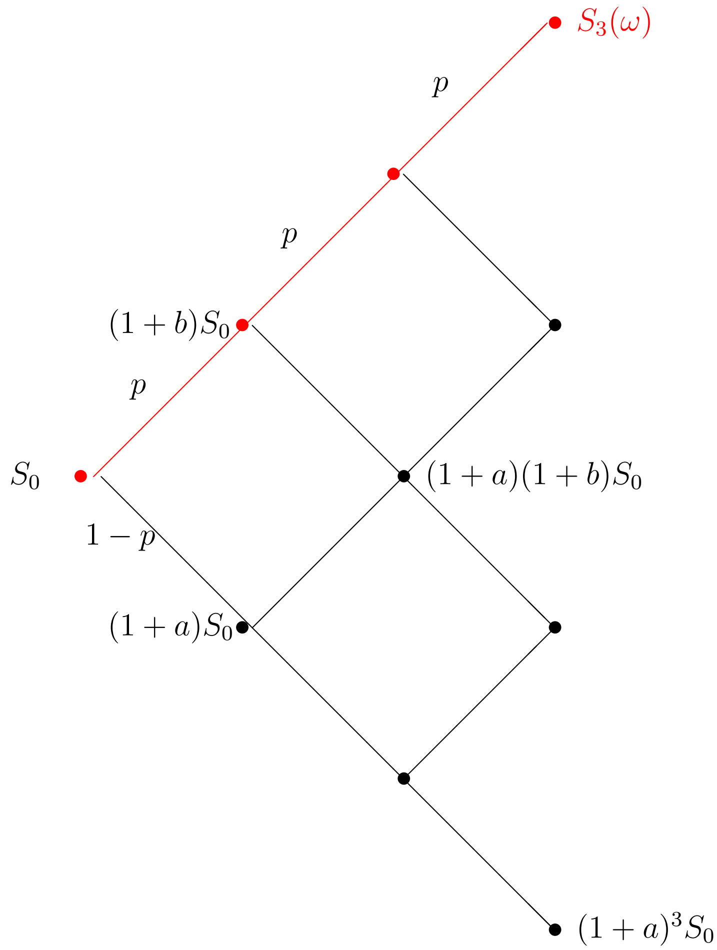
We have to check that

$$\mathbb{P}(\Omega) = 1.$$

$$\begin{aligned} \mathbb{P}(\Omega) &= \sum_{\epsilon_i \in \{1+a,1+b\}; i=1,...,T} \mathbb{P}\big(\{(\epsilon_1, ..., \epsilon_T)\}\big) \\ &= \sum_{k=0}^{T} \sum_{\substack{\omega \text{ with} \\ k = \#\{i, \epsilon_i = 1+b\}}} p^k(1-p)^{T-k} \\ &= \sum_{k=0}^{T} \binom{T}{k} p^k(1-p)^{T-k} \\ &= \big(p + (1-p)\big)^T = 1. \end{aligned}$$

Example 3.1. $\mathbb{T} = \{0, 1, 2, 3\}$

$$\omega = (1 + b, 1 + b, 1 + b)$$



1 Filtration

The investor does not know at time 0 how the values of $S_t, t = 1, ..., T$ will be. At time $t > 0$ he knows all about $S_0, S_1, ...S_t$ but nothing about $S_{t+1}, ..., S_T$.

We model the situation using a filtration.

Definition 3.2. A *filtration* is an increasing sequence of σ-fields:

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T$$

Definition 3.3. Assume $f : \Omega \to \{m_1, ...m_N\}$, and $\mathcal{G}$ is a σ-field on Ω. Then

$$f \text{ is } \mathcal{G}\text{-}measurable \;\Leftrightarrow f^{-1}(\{m_i\}) = \{\omega \in \Omega : f(\omega) = m_i\} \in \mathcal{G} \quad \forall m_i$$

If we have functions $f_1, f_2, \ldots, f_l : \Omega \to \{m_1, ..., m_N\}$ then $\mathcal{G} = \sigma(f_1, ...f_l)$ denotes the smallest σ-field, such that all functions $f_1, ..., f_l$ are $\mathcal{G}$-measurable.

Example 3.4. CRR model:

We assume $\mathcal{F}_t = \sigma\{S_0, ...S_t\}$ is the information which the investor has till time t.

$$\mathbb{T} = \{0, 1, 2, \} \qquad S_0 := 1.$$
$$\Omega = \big\{\omega = (\epsilon_1, \epsilon_2) : \; \epsilon_i \in \{1 + a, 1 + b\}\big\}$$

$$\mathcal{F}_1 := \sigma\{S_0, S_1\} \qquad S_0 \equiv 1$$

$$\begin{aligned} S_1(\omega) = 1 + a \quad &\Leftrightarrow \quad \omega = (1 + a, 1 + a) \text{ or } \omega = (1 + a, 1 + b) \\ S_1(\omega) = 1 + b \quad &\Leftrightarrow \quad \omega = (1 + b, 1 + a) \text{ or } \omega = (1 + b, 1 + b) \end{aligned}$$

Hence

$$\mathcal{F}_1 = \big\{\emptyset, \Omega, \{(1 + a, 1 + a), (1 + a, 1 + b)\}, \{(1 + b, 1 + a), (1 + b, 1 + b)\}\big\}.$$

But

$$\begin{aligned} S_2(\omega) = (1 + a)^2 \quad &\Leftrightarrow \quad \omega = (1 + a, 1 + a) \\ S_2(\omega) = (1 + a)(1 + b) \quad &\Leftrightarrow \quad \omega = (1 + a, 1 + b) \text{ or } \omega = (1 + b, 1 + a) \\ S_2(\omega) = (1 + b)^2 \quad &\Leftrightarrow \quad \omega = (1 + b, 1 + b) \end{aligned}$$

Consequently,

$$S_2 \text{ is not } \mathcal{F}_1\text{-measurable.}$$

We say that $(f_n)_{n=0}^T \quad (f_n : \Omega \to \mathbb{R})$ is *adapted* to $(\mathcal{F}_n)_{n=0}^T$ if it holds that f_n is $\mathcal{F}_n$-measurable $\forall n$. If f_n is $\mathcal{F}_{n-1}$-measurable $\forall n$ we say $(f_n)_{n=0}^T$ is *predictable*.

2 Martingales and conditional expectation

We assume we have a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence we can find a partition $A_1, ..., A_N$ of Ω with $\mathcal{F} = \sigma(A_1, ..., A_N)$.
If $f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable it can always be written as

$$f(\omega) = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}(\omega) \qquad \text{with } a_i \in \mathbb{R}$$

using *indicator functions* which are defined by

$$\mathbb{1}_A(\omega) := \begin{cases} 1 & \omega \in A, \\ 0 & \omega \in A^c. \end{cases}$$

We define the *expectation of f* by

$$\mathbb{E}f := \sum_{i=1}^{N} a_i \mathbb{P}(A_i).$$

Remark 3.5. Let $\Omega = \{\omega_1, ..., \omega_N\}$. Then

$$\mathbb{E}f := \sum_{i=1}^{N} f(\omega_i)\mathbb{P}(\{\omega_i\}).$$

Example 3.6. Rolling a die: $\Omega = \{1, ..., 6\}$.

$$f(i) = i, \quad i = 1, ..., 6.$$

The expectation is

$$\begin{aligned} \mathbb{E}f &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + ... + 6 \times \frac{1}{6} \\ &= \frac{1 + ... + 6}{6} = 3.5. \end{aligned}$$

Definition 3.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space and $f : \Omega \to \mathbb{R}$ an $\mathcal{F}$-measurable function. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-field of $\mathcal{F}$. If

1. $g : \Omega \to \mathbb{R}$ is $\mathcal{G}$-measurable and

2.
$$\mathbb{E}g\mathbb{1}_G = \mathbb{E}f\mathbb{1}_G \quad \forall G \in \mathcal{G}. \tag{3.1}$$

We say g is the *conditional expectation* of f with respect to $\mathcal{G}$ and write $g := \mathbb{E}[f|\mathcal{G}]$

Example 3.8. Let $\Omega = \{1, 2, ..., 2^N\}$

$$\mathcal{G}_N = 2^\Omega \quad , \quad f(\omega) = \omega \quad , \quad \mathbb{P}(\{\omega\}) = \frac{1}{2^N}$$

As sub-σ-field we choose

$$\mathcal{G}_{N-1} = \sigma\big\{\{1,2\}\{3,4\}, ..., \{2^N - 1, 2^N\}\big\}$$

We want to compute $\mathbb{E}[f|\mathcal{G}_{N-1}]$. Clearly, if (3.1) holds for all sets

$$G = \{2k-1, 2k\} \qquad k = 1, ..., 2^{N-1}$$

then (3.1) holds for all sets $G \in \mathcal{G}_{N-1}$.
By definition, if $g := \mathbb{E}[f|\mathcal{G}_{N-1}]$, then

$$g(2k-1, 2k) = g(2k) \ , \ \forall k$$

$$\Rightarrow \mathbb{E}g\mathbb{1}_{\{2k-1,2k\}} = \mathbb{E}f\mathbb{1}_{\{2k-1,2k\}}$$

$$\begin{aligned} \mathbb{E}g\mathbb{1}_{\{2k-1,2k\}} &= g(2k-1)\mathbb{E}\mathbb{1}_{\{2k-1,2k\}} \\ &= g(2k-1)\mathbb{P}\big(\{2k-1, 2k\}\big) = \tfrac{2}{2^N} g(2k-1) \end{aligned}$$

On the other hand

$$\begin{aligned} \mathbb{E}f\mathbb{1}_{\{2k-1,2k\}} &= f(2k-1)\mathbb{P}\big(\{2k-1\}\big) + f(2k)\mathbb{P}(2k) \\ &= \tfrac{2k-1+2k}{2^N} \end{aligned}$$

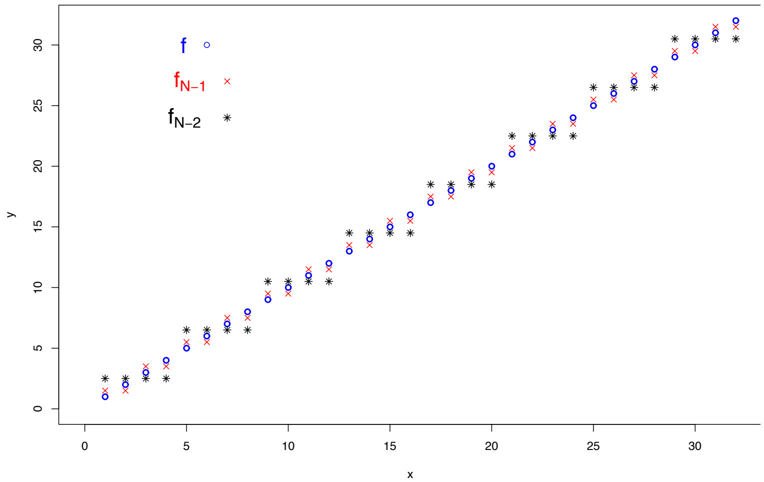
$$\Rightarrow g(2k-1) = g(2k) = \frac{2k-1+2k}{2}$$

Iteration:

$$\begin{aligned} \mathbb{G}_{N-2} &:= \sigma\big\{\{1,2,3,4\}, ..., \{2^N-3, 2^N-2, 2^N-1, 2^N\}\big\} \\ \mathbb{G}_0 &= \{\emptyset, \Omega\}. \end{aligned}$$

We define

$$\begin{aligned} \mathbb{E}[f|\mathcal{G}_{N-1}] &=: f_{N-1} \\ \mathbb{E}[f|\mathcal{G}_{N-2}] &=: f_{N-2} \\ \mathbb{E}f &=: f_0. \end{aligned}$$



Remark 3.9. $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq ... \subseteq \mathcal{G}_N$ is a filtration. Then $(f_k)_{k=0}^N$ with

$$f_k := \mathbb{E}[f|\mathcal{G}_k]$$

is an adapted sequence. Moreover it holds

$$\mathbb{E}[f_{k+1}|\mathcal{G}_k] = f_k \quad \forall k.$$

Definition 3.10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. An $(\mathcal{F}_n)_{n=0}^T$ adapted process $(M_n)_{n=0}^T$ is

1. a *a martingale* if $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n, \quad \forall 0 \leq n < T$,
2. a *a supermartingale* if $\mathbb{E}[M_{n+1}|\mathcal{F}_n] \leq M_n, \quad \forall 0 \leq n < T$,
3. a *a submartingale* if $\mathbb{E}[M_{n+1}|\mathcal{F}_n] \geq M_n, \quad \forall 0 \leq n < T$.

4. Finite market models and non-arbitrage pricing

1 The market model

Let $(\Omega, F, \mathbb{P})$ be a finite probability space where we agree on the convention $\mathbb{P}(A) > 0 \ \ \forall A \in \mathcal{F} \ , \ A \neq \emptyset$ i.e. 'every event is possible'.

Trading dates : $\mathbb{T} = \{0, 1, ..., T\}$

The information available to the investors at time t we model by the σ-field $\mathcal{F}_t$ where we assume

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T = \mathcal{F}.$$

The securities (assets) are modelled by a stochastic process in $\mathbb{R}^{d+1}$:

$$(S_t^0, S_t^1, ..., S_t^d)_{t \in \mathbb{T}}.$$

Here S_t^0 denotes the bond (or bank account) and is assumed to be nonrandom while $S_t^1, ..., S_t^d$ models the share prices at time t for d different shares and will be *random* (=depend on ω).

We want that S^i is $(\mathcal{F}_t)$-adapted for all $i = 1, ..., d$. This can be achieved by setting

$$\mathcal{F}_t := \sigma(S_u^1, ..., S_u^d \ : \ 0 \leq u \leq t)$$

The tuple $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t^0, ...S_t^d)\big)$ is the (securities) market model.

2 Strategies

Example 4.1. A *trading strategy* is a predictable process

$$\varphi = (\varphi_t^0, ... \varphi_t^d)_{t=1}^T$$

where φ_t^i denotes the number of shares of asset i the investor owns at time t. For fixed t the vector $(\varphi_t^0, ..., \varphi_t^d)$ is called the *portfolio* at time t.
The *wealth process* $V_t(\varphi)$ is given by

$$\begin{aligned} V_0(\varphi) &= \varphi_1 \cdot S_0, \qquad \text{the investor's initial wealth,} \\ V_t(\varphi) &= \varphi_t \cdot S_t = \sum_{i=0}^d \varphi_t^i S_t^i \quad \forall t \in \mathbb{T},\ t \geq 1. \end{aligned}$$

The investor trades at time $t-1$ which leads to the portfolio φ_t. At time t he will have $\varphi_t \cdot S_t = V_t(\varphi)$. If he uses exactly his wealth $V_t(\varphi)$ to trade at time t, then it must hold

$$V_t(\varphi) = \varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t.$$

where $\varphi_t \cdot S_t$ is the wealth which comes out from choosing φ_t at time $t-1$ and $\varphi_{t+1} \cdot S_t$ the needed wealth to buy the portfolio φ_{t+1} at time t.
We call φ *self-financing* if

$$\varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t\ , \quad t = 1, ..., T-1$$

Let us introduce *discounted prices:* S^0 models the bond, i.e. for example:,

$$S_t^0 = (1+r)^t$$

if we assume a constant interest rate $r \geq 0$, and it holds

$$S_t^0 > 0\ , \quad t = 0, ..., T.$$

Then

$$\tilde{S}_t = \left(1, \frac{S_t^1}{S_t^0}, ..., \frac{S_t^d}{S_t^0}\right)$$

is the vector of the *discounted prices.* (Clearly, in case of interest rate $r = 0$ the discounted price and the share price are equal). Now

$$\tilde{V}_t(\varphi) = \frac{1}{S_t^0}(\varphi_t \cdot S_t) = \varphi_t \cdot \tilde{S}_t$$

is the *discounted wealth* $V(\varphi)$ at t.

Proposition 4.2. The following assertions are equivalent.

1. φ is self-financing.
2. $V_t(\varphi) = V_0(\varphi) + \sum_{k=1}^{t} \varphi_k \cdot (S_k - S_{k-1}), \quad 1 \le t \le T.$
3. $\tilde{V}_t(\varphi) = V_0(\varphi) + \sum_{k=1}^{t} \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}), \quad 1 \le t \le T.$

Proof (1) $\Leftrightarrow$ (2) : We know that

$$V_t(\varphi) = \varphi_t \cdot S_t = \sum_{i=0}^{d} \varphi_t^i S_t^i$$

and this gives

$$\begin{aligned} V_t(\varphi) &= \big(V_t(\varphi) - V_{t-1}(\varphi)\big) + \cdots + \big(V_1(\varphi) - V_0(\varphi)\big) + V_0(\varphi) \\ &= (\varphi_t \cdot S_t - \varphi_{t-1} \cdot S_{t-1}) + \cdots + (\varphi_1 \cdot S_1 - \varphi_1 \cdot S_0) + \varphi_1 \cdot S_0 \\ &= \varphi_t \cdot (S_t - S_{t-1}) + \cdots + \varphi_1 \cdot (S_1 - S_0) + V_0(\varphi) \end{aligned}$$

if and only if φ is self-financing, i.e. it holds

$$\varphi_{t-1} \cdot S_{t-1} = \varphi_t \cdot S_{t-1}.$$

$$(1) \Leftrightarrow (3) : \quad \varphi_t \cdot S_t = \varphi_{t+1} \cdot S_t \Leftrightarrow \varphi_t \cdot \frac{S_t}{S_0} = \varphi_{t+1} \cdot \frac{S_t}{S_t^0}.$$

□

Example 4.3. A self-financing strategy φ

	bank account	first share	second share
time	S^0	S^1	S^2
0	1	20	50
1	1+0.05	25	40
2	$(1+0.05)^2$	23	45

Day 0: investors money: $V_0(\varphi) = 300\$$. The portfolio chosen at time 0

$$\begin{aligned} \varphi_1 &= (\varphi_1^0, \varphi_1^1, \varphi_1^2) \\ &= (100, 5, 2) \end{aligned}$$

$$V_0(\varphi) = \varphi_1 \cdot S_0 = 100 \times 1 + 5 \times 20 + 2 \times 50 = 300.$$

Day 1: investors value of φ_1: $\quad V_1(\varphi) = \varphi_1 \cdot S_1$,

$$V_1(\varphi) = 105 + 5 \times 25 + 2 \times 40 = 310$$

which is the amount that can be used for the new portfolio φ_2. It is *self-financing:*

$$\varphi_1 \cdot S_1 = 310 \overset{!}{=} \varphi_2 \cdot S_1.$$

If $\varphi_2 = \left(\frac{70}{1{,}05}\ ,\ 8\ ,\ 1\right)$, then

$$\varphi_2 \cdot S_1 = 70 + 8 \times 25 + 1 \times 40 = 310.$$

Day 2:

$$V_2(\varphi) = \frac{70}{1,05} \times (1,05)^2 + 8 \times 23 + 45 = 302,5.$$

Proposition 4.4. For any predictable process $(\varphi_t^1, ..., \varphi_t^d)_{t=1}^T$ and for any $V_0 \in \mathbb{R}$, there exists a unique predictable process $(\varphi_t^0)_{t=1}^T$ such that the strategy $\varphi = (\varphi^0, \varphi^1, ..., \varphi^d)$ is self-financing and $V_0(\varphi) = V_0$.

Proof. If φ is self-financing we get by Proposition 4.2 (3)

$$\begin{aligned} \tilde{V}_t(\varphi) &= V_0(\varphi) + \sum_{k=1}^t \varphi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) \\ &= V_0(\varphi) + \sum_{k=1}^t \varphi_k^0(\tilde{S}_k^0 - \tilde{S}_{k-1}^0) + \varphi_k^1(\tilde{S}_k^1 - \tilde{S}_{k-1}^1) + \cdots \\ &\qquad + \varphi_k^d(\tilde{S}_k^d - \tilde{S}_{k-1}^d). \end{aligned} \tag{4.1}$$

On the other hand,

$$\tilde{V}_t(\varphi) = \varphi_t \cdot \tilde{S}_t = \varphi_t^0 + \varphi_t^1 \tilde{S}_t^1 + \cdots + \varphi_t^d \tilde{S}_t^d. \tag{4.2}$$

From (4.1) and (4.2) we conclude

$$\begin{aligned} \varphi_t^0 &= V_0(\varphi) + \textstyle\sum_{k=1}^t \sum_{j=1}^d \varphi_k^j (\tilde{S}_k^j - \tilde{S}_{k-1}^j) - \sum_{j=1}^d \varphi_t^j \tilde{S}_t^j \\ &= V_0 + \textstyle\sum_{k=1}^{t-1} \sum_{j=1}^d \varphi_k^j (\tilde{S}_k^j - \tilde{S}_{k-1}^j) - \sum_{j=1}^d \varphi_t^j \tilde{S}_{t-1}^j \end{aligned}$$

From this it follows that $(\varphi_t^0)_{t=1}^T$ is uniquely defined. Moreover, it is predictable, i.e. φ_t^0 is $\mathcal{F}_{t-1}$-measurable because

- V_0 is a constant $\Leftrightarrow \mathcal{F}_0 \subseteq \mathcal{F}_{t-1}$ measurable,
- φ_t^j is $\mathcal{F}_{t-1}$-measurable for $j = 1, ..., d$,
- S_{t-1} is $\mathcal{F}_{t-1}$-measurable,
- addition and multiplication does keep the measurability.

□

Questions we want to answer

1. How can we get market models $\big(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), (S_t^0, S_t^1, ..., S_t^d)\big)$ where riskless profit is not possible?
2. Is there always a self-financing strategy φ to hedge the pay-off $V_T(\varphi) = f(S_T)$?
3. Is there a fair price for an option?

3 Properties of the conditional expectation

Assume $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. Then

$$\mathbb{P} : A \mapsto \mathbb{P}(A) \quad \forall A \in \mathcal{F} := \sigma(A_1, ..., A_N)$$

with the known properties of $\mathbb{P}$. If

$$f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$$

the *expectation of* f is defined by

$$\mathbb{E}f := \sum_{i=1}^{N} a_i \mathbb{P}(A_i).$$

Let us first recall some basic properties of the expectation.

Proposition 4.5. Assume $\Omega \neq \emptyset$ and $A_1, ..., A_N$ is a partition of Ω. Set $\mathcal{F} = \sigma(A_1, ..., A_N)$. Then it holds

1. A function $f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable
   $\Leftrightarrow$ f is constant on $A_1, ...A_N$, i.e. f can be represented by
   $f(\omega) = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}(\omega), \quad a_i \in \mathbb{R}, \omega \in \Omega.$

2. If f_1 and f_2 are $\mathcal{F}$-measurable and $a, b \in \mathbb{R}$ then $af_1 + bf_2$ and $f_1 \times f_2$ are $\mathcal{F}$-measurable.

3. $\mathbb{E}(af_1 + bf_2) = a\mathbb{E}f_1 + b\mathbb{E}f_2$, for f_1, f_2 $\mathcal{F}$-measurable and $a, b \in \mathbb{R}$.

Proof. We only prove 1. and leave 2. and 3. as an exercise.
(1) "$\Leftarrow$" Assume

$$f = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}, \quad a_i \in \mathbb{R}$$

If all the a_i's are different, then

$$f^{-1}(\{a_i\}) = A_i \in \mathcal{F}, \; i = 1, ..., N$$

If some a_i's are equal, we can arrange that

$$f = \sum_{i=1}^{n} b_j \mathbb{1}_{B_j} \; , \; b_j\text{'s different}$$

and $B_1, ..., B_n$ is a partition on Ω while all B_j's are unions of some A_i's

$$\Rightarrow B_j \in \mathcal{F} \quad \forall j.$$

Since $f^{-1}(\{b_j\}) = B_j$, $\Rightarrow f$ is $\mathcal{F}$-measurable.
"$\Rightarrow$" Assume f is *not* constant on all $A_1, ..., A_N$. We will show that then f is not $\mathcal{F}$-measurable: If there exists A_j such that f that is not constant on A_j then

$$\begin{array}{rl} \exists & \omega_1, \omega_2 \in A_i \quad a = f(\omega_1) \neq f(\omega_2) = b \\ \Rightarrow & \omega_1 \in f^{-1}(\{a\}) \\ & \omega_2 \in f^{-1}(\{b\}) \end{array}$$

Because f is a function we have

$$f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = \emptyset$$

But $\mathcal{F}$ consists only of unions of $A_1, ...A_N$, that means for any set $A \in \mathcal{F}$ it holds

$$\begin{array}{rl} \text{either} & \{\omega_1, \omega_2\} \subseteq A \\ \text{or} & \{\omega_1, \omega_2\} \subseteq A^c. \end{array}$$

Consequently, f is constant on any A_j. □

Example 4.6. If $f_1 = \mathbb{1}_A$, $f_2 = \mathbb{1}_B$ then

$$\begin{aligned} f_1 + f_2 &= \mathbb{1}_A + \mathbb{1}_B \\ &= \mathbb{1}_{A\cap B} + \mathbb{1}_{A\cup B} \\ &= \mathbb{1}_{(A\backslash B)\cup(B\backslash A)} + 2\mathbb{1}_{A\cap B} + 0\mathbb{1}_{(A\cup B)^c} \end{aligned}$$

and

$$f_1 f_2 = \mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A\cap B}.$$

Notice that

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F} \quad \text{and} \quad A\backslash B = A \cap B^c \in \mathcal{F}.$$

Proposition 4.7. Let $\mathcal{F} = \sigma(A_1, ..., A_N)$ like above. Then it holds

1. If $\mathcal{G}$ is a σ-field with $\mathcal{G} \subseteq \mathcal{F}$ and f is $\mathcal{G}$-measurable, then

$$\mathbb{E}[f|\mathcal{G}] = f.$$

2. "tower-property": f is $\mathcal{F}$-measurable, $\mathcal{G}_1$ and $\mathcal{G}_2$ are σ-fields such that $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ then

$$\mathbb{E}[\mathbb{E}[f|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[\mathbb{E}[f|\mathcal{G}_2]|\mathcal{G}_1] = \mathbb{E}[f|\mathcal{G}_1].$$

3. If g is $\mathcal{G}$-measurable and $\mathcal{G} \subseteq \mathcal{F}$ then

$$\mathbb{E}[fg|\mathcal{G}] = g\mathbb{E}[f|\mathcal{G}].$$

Proof: Exercise.

Example 4.8. If A_1, A_2, A_3 form a partition of Ω and

$$\mathbb{P}(A_1) = \frac{1}{10}, \quad \mathbb{P}(A_2) = \frac{7}{10}, \quad \mathbb{P}(A_3) = \frac{2}{10},$$

$$\mathcal{F} = \sigma(A_1, A_2, A_3),$$

$$f = a_1 \mathbb{1}_{A_1} + a_2 \mathbb{1}_{A_2} + a_3 \mathbb{1}_{A_3},$$

then

$$\mathbb{E}f = \frac{a_1}{10} + \frac{7a_2}{10} + \frac{2a_3}{10}.$$

Assume that

$$\mathcal{G} = \sigma(A_1 \cup A_2, A_3) \subseteq \sigma(A_1, A_2, A_3),$$

then $h = \mathbb{E}[fg|\mathcal{G}]$ is by definition $\mathcal{G}$-measurable, i.e. we have

$$h = b_1 \mathbb{1}_{A_1 \cup A_2} + b_2 \mathbb{1}_{A_3}.$$

We want to evaluate b_1 and b_2. By definition,

$$\mathbb{E}(h\mathbb{1}_B) \stackrel{!}{=} \mathbb{E}(f\mathbb{1}_B) \quad \forall B \in \mathcal{G}.$$

As it follows from the Lemma below it is sufficient to test only with $B \in \{A_1 \cup A_2, A_3\}$. We start with the condition

$$\mathbb{E}\big[(b_1 \mathbb{1}_{A_1 \cup A_2} + b_2 \mathbb{1}_{A_3})\mathbb{1}_{A_1 \cup A_2}\big] = \mathbb{E}(f\mathbb{1}_{A_1 \cup A_2}).$$

From the LHS we get

$$\mathbb{E}b_1 \mathbb{1}_{A_1 \cup A_2} = b_1 \mathbb{P}(A_1 \cup A_2)$$

and the RHS implies

$$\mathbb{E}a_1 \mathbb{1}_{A_1} + a_2 \mathbb{1}_{A_2} = a_1 \mathbb{P}(A_1) + a_2 \mathbb{P}(A_2).$$

This implies

$$b_1 = \frac{a_1 + 7a_2}{10}\frac{10}{8} = \frac{a_1 + 7a_2}{8}.$$

For $B = A_3$ we get from

$$\mathbb{E}(b_1 \mathbb{1}_{A_1 \cup A_2} + b_2 \mathbb{1}_{A_3})\mathbb{1}_{A_3} = \mathbb{E}(f \mathbb{1}_{A_3})$$

that it should hold

$$\mathbb{E} b_2 \mathbb{1}_{A_3} = \mathbb{E} a_3 \mathbb{1}_{A_3}$$

which implies $b_2 = a_3$. Hence

$$\mathbb{E}[f|\mathcal{G}] = \frac{a_1 + 7a_2}{8}\mathbb{1}_{A_1 \cup A_2} + a_3 \mathbb{1}_{A_3}.$$

Lemma 4.9. *Let $\mathcal{F}$ be a σ-field, f an $\mathcal{F}$-measurable function and $\mathcal{G} = \sigma(B_1, ..., B_n)$ where $B_1, ..., B_n$ is a partition of Ω. Assume that h is $\mathcal{G}$-measurable and*

$$\mathbb{E}(h \mathbb{1}_{B_j} = \mathbb{E}(f \mathbb{1}_{B_j}) \quad \forall B_j \ , j = 1, ..., n.$$

Then

$$\mathbb{E}(h \mathbb{1}_B) = \mathbb{E}(f \mathbb{1}_B) \quad \forall B \in \mathcal{G}.$$

4 Admissible strategies and arbitrage

If $\varphi_t^0 < 0$, we had borrowed the amount $|\varphi_t^0|$ from the bank at time $t-1$. If $\varphi_t^i < 0$ for $i \in \{1, ..., d\}$ we say that we are *short* a number φ_t^i of assets (shares) i. Borrowing and short-selling is allowed as long as the value of the portfolio $V_t(\varphi)$ is always non-negative.

Definition 4.10. 1. A strategy φ is *admissible* if it is self-financing and if

$$V_t(\varphi) \geq 0 \quad \forall t \in \mathbb{T}.$$

2. An *arbitrage opportunity* is an admissible strategy φ such that

$$V_0(\varphi) = 0 \text{ and } \mathbb{E} V_T(\varphi) > 0.$$

(Arbitrage means a possibility of riskless profit: 'free lunch'.)

3. The market is *viable* if it does not contain any arbitrage opportunities, i.e. if it holds

$$V_0(\varphi) = 0 \Rightarrow V_T(\varphi) = 0 \quad \forall \text{ admissible } \varphi.$$

Let us assume in the following that $\Omega = \{\omega_1, ..., \omega_N\}$. We can identify the space of all functions $f : \Omega \to \mathbb{R}$ with $\mathbb{R}^N$:

$$f \leftrightarrow \big(f(\omega_1), ..., f(\omega_N)\big) \in \mathbb{R}^N$$

Define

$$C := \big\{x = (x_1..., x_N) \in \mathbb{R}^N : \quad x_i \geq 0, i = 1, ..., N \text{ and there exists } i : x_i > 0\}$$

C is a *convex cone*.

Definition 4.11. *A subset* C *of a vector space is a* convex cone *if it holds:*

$$x, y \in C \Rightarrow x + y \in C$$
$$x \in C, a > 0 \Rightarrow ax \in C$$

Define $\Psi_a :=$ set of admissible strategies.

Recall that φ is self-financing if and only if

$$\tilde{V}_t(\varphi) = V_0(\varphi) + \sum_{k=1}^{t} \varphi_k \cdot (\tilde{S}_k - \tilde{S_{k-1}}).$$

The discounted gains process will be defined by

$$\tilde{G}_t(\varphi) := \sum_{k=1}^{t} \varphi_k \cdot (\tilde{S}_k - \tilde{S_{k-1}}).$$

Lemma 4.12. If the market $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$ is *viable* (does not admit any arbitrage opportunities) then it holds

$$\tilde{G}_T(\varphi) \notin C \quad \forall \text{ predictable } (\varphi_t^1, ..., \varphi_t^d)_{t=1}^T.$$

Proof. Assume $\tilde{G}_T(\varphi) \in C$. We will show that the market is not viable. First we prove that if

$$\tilde{G}_t(\varphi) \geq 0, \quad t = 0, ..., T$$

it follows that the market is not viable. Notice that

$$\begin{aligned} \tilde{G}_t(\varphi) &= \sum_{k=1}^{t} \varphi_k \cdot (\tilde{S}_k - \tilde{S_{k-1}}) \\ &= \sum_{k=1}^{t} \varphi_k^0(\tilde{S}_k^0 - \tilde{S}_{k-1}^0) + \varphi_k^1(\tilde{S}_k^1 - \tilde{S}_{k-1}^1) + \cdots + \varphi_k^d(\tilde{S}_k^d - \tilde{S}_{k-1}^d). \end{aligned}$$

Hence $\tilde{G}_T(\varphi)$ does not depend on φ^0. Proposition 4.4 implies that given $(\varphi^1, ..., \varphi^d)$ which is predictable and $V_0 = 0$ then there is a predictable and self-financing φ such that

$$\tilde{V}_t(\varphi) = V_0 + \tilde{G}_t(\varphi).$$

So we conclude that

$$\tilde{V}_0(\varphi) = 0, \quad \tilde{V}_t(\varphi) \geq 0 \quad t = 0, ..., T.$$

But $\tilde{G}_T(\varphi) \in C$ means $\tilde{G}_T(\varphi)(\omega_i) \geq 0 \quad i = 1, ..., N$ and there exists i_0 with $\tilde{G}_T(\varphi)(\omega_{i_0}) > 0$. Hence

$$\begin{aligned} \tilde{G}_T(\varphi) &= \sum_{i=1}^{N} \tilde{G}_T(\varphi)(\omega_i)\mathbb{P}(\{\omega_i\}) \\ &\geq G_T(\varphi)(\omega_{i_0}))\mathbb{P}(\{\omega_{i_0}\}) > 0. \end{aligned}$$

So there exists an arbitrage opportunity and the market is not viable.

Now we consider the general case i.e. $\tilde{G}_t(\varphi)$ can have negative values. Set

$$t_0 = \sup \left\{ t : \mathbb{P}(\{\omega : \tilde{G}_t(\varphi) < 0\}) > 0 \right\}$$

Clearly,

1. $t_0 \leq T - 1 \qquad$ (since $\tilde{G}_T(\varphi) \in C$),
2. $\mathbb{P}(\{\tilde{G}_{t_0}(\varphi) < 0\}) > 0$,
3. $\tilde{G}_t(\varphi) \geq 0, \qquad \forall t = t_0 + 1, ..., T$

We define a new strategy as follows. For $i = 1, ..., d$ put

$$\psi_t^i(\omega) := \begin{cases} 0, & \text{if } t \leq t_0, \\ \mathbb{1}_A(\omega)\varphi_t^i(\omega), & \text{if } t > t_0 \end{cases}$$

where $A = \{\omega : \tilde{G}_{t_0}(\varphi) < 0\} \in \mathcal{F}_{t_0}$ and hence ψ_t is predictable. It holds

$$\tilde{G}_t(\psi) = \sum_{k=1}^{t} \psi_k \cdot (\tilde{S}_k - \tilde{S}_{k-1}) = \begin{cases} 0 & t \leq t_0 \\ \mathbb{1}_A\big(\tilde{G}_t(\varphi) - \tilde{G}_{t_0}(\varphi)\big) & t > t_0 \end{cases}$$

so that by construction we have $\tilde{G}_t(\varphi) \geq 0$ and $-\tilde{G}_{t_0}(\varphi) > 0$ on A. Thus

$$\tilde{G}_t(\psi) \geq 0, \quad t = 0, ..., T, \quad \tilde{G}_T(\varphi) > 0 \text{ on } A.$$

Hence

$$\begin{aligned} \mathbb{E}\tilde{G}_T(\psi) &= \sum_{i=1}^{N} G_T(\psi)(\omega_i)\mathbb{P}(\{\omega_i\}) \\ &\geq \sum_{\omega_i \in A} G_T(\psi)(\omega_i)\mathbb{P}(\{\omega_i\}) > 0, \end{aligned}$$

i.e. the market is not viable which means $G_T(\psi) \notin C$. □

Remark

About the assumptions on our 'market': in contrary to reality we always assume here:

- a 'frictionless' market: no transaction costs,
- short sale and borrowing without any limit ($\varphi_t^i \in \mathbb{R}$),
- the securities are perfectly divisible: $S_t^i \in [0, \infty)$.

5. The fundamental theorem of asset pricing

1 Separation of convex sets in $\mathbb{R}^N$

Theorem 5.1. Let $C \in \mathbb{R}^N$ be a closed convex set and $(0, ..., 0) \notin C$. Then there exists a real linear functional $\xi : \mathbb{R}^N \to \mathbb{R}$ and $\alpha > 0$ such that

$$\xi(x) \geq \alpha \quad \forall x \in C.$$

Proof Let $B(0,r) = \{x \in \mathbb{R}^N : ||x|| := (x_1^2 + \cdots + x_N^2)^{\frac{1}{2}} \leq r\}$ which equals to a closed ball of radius r and center at the origin. Choose $r > 0$ such that

$$C \cap B(0,r) \neq \emptyset.$$

The map $x \mapsto ||x||$ is a continuous function and $C \cap B(0,r)$ is closed and bounded. Let $m_0 := \min_{x \in C \cap B(0,r)} ||x||$. Then there exists an $x_0 \in C \cap B(0,r)$ with $||x_0|| = m_0$.

Indeed, take $(x_n)_{n=1}^\infty$ with $||x_n|| \to m_0$ as $n \to \infty$. Then there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x_0$ for $k \to \infty$. The claim follows from

$$||x_0|| \leq ||x_0 - x_{n_k}|| + ||x_{n_k}||$$

because $||x_0 - x_{n_k}|| \to 0$ and $||x_{n_k}|| \to m_0$.
Hence,

$$||x|| \geq ||x_0|| \quad \forall x \in C \quad (x \notin B(0,r) \Rightarrow ||x|| > r)$$

Notice that

$$x \in C \Rightarrow \lambda x + (1-\lambda)x_0 \in C$$

for $\lambda \in [0,1]$ since C is convex. This implies

$$||\lambda x + (1-\lambda)x_0|| \geq ||x_0||$$

and therefore

$$\lambda^2 x \cdot x + 2\lambda(1-\lambda)x \cdot x_0 + (1-\lambda)^2 x_0 \cdot x_0 \geq x_0 \cdot x_0.$$

This gives

$$\lambda x \cdot x + 2(1-\lambda)x \cdot x_0 - 2x_0 \cdot x_0 + \lambda x_0 \cdot x_0 \geq 0$$

and

$$2x \cdot x_0 + \lambda(x \cdot x - 2x \cdot x_0 + x_0 \cdot x_0) \geq 2x_0 \cdot x_0 \quad \forall \lambda \in [0,1]$$

For $\lambda \to 0$ this inequality is only true if

$$x \cdot x_0 \geq x_0 \cdot x_0 = ||x_0||^2 > 0.$$

If we define $\xi(x) := x_0 \cdot x$ we get a linear functional

$$\xi(x) \geq m_0^2 = \alpha \quad \text{for} \quad x \in C.$$

□

Theorem 5.2. *Let K be a compact convex subset in $\mathbb{R}^N$ and V a linear subspace of $\mathbb{R}^N$. If $V \cap K = \emptyset$, then there exists a linear functional*

$$\xi : \mathbb{R}^N \to \mathbb{R}$$

such that

1. $\xi(x) > 0, \quad \forall x \in K,$
2. $\xi(x) = 0, \quad \forall x \in V.$

Therefore, the subspace V is included in a hyperplane that does not intersect K.

Proof. The set

$$C := K - V = \{x \in \mathbb{R}^N : \exists (k,v) \in K \times V,\ x = k - v\}$$

is convex since for $x_1, x_2 \in C$ we have

$$\lambda x_1 + (1-\lambda)x_2 \quad = \quad \lambda(k_1 - v_1) + (1-\lambda)(k_2 - v_2)$$

$$= \lambda k_1 + (1-\lambda)k_2 - (\lambda v_1 + (1-\lambda)v_2).$$

Now $\lambda k_1 + (1-\lambda)k_2 \in K$ and $\lambda v_1 + (1-\lambda)v_2 \in V$ and its difference is in C. The set C is closed because V is closed and K is compact. We have $(0,\dots,0) \notin C$ since $V \cap K = \emptyset$. Hence we can apply Theorem 5.1 and find a linear functional $\xi : \mathbb{R}^N \to \mathbb{R}$ and a constant $\alpha > 0$ with

$$\xi(x) \geq \alpha \quad \forall x \in C.$$

This implies

$$\xi(k - v) = \xi(k) - \xi(v) \geq \alpha \quad \forall k \in K, v \in V.$$

Especially, it holds for fixed $k_0 \in K$ and and $v_0 \in V$ and all $\lambda \in \mathbb{R}$ that

$$\xi(k_0) - \xi(\lambda v_0) \geq \alpha,$$

and because ξ is linear also

$$\xi(k_0) - \lambda \xi(v_0) \geq \alpha.$$

Consequently, $\xi(v) = 0$ for all $v \in V$ and $\xi(k) \geq \alpha$ for all $k \in K$. □

2 Martingale transforms

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space.

Lemma 5.3. Let $(\mathcal{F}_n)_{n=0}^T$ be a filtration $(\varphi_n)_{n=1}^T$ a predictable sequence and $(M_n)_{n=0}^T$ a martingale. Then the process

$$X_0 := 0$$

$$X_n := \varphi_1(M_1 - M_0) + \varphi(M_2 - M_1) + ... + \varphi_n(M_n - M_{n-1}), \quad n = 1, ..., T$$

is a martingale with respect to $(\mathcal{F}_n)_{n=0}^T$. The sequence $(X_n)_{n=0}^T$ is called a *martingale transform* of (M_n) by (φ_n).

Proof. We have that X_n is $\mathcal{F}_n$-measurable for all $n = 0, ..., T$. We check the martingale property:

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}\Big[\sum_{t=1}^{n+1} \varphi_t(M_t - M_{t-1})\big|\mathcal{F}_n\Big]$$

$$\begin{aligned} &= \sum_{t=1}^{n} \varphi_t(M_t - M_{t-1}) + \mathbb{E}[\varphi_{n+1}(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= X_n \end{aligned}$$

where we used that

$$\begin{aligned} \mathbb{E}[\varphi_{n+1}(M_{n+1} - M_n)|\mathcal{F}_n] &= \varphi_{n+1}\mathbb{E}[(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= \varphi_{n+1}\mathbb{E}[M_{n+1}|\mathcal{F}_n] - \varphi_{n+1}M_n \end{aligned}$$

because (φ_n) is predictable and (M_n) is adapted.

□

Theorem 5.4. An adapted real-value process $(M_n)_{n=0}^T$ is a martingale if and only if

$$\mathbb{E}\sum_{n=1}^{t} \varphi_n(M_n - M_{n-1}) = 0 \quad \forall t = 1, ..., T \tag{5.1}$$

for all predictable processes $(\varphi_n)_{n=1}^T$.

Proof. " $\Rightarrow$"
If $(M_n)_{n=0}^T$ is a martingale, $X_t = \sum_{n=1}^t \varphi_n(M_n - M_{n-1})$ is a martingale transform. Hence by the previous Lemma

$$\mathbb{E}X_t = 0 \;\; \forall t = 1, ..., T.$$

"$\Leftarrow$" Assume (5.1) holds. Let $A \in \mathcal{F}_{n_0}$ and define

$$\varphi_n(\omega) := \begin{cases} 0 \quad n \neq n_0 + 1 \\ \mathbb{1}_A(\omega) \quad n = n_0 + 1. \end{cases}$$

Then

$$\mathbb{E}X_T = \mathbb{E}\mathbb{1}_A(M_{n_0+1} - M_{n_0}) = 0 \quad \forall A \in \mathcal{F}_{n_0}$$

and consequently

$$\mathbb{E}[M_{n_0+1}|\mathcal{F}_{n_0}] = M_{n_0} \quad n_0 = 0, ..., T.$$

□

Definition 5.5. (Independence)

1. The *sets* $A, B \in \mathcal{F}$ are called *independent* $:\Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

2. The *σ-fields* $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{F}$ are called *independent*

$$:\Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \quad \forall A \in \mathcal{G}_1, B \in \mathcal{G}_2$$

   (every set of $\mathcal{G}_1$ is independent of every set of $\mathcal{G}_2$).

3. If $f_1, ..., f_n : \Omega \to \{a_1, ..., a_M\}$ ($a_i \in \mathbb{R}$) are $\mathcal{F}$-measurable then $f_1, ..., f_n$ are called *independent (random variables)* $:\Leftrightarrow$

$$\mathbb{P}(\{\omega : f_1(\omega) = x_1, ..., f_N(\omega) = x_n\})) = \Pi_{k=1}^n \mathbb{P}(\{\omega : f_i(\omega) = x_i)$$

   $\forall x_i \in \{a_1, ..., a_M\}$. In other words all the pre-images of $f_1, ..., f_n$ are independent sets.

4. An $\mathcal{F}$-measurable *function* f is called *independent* from a *σ-field* $\mathcal{G}$ ($\mathcal{G} \subseteq \mathcal{F}$) $:\Leftrightarrow$

$$f \text{ and } \mathbb{1}_G \text{ are independent } \forall G \in \mathcal{G}.$$

Remark to (3)

$$\begin{aligned}\{\omega : f_1(\omega) = x_1, ..., f_N(\omega) = x_n\} &= \{\omega : f_1(\omega) = x_1 \text{ and ... and } f_n(\omega) = x_n\} \\ &= \bigcap_{k=1}^n f_k^{-1}(\{x_k\}).\end{aligned}$$

Example 5.6. 1. Tossing a coin 2 times:

$$\begin{aligned}&\mathbb{P}(\text{1st toss = 'heads' and 2nd toss = 'tails'}) \\ &\quad = \mathbb{P}(\text{1st toss = 'heads'})\mathbb{P}(\text{ 2nd toss = 'tails'})\end{aligned}$$

2.

CRR model	Tossing a coin T-times
$\Omega = \{\omega = (\epsilon_1, ..., \epsilon_T), \epsilon_i \in \{(1+a), (1+b)\}\}$	Write for each toss
$\mathbb{P}(\{\omega\}) = p^k(1-p)^{T-k}$	$\begin{cases} 1+a \text{ if 'tails'} \\ 1+b \text{ if 'heads'} \end{cases}$
if ω contains k times $1+b$ and $T-k$ times $1+a$	$\mathbb{P}$(tossing 'heads') $= p$
	$\Rightarrow \mathbb{P}$(tossing 'tails')$=1-p$

If

$$S_t(\omega) = S_0 \varepsilon_1 \varepsilon_2 \cdots \varepsilon_t$$

then

$$\frac{S_{t+1}(\omega)}{S_t(\omega)} = \varepsilon_{t+1}.$$

The functions

$$\frac{S_1}{S_0}, \quad \frac{S_2}{S_1}, ..., \frac{S_T}{S_{T-1}}$$

are independent: for $x_i \in \{(1+a),(1+b)\}$

$$\begin{aligned} &\mathbb{P}\big(\{\omega : \frac{S_1(\omega)}{S_0(\omega)} = x_1, ..., \frac{S_T(\omega)}{S_{T-1}(\omega)} = x_T\}\big) \\ &= \mathbb{P}\big(\{\omega = (\varepsilon_1, ..., \varepsilon_T) : \varepsilon_1 = x_1, ..., \varepsilon_T = x_T\}\big) \\ &= p^k(1-p)^{T-k} \\ &= \Pi_{t=1}^T \mathbb{P}\big(\{\omega : \frac{S_t}{S_{t-1}} = \varepsilon_t = x_t\}\big) \end{aligned}$$

if k of the x_i's are $1+b$ and $T-k$ are equal to $1+a$. We have for $t = 1, \dots, T$

$$\mathbb{P}\big(\{\omega = (\varepsilon_1, ..., \varepsilon_T) : \varepsilon_t = 1+b\}\big) = p$$

where p is the probability that one tosses 'heads' the t-th time if one tosses T-times altogether. The outcome of the other times does not influence that of time t.

Theorem 5.7. Let f, g be $\mathcal{F}$-measurable.

1. If f and g are independent then

$$\mathbb{E}fg = \mathbb{E}f\mathbb{E}g$$

2. If f is independent from the σ-field $\mathcal{G}(\mathcal{G} \subseteq \mathcal{F})$ then $\mathbb{E}[f|\mathcal{G}] = \mathbb{E}f$

Proof

1. Let

$$f = \sum_{i=1}^n x_i \mathbb{1}_{F_i} \; , \; g = \sum_{j=1}^m y_j \mathbb{1}_{G_j}$$

where we assume that all x_i's are different and all y_j's are different.

$$\begin{aligned}
\mathbb{E}fg &= \mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \mathbb{1}_{F_i}\mathbb{1}_{G_j} \\
&= \mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \mathbb{1}_{F_i\cap G_j} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \mathbb{P}(F_i\cap G_j) \\
&= \big(\sum_{i=1}^{n} x_i \mathbb{P}(F_i)\big)\big(\sum_{j=1}^{m} y_i \mathbb{P}(G_j)\big) = \mathbb{E}f\mathbb{E}g
\end{aligned}$$

where we used $\mathbb{P}(F_i \cap G_j) = \mathbb{P}\big(\{\omega : f(\omega) = x_i\} \cap \{g(\omega) = y_j\}\big) = \mathbb{P}\big(\{\omega : f(\omega) = x_i\}\big)\mathbb{P}\big(\{\omega : g(\omega) = y_j\}\big) = \mathbb{P}(F_i)\mathbb{P}(G_j)$.

2. Exercise.

□

Proposition 5.8. Assume $f_1, ..., f_n$ are independent and $\mathcal{F}$-measurable. Let $\mathcal{F}_k = \sigma(f_1, ..., f_k)$.

1. Then $f_l, l > k$ is independent from $\mathcal{F}_k$.
2. If $\mathbb{E}f_k = 0$, for $k = 1, ..., n$ then (M_t) with $M_t := \sum_{k=1}^{t} f_k$ for $t \geq 1$ and $M_0 = 0$ is a martingale with respect to $(\mathcal{F}_t)$.
3. If $\mathbb{E}f_k = 1$, for $k = 1, ..., n$ then (N_t) with $N_t := \Pi_{k=1}^{t} f_k$ for $t \geq 1$ and $N_0 = 1$ is a martingale with respect to $(\mathcal{F}_t)$.

Proof
1. The idea is to use $G \in \mathcal{F}_k$ which can be represented by $G = \{f_1 = x_1, ..., f_k = x_k\}$ and to show that

$$\mathbb{P}(f_l = x_l,\ \mathbb{1}_G = x) = \mathbb{P}(f_l = x_l)\mathbb{P}(\mathbb{1}_G = x).$$

2. and 3. are Exercises.

Remark 5.9. *One can show that*

$$\sigma\left(\frac{S_1}{S_0}, ..., \frac{S_t}{S_{t-1}}\right) = \sigma(S_1, ...S_t) = \mathcal{F}_t.$$

From (3) it follows now that $(S_t)_{t=0}^T$ *with* $S_t = S_0 \frac{S_1}{S_0}\frac{S_2}{S_1} \cdots \frac{S_t}{S_{t-1}}$ *is a martingale*

$$\Leftrightarrow \mathbb{E}\frac{S_t}{S_{t-1}} = 1 \quad \forall t.$$

3 The fundamental theorem of asset pricing

With the results of the previous sections we will get a characterization of the 'no arbitrage' condition.

Definition 5.10. Let $\mathbb{P}, \mathbb{Q} : (\Omega, \mathcal{F}) \to [0,1]$ be probability measures. Then $\mathbb{P}$ *is said to be equivalent to* $\mathbb{Q}$ (notation $\mathbb{P} \sim \mathbb{Q}$) if and only if

$$\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0 \quad \text{ for any } A \in \mathcal{F}.$$

If $\Omega := \{\omega_1, ..., \omega_N\}$, $\mathcal{F} := 2^\Omega$ and $\mathbb{P}(\{\omega_i\}) > 0$, $i = 1, ..., N$ then $\mathbb{Q} \sim \mathbb{P}$ iff $\mathbb{Q}(\{\omega_i\}) > 0$, $i = 1, ..., N$.

Theorem 5.11. [Fundamental Theorem of Asset pricing]
The market $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$ is viable if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\tilde{S_t^i} = \frac{1}{S_t^0} S_t^i, t \in \mathbb{T}$ are $\mathbb{Q}$-martingales for $i = 1, ..., d$ ($\mathbb{Q}$ is called the *equivalent martingale measure: EMM*).

Proof. The proof for (our case, namely $\#\Omega < \infty$) was done by Harrison, Kreps and Pliska between 1979 and 1981. For general Ω this theorem was proved by Dalang, Morton and Willinger in 1990.

" $\Leftarrow$ "
Assume $\mathbb{Q} \sim \mathbb{P}$ and $\tilde{S_t^i}$, $i = 1, ..., d$ are $\mathbb{Q}$-martingales. By Proposition 4.2(3), if φ is self-financing then

$$\tilde{V_t}(\varphi) = V_0(\varphi) + \sum_{k=1}^{t} \varphi_k \cdot (\tilde{S_k} - \tilde{S_{k-1}}).$$

We denote by $\mathbb{E}_{\mathbb{Q}}$ the expectation with respect to $\mathbb{Q}$. Hence by Theorem 5.4

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\tilde{V_T}(\varphi) &= \mathbb{E}_{\mathbb{Q}}V_0(\varphi) + \mathbb{E}_{\mathbb{Q}}\sum_{k=1}^{T}\varphi_k\cdot(\tilde{S_k} - \tilde{S_{k-1}}) \\
&= \mathbb{E}_{\mathbb{Q}}V_0(\varphi) + \mathbb{E}_{\mathbb{Q}}\sum_{i=1}^{d}\sum_{k=1}^{T}\varphi_k^i(\tilde{S_k^i} - \tilde{S_{k-1}^i}) \\
&= \mathbb{E}_{\mathbb{Q}}V_0(\varphi) + \sum_{i=1}^{d}\mathbb{E}_{\mathbb{Q}}\sum_{k=1}^{T}\varphi_k^i(\tilde{S_k^i} - \tilde{S_{k-1}^i}) \\
&= \mathbb{E}_{\mathbb{Q}}V_0(\varphi). \qquad (5.2)
\end{aligned}
$$

If $V_0(\varphi) = 0$ then $\mathbb{E}_{\mathbb{Q}}\tilde{V_T}(\varphi) = 0$. Now assume that

$$\mathbb{E}_{\mathbb{Q}}\tilde{V_T}(\varphi) = \sum_{i=1}^{N}\tilde{V_T}(\varphi)(\omega_i)\mathbb{Q}(\{\omega_i\}) = 0. \qquad (5.3)$$

If φ is admissible, then $\tilde{V_T}(\varphi)(\omega_i) \geq 0$, $i = 1, ..., N$. So (5.3) implies that $\tilde{V_T}(\varphi)(\omega_i) = 0,\ i = 1, ..., N$.
Consequently,

$$V_0(\varphi)(\omega_i) = 0, \qquad i = 1, ..., N \Rightarrow V_T(\varphi)(\omega_i) = 0,\ i = 1, ..., N$$

for all admissible φ. Hence the market does not admit arbitrage opportunities and is not viable.
" $\Rightarrow$ "
By Lemma 4.12 we have : If the market is viable then

$$\tilde{G}_T(\varphi) \notin C = \{x = (x_1, ..., x_N) \in \mathbb{R}^N, x_i \geq 0, i = 1, ..., N, \exists i\ x_i > 0\} \quad \forall(\varphi^1, ..., \varphi^d) \text{ predictable}$$

where

$$\tilde{G}_t(\varphi) = \sum_{k=1}^{t}\left(\varphi_k^1(\tilde{S_k^1} - \tilde{S_{k-1}^1}) + \cdots + \varphi_k^d(\tilde{S_k^d} - \tilde{S_{k-1}^d})\right)$$

is the discounted gains process. We define

$$V := \{\tilde{G}_T(\varphi) : (\varphi^1, ..., \varphi^d) \text{ predictable }.\}$$

Then V is a linear subspace of $\mathbb{R}^N$ and

$$\big(\tilde{G}_T(\varphi)(\omega_1), ..., \tilde{G}_T(\varphi)(\omega_N)\big) \in \mathbb{R}^N.$$

By the Lemma we have $V \cap C = \emptyset$. We define

$$K := \{f = (f(\omega_1), ..., f(\omega_N)) \in C : \sum_{i=1}^{N} f(\omega_i) = 1\}.$$

We have that

$$V \cap K = \emptyset$$

and K is convex: If $f, g \in K$ then

1. $\lambda f + (1-\lambda) g \in C$ (since C is convex)
2. $\sum_{i=1}^N \big(\lambda f(\omega_i) + (1-\lambda) g(\omega_i)\big) = \lambda + (1-\lambda) = 1$

K is compact because it is bounded ($||f|| = \sum_{i=1}^N |f(\omega_i)| = 1$) and closed. Therefore, by Theorem 5.2 there exists a linear functional $\xi(x) = \xi_1 x_1 + \cdots + \xi_N x_N$ with

1. $\sum_{i=1}^N \xi_i f(\omega_i) > 0 \quad \forall f \in K$
2. $\sum_{i=1}^N \xi_i \tilde{G}_T(\varphi)(\omega_i) = 0 \quad \forall (\varphi^1, ..., \varphi^d)$ predictable.

Now, if $f := (0, ..., 0, 1, 0, ..., 0)$, then $f \in K$ and "1." implies $\xi_i > 0$ for all $i = 1, ..., N$. We define

$$\mathbb{Q}(\{\omega_i\}) := \frac{\xi_i}{\sum_{i=1}^N \xi_i}.$$

Then $\mathbb{Q}$ is a probability measure, $\mathbb{Q} \sim \mathbb{P}$ and by "2."

$$\mathbb{E}_{\mathbb{Q}} \tilde{G}_T(\varphi) = \sum_{i=1}^{N} \tilde{G}_N(\varphi)(\omega_i) \mathbb{Q}(\{\omega_i\}) = 0 \quad \forall (\varphi_i, ..., \varphi^d) \text{ predictable.}$$

In other words,

$$\mathbb{E}_{\mathbb{Q}} \sum_{t=1}^{T} \sum_{i=1}^{d} \varphi_t^i (\tilde{S}_t^i - \tilde{S}_{t-1}^i) = 0$$

or in short form

$$\mathbb{E}_{\mathbb{Q}} \sum_{i=1}^{N} \varphi_t \cdot (\tilde{S}_t - \tilde{S}_{t-1}) = 0 \quad \forall i = 1, ..., d \ (\varphi_t^i) \text{ predictable.}$$

Hence by Theorem 5.4 we have that $(\tilde{S}^1, \ldots, \tilde{S}^d)$ are $\mathbb{Q}$-martingales.

□

Remark 5.12. The *scalar product* (or inner product) on a vector space $V(= \mathbb{R}_N)$ is a function

$$(\ ,\ ) : V \times V \to \mathbb{R} \text{ such that } \forall \alpha, \beta \in \mathbb{R}$$

and $v_1, v_2, v \in V$

1. $(\alpha v_1 + \beta v_2, v) = \alpha(v_1, v) + \beta(v_2, v)$ linearity
2. $(v1, v) = (v, v_1)$ symmetry
3. $(v, v) \geq 0$ and $(v, v) = 0 \Leftrightarrow v = 0$ positive definite.

For $(x_1, ..., x_N) \in \mathbb{R}_N$, $v, w \in V$ the expression
$(v, w) := \sum_{i=1}^{N} v_i w_i x_i$ defines a scalar product iff $x_i > 0 \quad \forall i = 1, ..., N$. (To see this assume $x_1 = 0$. Then $v = (1, 0, ..., 0)$ implies $(v, v) = 0$ which is a contadiction. In the same way it follows that $x_i < 0$ is not possible.)

Orthogonality: We define $V \perp W$ orthogonal $\Leftrightarrow (v, w) = 0$ for all $v \in V$ and $w \in W$.

4 Complete markets and option pricing

Let us assume the market model $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$. We already know the European call-option $H = (S_T^1 - K)^+$ and the European put-option $H = (K - S_T^1)^+$.
Options can also depend on the whole path of the underlying security. For example,

$$H = \left(S_T^1 - \frac{S_1^1 + S_2^1 + \cdots + S_T^1}{T} \right)^+$$

would be one type of a so-called *Asian option*. In general we define a European option (or a *contingent claim*) to be a non-negative function $H : \Omega \to [0, \infty)$ which is $\mathcal{F}$-measurable. We say the contingent claim H is *attainable* if there exists an admissible strategy φ with

$$H = V_T(\varphi).$$

If the market is viable, then there exists a $\mathbb{Q} \sim \mathbb{P}$ such that $(\tilde{S}_t)_{t=0}^T$ is a $\mathbb{Q}$-martingale and if we find a self-financing strategy φ such that

$$H = V_T(\varphi) \quad \Big(\text{resp. } \frac{H}{S_T^0} = \tilde{V}_T(\varphi) \Big).$$

It follows

$$\mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi)) = V_0(\varphi)$$

is a no-arbitrage price. This implies that $\mathbb{E}_{\mathbb{Q}} \frac{H}{S_T^0}$ is a no-arbitrage price if H is attainable and $\mathbb{Q} \sim \mathbb{P}$. In general

$$\tilde{V}_t(\varphi) = \mathbb{E}_{\mathbb{Q}} \left[\frac{H}{S_T^0} \middle| \mathcal{F}_t \right]$$

is the *discounted no-arbitrage price at time t.*

Definition 5.13. The market is *complete* if every contingent claim is attainable; i.e. for any $\mathcal{F}_T$-measurable $H \geq 0$ there exists an admissible strategy φ such that $H = V_T(\varphi)$.

Remark 5.14. Completeness is a restrictive assumption: a lot of market models are not complete. But there is a nice mathematical characterization of completeness.

Theorem 5.15. A viable market is complete if and only if there exists a *unique* $\mathbb{Q} \sim \mathbb{P}$ such that $(\tilde{S}^1, ..., \tilde{S}^d)$ are $\mathbb{Q}$-martingales.

Proof" $\Rightarrow$ "
Assume the market is viable and complete. Let H be $\mathcal{F}_T$-measurable and $H \geq 0$. Completeness implies that there exists an admissible strategy such that

$$H = V_T(\varphi) \tag{5.4}$$

where φ is self-financing so that

$$\tilde{V}_T(\varphi) = V_0(\varphi) + \sum_{t=1}^{T} \varphi_t(\tilde{S}_t - \tilde{S}_{t-1}).$$

A viable market implies that there exists a $\mathbb{Q} \sim \mathbb{P}$ such that the $(\tilde{S}_t)_t$ are $\mathbb{Q}$-martingales. We have to show that $\mathbb{Q}$ is unique. Let $\hat{\mathbb{Q}}$ be another probability measure such that $\hat{\mathbb{Q}} \sim \mathbb{P}$ and $(\tilde{S}_t)_t$ are $\hat{\mathbb{Q}}$-martingales. Then

$$\mathbb{E}_{\mathbb{Q}} \tilde{V}_T(\varphi) = V_0(\varphi) = \mathbb{E}_{\hat{\mathbb{Q}}} \tilde{V}_T(\varphi).$$

Hence

$$\mathbb{E}_{\mathbb{Q}} \frac{H}{S_T^0} = V_0(\varphi) = \mathbb{E}_{\hat{\mathbb{Q}}} \frac{H}{S_T^0}.$$

By assumption H is attainable, so the no arbitrage price is the *same* for any $\mathbb{Q}$.
Choose

$$H = \mathbb{1}_A S_T^0 \text{ for } A \in \mathcal{F}_T.$$

Then $\mathbb{Q}(A) = \hat{\mathbb{Q}}(A) \quad \forall A \in \mathcal{F}_T$ and $\mathbb{Q} = \hat{\mathbb{Q}}$. □

" $\Leftarrow$ "
Assume the market is viable and incomplete. Then there exists $H \geq 0$,and H is $\mathcal{F}_T$-measurable and not attainable. Defining

$$V := \left\{ V_0 + \sum_{t=1}^{T} \varphi_t(\tilde{S}_t - \tilde{S}_{t-1}) \ , \quad V_0 \in \mathbb{R}, (\varphi^1, ..., \varphi^d) \text{ predictable} \right\}$$

implies

$$\frac{H}{S_T^0} \notin V.$$

Let

$$W = \big\{ f = \big(f(\omega_1), ..., f(\omega_N)\big) : \ f : \Omega \to \mathbb{R} \big\} = \mathbb{R}^N.$$

Hence we get

$$V \subsetneq W.$$

We introduce the scalar product

$$(f,g) := \mathbb{E}_{\mathbb{Q}} fg = \sum_{i=1}^{N} f(\omega_i) g(\omega_i) \mathbb{Q}(\{\omega_i\})$$

We take a basis $v_1, ..., v_M \in V, \quad x := \frac{H}{S_T^0} \notin V$ then

$$\hat{x} := x - \sum_{i=1}^{M} (x, v_i) v_i \perp V.$$

Indeed, for any $v = \sum_{k=1}^{M} \alpha_k v_k \in V$ it holds

$$(\hat{x}, v) = \sum_{k=1}^{M} \alpha_k (x, v_k) - \sum_{n=1}^{M} \alpha_n (x, v_n) = 0.$$

Define

$$\hat{\mathbb{Q}}(\{\omega\}) := \left(1 + \frac{\hat{x}(\omega)}{2 \sup_{\tilde{\omega}} |\hat{x}(\tilde{\omega})|}\right) \mathbb{Q}(\{\omega\}).$$

Obviously $\hat{\mathbb{Q}}(\{\omega\}) > 0 \quad \forall \omega \in \Omega$ and

$$\hat{\mathbb{Q}}(\Omega) = \sum_{i=1}^{N} \mathbb{Q}(\omega_i) + \sum_{i=1}^{N} \frac{\hat{x}(\omega_i)}{2 \sup_{\hat{\omega}} |\hat{x}(\hat{\omega})|} \mathbb{Q}(\{\omega_i\}) = 1.$$

Indeed, since $1 \in V$ we have

$$(\hat{x}, 1) = \mathbb{E}_{\mathbb{Q}} \hat{x} = \sum_{i=1}^{N} \hat{x}(\omega_i) \mathbb{Q}(\{\omega_i\}) = 0.$$

Hence $\hat{\mathbb{Q}}$ is a probability measure and $\hat{\mathbb{Q}} \sim \mathbb{P}$ and $\hat{\mathbb{Q}} \neq \mathbb{Q}$ by definition. Finally, we show that $(\tilde{S}_t)$ is also a $\hat{\mathbb{Q}}$-martingale. Setting $v = \sum_{t=1}^{N} \varphi_t (\tilde{S}_t - \tilde{S}_{t-1})$ we have

$$\begin{aligned}
\mathbb{E}_{\hat{\mathbb{Q}}} \textstyle\sum_{t=1}^{N} \varphi_t (\tilde{S}_t - \tilde{S}_{t-1}) &= \textstyle\sum_{i=1}^{N} v(\omega_i) \hat{\mathbb{Q}}(\{\omega_i\}) \\
&= \textstyle\sum_{i=1}^{N} v(\omega_i) \big(1 + \frac{\hat{x}(\omega_i)}{2 \sup_{\omega} |\hat{x}(\omega)|}\big) \mathbb{Q}(\{\omega_i\}) \\
&= \mathbb{E}_{\mathbb{Q}} v + \frac{1}{2 \sup_{\omega} |\hat{x}(\omega)|} \mathbb{E}_{\mathbb{Q}} v \hat{x} \\
&= 0
\end{aligned}$$

$\forall (\varphi^1, ..., \varphi^d)$ predictable. Hence by the Theorem 5.4 $(\tilde{S}_t)_{t=0}^{T}$ is a $\hat{\mathbb{Q}}$-martingale. $\square$

6. American Options

1 Stopping Times

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0}^T, (S_t)_{t=0}^T)$ a market model as before.
An American option can be exercised **at any time** t $\in \{0, 1, ..., T\} =: \mathbb{T}$.
For example, the American call option with strike price K:
$Z_t = (S_t^1 - K)^+$, t = 0,1,..., T is then a sequence adapted $(\mathcal{F}_T)$. The random variable Z_t stands for the profit made by exercising the option at time t.
For the decision to exercise or not at time t the trader can only use the information available until time t, i.e. the information is given by $\mathcal{F}_t$. We describe this using stopping times.

Definition 6.1. A random variable τ: $\Omega \to \mathbb{T}$ is a **stopping time** if

$$\{\tau = t\} \in \mathcal{F}_t \quad \forall t = 0, ..., T.$$

$(\{\tau = t\} = \{\omega \in \Omega : \tau(\omega) = t\})$

Remark 6.2. It holds

$$\begin{aligned} \{\tau = t\} = \{\tau \leq t\} \setminus \{\tau \leq t-1\} \in \mathcal{F}_t \forall t = 1, ..., T \\ \iff \{\tau \leq t\} = \{\tau = 0\} \cup \{\tau = 1\} \cup ... \cup \{\tau = t\} \in \mathcal{F}_t \forall t = 0, ..., T \end{aligned}$$

Definition 6.3. Let $(X_t)_{t=o}^T$ be an adapted sequence and τ a stopping time. We define
$X_t^\tau(\omega) := X_{t \wedge \tau(\omega)}(\omega)$ where $(a \wedge b := \min\{a, b\})$.

This means on the set $\{\omega : \tau(\omega) = k\}$ it holds

$$X_t^\tau = \begin{cases} X_k \text{ if } t \geq k \\ X_t \text{ if } t < k. \end{cases}$$

Theorem 6.4. Let τ be a stopping time.
(1) (X_t) is $(\mathcal{F}_t)$ adapted $\Rightarrow$ (X_t^τ) is $(\mathcal{F}_t)$ adapted.
(2) (X_t) is a martingale $\Rightarrow$ (X_t^τ) is a martingale.
(3) (X_t) is a supermartingale $\Rightarrow$ (X_t^τ) is a supermartingale.

Proof
(1)

$$X_{t\wedge\tau} = X_0 + \sum_{k=1}^{t} \mathbb{1}_{\{k\le\tau\}}(X_k - X_{k-1})$$

It holds $\{k \le \tau\} = \{k > \tau\}^c$. But $\{\tau < k\} = \{\tau \le k-1\} \in \mathcal{F}_{k-1}$. Hence φ(k):=$\mathbb{1}_{\{k\le\tau\}}$ is a predictable sequence. Clearly, $(X_{t\wedge\tau})_{t=0}^T$ is adapted.

(2) Let (X_t) be a martingale. Since $(X_t^\tau - x_0)$ is a martingale transform of (X_t) by $(\varphi(t))$ it follows by Lemma 5.3 that (X_t^τ) is a martingale.

(3) Can be shown similarly. □

2 The Snell Envelope

We want to define the price of an American option, for example, for

$$Z_t = (S_t - K)^+, \qquad t = 0, ..., T.$$

We use a backward in induction. Let $t = T$. Then for the option price U_T it should hold

$$U_T = Z_T.$$

For $t = T-1$ the option holder has 2 possibilities:
(1) Trading at once $(t = T-1)$ implies that the writer must pay Z_{T-1}
(2) Trading at time $t = T$. The writer must be able to pay Z_T which means that he needs an admissible strategy with the price

$$S^0_{T-1}\mathbb{E}_{\mathbb{Q}}\left[\frac{Z_T}{S^0_T}\middle|\mathcal{F}_{T-1}\right] = S^0_{T-1}\tilde{V}_{T-1} = V_{T-1}.$$

Here $\mathbb{Q}$=EMM and we assume that the market is complete. Then for the option price it should hold

$$U_{T-1} = \max\left\{Z_{T-1}, S^0_{T-1}\mathbb{E}_{\mathbb{Q}}\left[\frac{Z_T}{S^0_T}\middle|\mathcal{F}_{T-1}\right]\right\}$$

By induction we have

$$U_{t-1} = \max\left\{Z_{t-1}, S^0_{t-1}\mathbb{E}_{\mathbb{Q}}\left[\frac{U_t}{S^0_t}\middle|\mathcal{F}_{t-1}\right]\right\}.$$

Theorem 6.5. *Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a finite probability space, $(X_t)_{t=0}^T$ an $(\mathcal{F}_t)$ adapted sequence with $X_t \geq 0$, $t = 0, ..., T$. Then the process (U_t) with*

$$U_T := X_T$$
$$U_{t-1} := \max\{X_{t-1}, E_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}]\}$$

is a supermartingale. It is the smallest supermartingale dominating (X_t), *i.e. it holds*

$$U_t \geq X_t, \qquad t = 0, ..., T.$$

Remark 6.6. *The process (U_t) is called the* **Snell envelope** *of (X_t).*

Proof (of the Theorem)
Clearly, $U_t = \max\,\{X_t, \mathbb{E}_{\mathbb{Q}}[U_{t+1}|\mathcal{F}_t]\} \geq X_t, \;\; t = 0, ..., T$. So (U_t) is dominating (X_t). Moreover, (U_t) is adapted. From

$$\mathbb{E}[U_t|\mathcal{F}_{t-1}] \leq \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}]\} = U_{t-1}$$

we conclude that (U_t) is a supermartingale.

We have to show that (U_t) is the smallest one. Suppose (Y_t) is a supermartingale dominating (X_t). Then $Y_T \geq X_T = U_T$.
Backward induction: Assume for some $t \leq T$ that $Y_t \geq U_t$. Then it follows by the supermartingale property of (Y_t) that

$$Y_{t-1} \geq \mathbb{E}_{\mathbb{Q}}[Y_t|\mathcal{F}_{t-1}] \geq \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}].$$

But $Y_{t-1} \geq X_{t-1}$ holds also. Consequently,

$$Y_{t-1} \geq \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}]\} = U_{t-1}.$$

□

Theorem 6.7. *1. $\tau^* = \min\{t \geq 0 : U_t = X_t\}$ is a stopping time.*

2. The stopped process $(U_t^{\tau^*})$ is a martingale.

Proof
1. $\{\tau^* = 0\} = \{U_0 = X_0\} \in \mathcal{F}_0$ since U_0 and X_0 are $\mathcal{F}_0$-measurable.

$$\{\tau^* = t\} = \bigcap_{s=0}^{t-1} \{U_s > X_s\} \cap \{U_t = X_t\}$$

Since $\{U_s > X_s\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$ and $\{U_t = X_t\} \in \mathcal{F}_t$ we have $\{\tau^* = t\} \in \mathcal{F}_t$.
2. Define $\varphi(t) := 1\!\!1_{\{\tau^* \geq t\}}$. We know that $\varphi(t)$ is predictable. It holds

$$U_t^{\tau^*} = U_0 + \sum_{s=1}^{t} \varphi(s)(U_s - U_{s-1})$$

and

$$\begin{aligned} U_t^{\tau^*} - U_{t-1}^{\tau^*} &= \varphi(t)(U_t - U_{t-1}) \\ &= 1\!\!1_{\{\tau^* \geq t\}}(U_t - U_{t-1}) \\ &= 1\!\!1_{\{\tau^* \geq t\}}(U_t - \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}]) \end{aligned}$$

since on the set $\{\tau^* \geq t\} = \{\tau^* > t-1\}$ it holds $U_{t-1} > X_{t-1}$ and hence

$$\begin{aligned} U_{t-1} &= \max\{X_{t-1}, \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}]\} \\ &= \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}]. \end{aligned}$$

So it follows

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[U_t^{\tau^*} - U_{t-1}^{\tau^*}|\mathcal{F}_{t-1}] &= \mathbb{E}_{\mathbb{Q}}[1\!\!1_{\{\tau^* \geq t\}}(U_t - \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}])|\mathcal{F}_{t-1}] \\ &= 1\!\!1_{\{\tau^* \geq t\}}(\mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}] - \mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_{t-1}] \\ &= 0 \end{aligned}$$

i.e. $(U_t^{\tau^*})$ is a martingale. □

Definition 6.8. *A stopping time $\sigma : \Omega \to \mathbb{T}$ is* **optimal** *for (X_t) if*

$$\mathbb{E}_{\mathbb{Q}} X_\sigma = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$$

where $\mathcal{T}$ denotes the set of stopping times $\tau : \Omega \to \mathbb{T}$.

Interpretation: If we think of X_n as the total winnings at time n, then stopping at time σ would maximize the expected gain.

Corollary 6.9. $\tau^* = \min\{t \geq 0 : U_t = X_t\}$ *is an optimal stopping time for* (X_t) *and*

$$U_0 = \mathbb{E}_{\mathbb{Q}} X_{\tau^*} = \sup_{\tau \in \mathcal{T}} \mathbb{E} X_\tau.$$

Proof
The process $(U_t^{\tau^*})$ is a martingale. It holds

$$U_0 = U_0^{\tau^*} = \mathbb{E}_{\mathbb{Q}} U_T^{\tau^*} = \mathbb{E}_{\mathbb{Q}} U_{T \wedge \tau^*} = \mathbb{E}_{\mathbb{Q}} X_{\tau^*}$$

because $T \wedge \tau^* = \tau^*$ and $U_{\tau^*} = X_{\tau^*}$ by definition.
On the other hand (U_t^τ) is a supermartingale for any $\tau \in \mathcal{T}$ by Theorem 6.4. So it follows because (U_t^τ) is a supermartingale and (U_t) dominates (X_t) that

$$U_0 = U_0^\tau \geq \mathbb{E}_{\mathbb{Q}} U_\tau \geq \mathbb{E}_{\mathbb{Q}} X_\tau.$$

Which implies $U_0 \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$, and since $\tau^* \in \mathcal{T}$ and $\mathbb{E}_{\mathbb{Q}} X_{\tau^*} = U_0$ we get $U_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$. □

There is the following characterization for optimal stopping times:

Theorem 6.10. *A stopping time* σ *is optimal for* (X_t) *iff*

1. $U_\sigma = X_\sigma$,
2. U^σ *is a* $((\mathcal{F}_t), \mathbb{Q})$ *- martingale (*U *denotes the Snell envelope of* (X_t)*).*

Proof "$\Leftarrow$"
If U^σ is a martingale, it holds

$$U_0 = \mathbb{E}_{\mathbb{Q}} U_T^\sigma = \mathbb{E}_{\mathbb{Q}} U_\sigma = \mathbb{E}_{\mathbb{Q}} X_\sigma.$$

On the other hand (U_t^τ) is a supermartingale for any $\tau \in \mathcal{T}$ (since (U_t) is a supermartingale, see Theorem 6.4.) Hence

$$U_0 = U_0^\tau \geq \mathbb{E}_{\mathbb{Q}} U_T^\tau = \mathbb{E}_{\mathbb{Q}} U_\tau \geq \mathbb{E}_{\mathbb{Q}} X_\tau$$

because (U_t) dominates (X_t). From $\sigma \in \mathcal{T}$ we get

$$\mathbb{E}_{\mathbb{Q}} X_\sigma = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau, \text{ i.e. } \sigma \text{ is optimal.}$$

"$\Rightarrow$" Assume σ is optimal; i.e. $\mathbb{E}_{\mathbb{Q}} X_\sigma = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$. By Collary 6.9 we have that $U_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} X_\tau$. Hence

$$U_0 = \mathbb{E}_{\mathbb{Q}} X_\sigma \leq \mathbb{E}_{\mathbb{Q}} U_\sigma$$

because (U_t) dominates (X_t). The process (U_t) is a supermartingale, therefore (U_t^σ) is a supermartingale so that also

$$U_0 = U_0^\sigma \geq \mathbb{E}_{\mathbb{Q}} U_\sigma$$

and therefore

$$\mathbb{E}_{\mathbb{Q}} X_\sigma = U_0 = \mathbb{E}_{\mathbb{Q}} U_\sigma. \tag{6.1}$$

Hence

$$U_t \geq X_t \; \forall t, \omega \Rightarrow X_\sigma = U_\sigma.$$

Since (U_t^σ) is a supermartingale,

$$\mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t] \leq U_t^\sigma \tag{6.2}$$

and

$$\begin{aligned} U_0 = U_0^\sigma \geq \mathbb{E}_{\mathbb{Q}} U_t^\sigma &\geq \mathbb{E}_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t] \\ &= \mathbb{E}_{\mathbb{Q}} U_\sigma = U_0 \end{aligned}$$

because of the relations (6.1) and (6.2). Since

$$U_t^\sigma \geq \mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t]$$

and

$$\mathbb{E}_{\mathbb{Q}} U_t^\sigma = \mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t]$$

we conclude that

$$\mathbb{E}_{\mathbb{Q}}[U_T^\sigma | \mathcal{F}_t] = U_t^\sigma,$$

i.e. (U_t^σ) is a martingale.

Remark 6.11. τ^* from the Corollary is the *smallest* optimal stopping time for (X_t).

3 Decomposition of Supermartingales

We will consider the so called "Doob decomposition" which we will use to find trading strategies for American options. Doob decomposition is also used to model trading strategies with consumption.

Theorem 6.12. *Every supermartingale* $(U_t)_{t=0}^T$ *has the following unique decomposition*

$$U_t = M_t - A_t,$$

where (M_t) *is a martingale an* (A_t) *is a non-decreasing predictable process with* $A_0 = 0$.

Proof Induction $\underline{t = 0}$: From $A_0 = 0$ we conclude that $M_0 = U_0$ is uniquely determined.
$\underline{t \to t+1}$: Consider

$$U_{t+1} - U_t = M_{t+1} - M_t - (A_{t+1} - A_t). \tag{6.3}$$

We take the conditional expectation on both sides and assume (M_t) is a martingale and that (A_t) is predictable. Then

$$\mathbb{E}[U_{t+1}|\mathcal{F}_t] - U_t = \mathbb{E}[M_{t+1}|\mathcal{F}_t] - M_t - (A_{t+1} - A_t)$$

implies

$$-(A_{t+1} - A_t) = \mathbb{E}[U_{t+1}|\mathcal{F}_t] - U_t \leq 0 \tag{6.4}$$

and therefore

$$A_t \leq A_{t+1}$$

i.e. (A_t) is non-decreasing. □

Remark 6.13. From (6.3) and (6.4) one gets

$$M_{t+1} - M_t = U_{t+1} - \mathbb{E}[U_{t+1}|\mathcal{F}_t]$$

and

$$A_{t+1} - A_t = U_t - \mathbb{E}[U_{t+1}|\mathcal{F}_t].$$

One can find also the **largest** optimal stopping time for (X_t):

Definition 6.14. Define $\sigma : \Omega \to \{0, 1, ..., T\}$ by setting

$$\sigma(\omega) := \begin{cases} T & \text{if} \quad A_T(\omega) = 0, \\ \min\{t \geq 0; A_{t+1} > 0\} & \text{if} \quad A_T(\omega) > 0. \end{cases}$$

if (U_t) is the Snell envelope of (X_t) and $U_t = M_t - A_t$ (Doob decomposition).

Theorem 6.15. *σ is the largest optimal stopping time for (X_t)*

Proof
(1) σ is **a stopping time**:
$\{\sigma = T\} = \{A_T = 0\} \in \mathcal{F}_T$ and for $0 \leq t \leq T-1$

$$\{\sigma = t\} = \cap_{s \leq t}\{A_s = 0\} \cap \{A_{t+1} > 0\} \text{ is } \mathcal{F}_t - \text{measurable}$$

because $\{A_s = 0\} \in \mathcal{F}_{s-1} \subseteq \mathcal{F}_{t-1}$ for $1 \leq s \leq t$ and $\{A_{t+1} > 0\} \in \mathcal{F}_t$ since A is predictable.

(2) σ is **optimal**:
We conlude from

$$U_t = M_t - A_t \quad \text{and} \quad U_t^\sigma = M_t^\sigma$$

that (U_t^σ) is a martingale. This gives us property (2) of Theorem 6.10 i.e. σ is optimal. We still have to show $U_\sigma = X_\sigma$.

$$\begin{aligned} U_\sigma &= \Sigma_{s=0}^{T-1} \mathbb{1}_{\{\sigma=s\}} U_s + \mathbb{1}_{\{\sigma=T\}} U_T \\ &= \Sigma_{s=0}^{T-1} \mathbb{1}_{\{\sigma=s\}} \max\{X_s, \mathbb{E}[U_{s+1}|\mathcal{F}_s]\} + \mathbb{1}_{\{\sigma=s\}} U_T \end{aligned}$$

We have $\mathbb{E}[U_{s+1}|\mathcal{F}_s] = \mathbb{E}[M_{s+1} - A_{s+1}|\mathcal{F}_s] = M_s - A_{s+1}$ and $A_{s+1} > 0$ on $\{\sigma = s\}$. On the other hand

$$U_s = M_s - A_s \text{ and } A_s = 0 \text{ on } \{\sigma = s\}.$$

This gives

$$\mathbb{E}[U_{s+1}|\mathcal{F}_s] < U_s$$

and therefore

$$U_s = \max\{X_s, \mathbb{E}[U_{s+1}|\mathcal{F}_s]\} = X_s.$$

We get

$$U_\sigma = \Sigma_{s=0}^{T-1} \mathbb{1}_{\{\sigma=s\}} X_s + \mathbb{1}_{\{\sigma=T\}} U_T = X_\sigma,$$

because $U_T = X_T$ by construction, i.e. σ is optimal.

(3) σ is the **largest**:
Assume $\tau \geq \sigma$ and $\mathbb{Q}(\tau > \sigma) > 0$. Then

$$\mathbb{E}U_\tau = \mathbb{E}M_\tau - \mathbb{E}A_\tau = \mathbb{E}M_0 - \mathbb{E}A_\tau < \mathbb{E}U_0 = U_0$$

which means τ is not optimal. □

4 Pricing and hedging of American options

We assume the market $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (S_t))$ is viable and complete ($(\Omega, \mathcal{F}, \mathbb{P})$ finite probability space) and $\mathbb{Q}$ is the EMM. An **American option** is an adapted sequence $(Z_t)_{t=0}^T$ with $Z_t \geq 0$. In Section 2 of this chapter we saw: Given an American option $(Z_t)_{t=0}^T$ its value process $(U_t)_{t=0}^T$ can be described by

$$\begin{cases} U_T = Z_T, \\ U_t = \max\{Z_t, S_t^0 \mathbb{E}_{\mathbb{Q}}[\frac{U_{t+1}}{S_{t+1}} | \mathcal{F}_t]\}, \quad 0 \leq t \leq T-1. \end{cases}$$

That means, **the discounted price of the option** $\tilde{U}_t := \frac{U_t}{S_t^0}$, t=0,...,T **is the Snell envelope under** $\mathbb{Q}$ of $(\tilde{Z}_t)_{t=0}^T$. Like in Section 2 one can show

$$\tilde{U}_t = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}}[\frac{Z_\tau}{S_\tau^0} | \mathcal{F}_t],$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping time $\tau : \Omega \to \{t, \ldots, T\}$. Consequently, the price U_t of the option $(Z_t)_{t=0}^T$ is

$$U_t = S_t^0 \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}}[\frac{Z_\tau}{S_\tau^0} | \mathcal{F}_t].$$

Now we use the Doob decomposition:

$$\tilde{U}_t = \tilde{M}_t - \tilde{A}_t$$

where $\tilde{M}_t$ is a $\mathbb{Q}$-martingale and $\tilde{A}_t$ is non-decreasing, predictable and $A = 0$. By assumption, the market is complete. This implies the existence of a self-financing strategy φ, such that for $H = S_T^0 \tilde{M}_T$ it holds

$$\vee_T(\varphi) = S_T^0 \tilde{M}_T.$$

This implies

$$\tilde{V}_T(\varphi) = \tilde{M}_T \Rightarrow \tilde{V}_t(\varphi) = \tilde{M}_t, t = 0, ..., T.$$

because $(\tilde{V}_t(\varphi))$ is a $\mathbb{Q}$-martingale. Hence the Doob decomposition can be written as

$$\tilde{U}_t = \tilde{V}_t(\varphi) - \tilde{A}_t.$$

which implies $U_t = V_t(\varphi) - A_t$.

Optionprice $V_0(\varphi)$	$=$	$U_0 = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}} \frac{Z_\tau}{S^0_\tau}$
	$=$	investment needed for a hedging strategy
	$=$	rational (or "fair") price or "no-arbitrage price".

$\vee_0(\varphi)$ is the minimal investment capital to hedge (using a self-financing strategy) the option

$$Z_t, \quad t = 1, ..., T.$$

Theorem 6.16. *A stopping time* $\sigma \in \mathcal{T}$ *is* **an optimal exercise time** *for the American Option* $(Z_t)_{t=0}^T$ *iff*

$$\mathbb{E}_{\mathbb{Q}} \frac{Z_\sigma}{S^0_\sigma} = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}} \frac{Z_\tau}{S^0_\tau}. \tag{6.5}$$

Proof
An optimal date to exercise the option has to be a stopping time (traders do not have information about the future). An option holder would not exercise (=use) the option in case

$$U_t > Z_t$$

because he would trade an asset worth U_t (=the price of the option at time t) for an amount Z_t (= profit he gets from exercising the option at time t). We know $(\tilde{U}_t)$ dominates $(\tilde{Z}_t)$ which means $U_t \geq Z_t \; \forall t$. So the option holder waits till $U_{\tau*} = Z_{\tau*}$. But this is property (1) of Theorem 6.10 for an optimal stopping time. Also

$$\sigma_{\max} = \begin{cases} \inf\{0 \leq t \leq T-1, A_{t+1} \neq 0\} & \\ T, & A_T = 0 \end{cases}$$

is an optimal exercise time. **After** $\sigma_{\max}$ one should not exercise: From

$$U_t = \vee_t(\varphi) - A_t$$

It follows $U_t < \vee_t(\varphi)$ on $\{t > \sigma_{\max}\}$. If the holder would sell the option at the time $\sigma_{\max}$ he would get $U_{\sigma_{\max}}$. By using $U_{\sigma_{\max}} = \vee_{\sigma_{\max}}(\varphi)$ and trading with the trading strategy φ he creates a parfolio φ_t such that

$$\vee_{\sigma_{\max}+1}(\varphi) > U_{\sigma_{\max}+1}, ..., \vee_T(\varphi) > U_T$$

Hence, for an optimal exercise time τ it holds (U_t^τ) is a martingale, so also property (2) of an optimal stopping time must hold (compare Theorem 6.10)
Remark:
Or, from the writer's point of view: If he uses φ as defined above and the buyer exercises at $\tau \neq$ 'optimal' then

$$U_\tau > Z_\tau \text{ or } A_\tau > 0.$$

The writer can make riskless profit: From

$$U_t = \vee_t(\varphi) - A_t$$

it follows

$$\vee_\tau(\varphi) - Z_\tau = U_\tau + A_\tau - Z_\tau > 0$$

if $U_\tau > Z_\tau, A_\tau \geq 0$ or $A_\tau > 0, U_\tau \geq Z_\tau$. The writer gets the amount $V_\tau(\varphi)$ by hedging while Z_τ is the amount the writer has to pay to the holder.

5 American options and European options

Theorem 6.17. *Let C_t^A be the value of an American option at time t described by an adapted sequence $(Z_t)_{t=0}^T$ and let C_t^E be the value of an European option at time t defined by the $\mathcal{F}_T$ measurable random variable $H = Z_T$. Then it holds*

$$C_t^A \geq C_t^E.$$

Moreover, if $C_t^E \geq Z_t$ for all t, then

$$C_t^A = C_t^E \quad \forall t = 1, \ldots, T.$$

Remark 6.18. One can imagine that $C_t^A \geq C_t^E$ should be true because the American holder has more choices to exercise than the European.

Proof Put $C_t^E := S_t^0 \mathbb{E}_{\mathbb{Q}}[\frac{H}{S_T^0}|\mathcal{F}_t]$. Then $\tilde{C}_t^A = \frac{C_t^A}{S_t^0}$, $t = 0, 1, ..., T$ is a $\mathbb{Q}-$ supermartingale and because of the assumption $C_T^A = C_T^E$ we have

$$\tilde{C}_t^A \geq \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T^A|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T^E|\mathcal{F}_t] = \tilde{C}_t^E.$$

If $C_t^E \geq Z_t$ for all t then $\tilde{C}_t^E$ is a $\mathbb{Q}-$ martingale (and as any martingale is a supermartingale) it is a $\mathbb{Q}-$ supermartingale which dominates $\tilde{Z}_t$. Hence

$$\tilde{C}_t^A \leq \tilde{C}_t^E \quad t = 0, \dots, T.$$

□

Remark 6.19. *The price of the European call and the American call are equal:*

$$\begin{aligned} \tilde{C}_t^E &= \mathbb{E}_{\mathbb{Q}}[\tilde{C}_T^E|\mathcal{F}_t] \\ &= (1+r)^{-T}\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+|\mathcal{F}_t] \\ &\geq \mathbb{E}_{\mathbb{Q}}[\tilde{S}_T - K(1+r)^{-T}|\mathcal{F}_t] \\ &= \tilde{S}_t - K(1+r)^{-T}. \end{aligned}$$

Thus

$$C_t^E \geq S_t - K(1+r)^{-(T-t)} \geq S_t - K, \quad r \geq 0.$$

Because $C_t^E \geq 0$ *we have* $C_t^E \geq (S_t - K)^+ = Z_t$.

7. Some stochastic calculus

Mathematical finance in continuous time is described in the language of stochastic integrals and stochastic differential equations. Therefore the course begins by introducing

- Brownian motion,
- martingales,
- Itô integral and
- Itô's formula.

1 Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e.

(1) Ω is a non-empty set.

(2) $\mathcal{F}$ is a σ-algebra on Ω.

(3) $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **complete**, if $B \in \mathcal{F}$ with $\mathbb{P}(B) = 0$ and $A \subseteq B$ imply $A \in \mathcal{F}$, in other words '$\mathcal{F}$ contains all $\mathbb{P}$-null sets'. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a **filtration**, i.e.

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F},\ 0 \leq s \leq t < \infty,$$

where $\mathcal{F}_s$ and $\mathcal{F}_t$ are σ-algebras. A σ-algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all open intervals of $\mathbb{R}$, see [4],
In the future, it is assumed, that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the **'usual conditions'**, namely

(1) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,

(2) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$,

(3) $\mathbb{F}$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$.

To model random phenomena in finance, the Brownian motion will be used.

Definition 7.1 (Brownian motion)**.**
Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)$. A family of random variables $W = (W_t)_{t\geq 0}$ is called **(standard) Brownian motion** with respect to $(\mathcal{F}_t)$, if

(a) for all $\omega \in \Omega$, $t \mapsto W_t(\omega) : [0, \infty) \to \mathbb{R}$ is a continuous function with $W_0(\omega) = 0$.

(b) (W_t) is $(\mathcal{F}_t)$-adapted and for $0 \leq s < t$ it holds that $W_t - W_s$ is independent from $\mathcal{F}_s$
(i.e. $\forall A \in \mathcal{F}_s$, $\forall B \in \mathcal{B}(\mathbb{R})$:
$\mathbb{P}(A \cap \{W_t - W_s \in B\}) = \mathbb{P}(A)\mathbb{P}(W_t - W_s \in B)$).

(c) W_t is normally distributed for all $t > 0$ with $\mathbb{E}W_t = 0$ and $\mathbb{E}W_t^2 = t$, i.e.

$$\mathbb{P}(W_t \leq x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{z^2}{2t}} dz.$$

(d) (W_t) is homogeneous:

$$\mathbb{P}(W_{t-s} \leq x) = \mathbb{P}(W_t - W_s \leq x).$$

1.1 Some properties of the Brownian motion

1. The Brownian motion exists. The space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ can be chosen to satisfy the "usual conditions".

2. The Brownian motion can only be sketched but not drawn: The length of the path on the interval $[0, 1]$ is ∞ almost surely:

$$\mathbb{P}\left(\left\{\omega : \lim_{N\to\infty} \sum_{k=1}^{N} \left| W_{\frac{k}{N}}(\omega) - W_{\frac{k-1}{N}}(\omega) \right| = \infty \right\}\right) = 1$$

3. The paths $t \mapsto W_t(\omega)$ of the Brownian motion are for almost all $\omega \in \Omega$ nowhere differentiable.

4. For any $0 = t_0 < t_1 < ... < t_n$ the random variables $W_{t_n} - W_{t_{n-1}}$, $W_{t_{n-1}} - W_{t_{n-2}}, ..., W_{t_1}$ are independent.

5. Because W is homogeneous,

$$\mathbb{E}W_t - W_s = \mathbb{E}W_{t-s} = 0 \text{ and } \mathbb{E}(W_t - W_s)^2 = t - s.$$

2 Conditional expectation and martingales

The main properties of conditional expectation are recalled for later use.
Definition 7.2 (Conditional expectation)**.**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra, and X a random variable such that $\mathbb{E}|X| < \infty$. If Y is $\mathcal{G}$-measurable and

$$\mathbb{E}\left(X \mathbb{1}_A\right) = \mathbb{E}\left(Y \mathbb{1}_A\right) \quad \text{for all } A \in \mathcal{G},$$

then Y is called **conditional expectation of X given $\mathcal{G}$**. The conditional expectation is denoted by $\mathbb{E}[X|\mathcal{G}] := Y$.

Remark: The conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is only almost surely unique.

Example 7.3. If for example, $X(\omega) = \omega^2$, $\omega \in [0,1]$, then by choosing $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and $\mathbb{P} = \lambda$, where λ is the Lebesgue-measure on $[0,1]$, and a σ-algebra

$$\mathcal{G} = \left\{ \left[0, \frac{1}{4}\right], \left(\frac{1}{4}, 1\right], \emptyset, \Omega \right\},$$

$\mathbb{E}[X|\mathcal{G}]$ can be determined in the following way: Any $\mathcal{G}$-measurable random variable Y is of the form $Y = a\mathbb{1}_{[0,\frac{1}{4}]} + b\mathbb{1}_{(\frac{1}{4},1]}$, $a, b \in \mathbb{R}$. Now, a and b need to be chosen such that

$$\begin{aligned} \mathbb{E}X\mathbb{1}_{[0,\frac{1}{4}]} &= \mathbb{E}Y\mathbb{1}_{[0,\frac{1}{4}]} \text{ and} \\ \mathbb{E}X\mathbb{1}_{(\frac{1}{4},1]} &= \mathbb{E}Y\mathbb{1}_{(\frac{1}{4},1]}. \end{aligned}$$

Since

$$\mathbb{E}X\mathbb{1}_{[0,\frac{1}{4}]} = \int_0^1 \omega^2 \mathbb{1}_{[0,\frac{1}{4}]}(\omega) d\omega = \frac{1}{3 \cdot 4^3}$$

and

$$\mathbb{E}Y\mathbb{1}_{[0,\frac{1}{4}]} = \int_0^1 Y\mathbb{1}_{[0,\frac{1}{4}]}(\omega)d\omega = \frac{a}{4}$$

implying that $a = \frac{1}{48}$. Similarly,

$$\mathbb{E}X\mathbb{1}_{(\frac{1}{4},1]} = \int_0^1 \omega^2\mathbb{1}_{(\frac{1}{4},1]} = \frac{1}{3}\left(1 - \frac{1}{4^3}\right)$$

and

$$\mathbb{E}Y\mathbb{1}_{(\frac{1}{4},1]} = \int_0^1 Y\mathbb{1}_{[0,\frac{1}{4}]}(\omega)d\omega = \frac{3b}{4},$$

so $b = \frac{4^3-1}{4^2\cdot 9} = \frac{7}{16}$. Hence

$$\mathbb{E}[X|\mathcal{G}](\omega) = \frac{1}{48}\mathbb{1}_{[0,\frac{1}{4}]}(\omega) + \frac{7}{16}\mathbb{1}_{(\frac{1}{4},1]}(\omega) \text{ almost surely.}$$

Proposition 7.4 (Properties of the conditional expectation)**.**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$.

1. *If $\mathbb{E}|X| < \infty$ or $X \geq 0$ a.s. then $\mathbb{E}[X|\mathcal{G}]$* **exists**.
2. *Let $\mathbb{E}|X| < \infty$ or $X \geq 0$ a.s. then*
   - *(a) If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$ almost surely.*
   - *(b) If X and $\mathcal{G}$ are independent, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$ almost surely.*
   - *(c)* **Tower property***: If $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ are sub-σ-algebras, then*

     $$\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}\big] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}\big] = \mathbb{E}[X|\mathcal{G}] \text{ a.s.}$$

   - *(d) Linearity: If $\mathbb{E}|X| < \infty$ and $\mathbb{E}|Z| < \infty$, then*

     $$\mathbb{E}\big[\alpha X + \beta Z|\mathcal{G}\big] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Z|\mathcal{G}] \text{ a.s.}$$

     for all $\alpha, \beta \in \mathbb{R}$.
   - *(e)* **'Take out what is known'***: If $\mathbb{E}|X| < \infty$ and Y is bounded (***or*** if $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$ for $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$) and Y is $\mathcal{G}$-measurable, then*

     $$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}] \text{ almost surely.}$$

Definition 7.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration.

(a) A **stochastic process** $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of random variables $(X_t)_{t\geq 0}$, (i.e. X_t is $\mathcal{F}$-measurable for all $t \geq 0$.)

(b) A stochastic process $X = (X_t)_{t\geq 0}$ is **adapted** if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(c) An adapted stochastic process $(X_t)_{t\geq 0}$ is called a **martingale** with respect to $(\mathcal{F}_t)_{t\geq 0}$, if

   (1) $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$, i.e. X_t is integrable.

   (2) for $0 \leq s \leq t$

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s.$$

(d) A stochastic process $(X_t)_{t\geq 0}$ is called **square integrable** if

$$\mathbb{E}X_t^2 < \infty \text{ for all } t \geq 0.$$

Proposition 7.6. *The Brownian motion $(W_t)_{t\geq 0}$ is a martingale.*

Proof
Exercise. □

3 Itô's integral for simple integrands

We assume that $W = (W_t)_{t\geq 0}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and want to define the stochastic integral (=**Itô integral**)

$$\int_0^T L_t dW_t \quad \text{for } T > 0.$$

In this section it is assumed that the stochastic process $(L_t)_{t\geq 0}$ is a **simple process**, i.e. there exists a sequence $0 = t_0 < t_1 < ... < t_n = T$ and random variables $\xi_i,\ i = 0, 1, ..., n$ with the properties

(i) ξ_i is $\mathcal{F}_{t_i}$-measurable

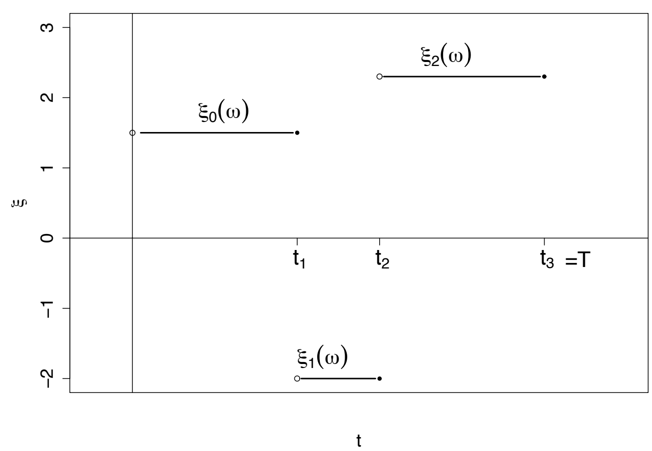
(ii) $\sup_{\omega\in\Omega} |\xi_i(\omega)| < C$ for some $C > 0$ for all $i = 1, ..., n$ such that $(L_t)_{t\geq 0}$ can be represented by

$$L_t = \sum_{i=1}^{n} \xi_{i-1} \mathbb{1}_{(t_{i-1},t_i]}(t)$$

The space of simple processes is denoted by $\mathcal{L}_0$.

Remark 7.7.

1. $(L_t)_{0\leq t\leq T}$ is a stochastic process which has piece-wise constant paths for each $\omega \in \Omega$.



2. (L_t) is an adapted process:

$$L_t = \xi_{i-1} \text{ for } t \in (t_{i-1}, t_i],\ L_0 = 0.$$

Then ξ_{i-1} is $\mathcal{F}_{t_{i-1}} \subseteq \mathcal{F}_t$-measurable, hence L_t is $\mathcal{F}_t$-measurable.

If b is a continuously differentiable function on $[0, T]$, then

$$\int_0^T L_t(\omega)db(t) \quad = \quad \int_0^T L_t(\omega)b'(t)dt$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \xi_{i-1}(\omega) b'(t) dt$$

$$= \sum_{i=1}^{n} \xi_{i-1}(\omega)(b(t_i) - b(t_{i-1})).$$

This relation is the motivation for the definition of $\int_0^T L_t dW_t$ (but $(W_t(\omega)$ is **not** differentiable!).

Definition 7.8 (Itô integral on $\mathcal{L}_0$)**.**
The **Itô integral** for $L = (L_t)_{t\geq 0} \in \mathcal{L}_0$ is defined by

$$I_t(L) := \sum_{i=1}^{k-1} \xi_{i-1}(W_{t_i} - W_{t_{i-1}}) + \xi_k(W_t - W_{t_k}),$$

if $t_{k-1} < t \leq t_k$ and $L_t = \sum\limits_{i=1}^{n} \xi_{i-1} 1\!\!1_{(t_{i-1}, t_i]}(t)$. This can also be written as

$$I_t(L) = \sum_{i=1}^{n} \xi_{i-1}(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}),\ t \in [0, T],$$

where $a \wedge b := \min\{a, b\}$.

Notation:

$$I_t(L) = \int_0^t L_s dW_s$$

Proposition 7.9 (Properties of $I_t(L),\ L \in \mathcal{L}_0$)**.**

(a) **Itô isometry:**

$$\mathbb{E}(I_T(L))^2 = \mathbb{E} \int_0^T L_t^2 dt.$$

(b) $(I_t(L))_{t\geq 0}$ is square integrable and a continuous martingale.

(c) $I_t(\alpha L + \beta K) = \alpha I_t(L) + \beta I_t(K)$ for all $L, K \in \mathcal{L}_0$ and $\alpha, \beta \in \mathbb{R}$.

Proof:

(a) By a direct computation,

$$\begin{aligned}\mathbb{E}(I_T(L))^2 &= \mathbb{E}\left(\sum_{i=1}^n \xi_{i-1}(W_{t_i}-W_{t_{i-1}})\right)^2 \\ &= \sum_{i=1}^n\sum_{k=1}^n \mathbb{E}\Big(\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})(W_{t_k}-W_{t_{k-1}})\Big) \\ &= \sum_{i=1}^n \mathbb{E}\xi_{i-1}^2(t_i-t_{i-1})+0,\end{aligned}$$

because if $i \neq k$, for example $i < k$, then by using the tower property and taking out what is known

$$\begin{aligned}&\mathbb{E}\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})(W_{t_k}-W_{t_{k-1}}) \\ =\ &\mathbb{E}\mathbb{E}\left[\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})(W_{t_n}-W_{t_{n-1}})|\mathcal{F}_{t_{k-1}}\right] \\ =\ &\mathbb{E}\xi_{i-1}\xi_{k-1}(W_{t_i}-W_{t_{i-1}})\mathbb{E}\left[W_{t_k}-W_{t_{k-1}}|\mathcal{F}_{t_{k-1}}\right]=0,\end{aligned}$$

since $W_{t_k}-W_{t_{k-1}}$ is independent from $\mathcal{F}_{t_{k-1}}$ and $\mathbb{E}(W_{t_k}-W_{t_{k-1}})=0$. If $i = k$, then

$$\begin{aligned}\mathbb{E}\xi_{i-1}^2(W_{t_i}-W_{t_{i-1}})^2 &= \mathbb{E}\mathbb{E}\left[\xi_{i-1}^2(W_{t_i}-W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right] \\ &= \mathbb{E}\xi_{i-1}^2\mathbb{E}\left[(W_{t_i}-W_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}\right] \\ &= \mathbb{E}\xi_{i-1}^2(t_i-t_{i-1}). \qquad (7.1)\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\int_0^T L_t^2dt &= \mathbb{E}\int_0^T\left(\sum_{i=1}^n \xi_{i-1}\mathbb{I}_{(t_{i-1},t_i]}(t)\right)^2dt \\ &= \mathbb{E}\int_0^T\left(\sum_{i=1}^n \xi_{i-1}^2\mathbb{I}_{(t_{i-1},t_i]}(t)\right)dt \\ &= \mathbb{E}\sum_{i=1}^n \xi_{i-1}^2\int_0^T \mathbb{I}_{(t_{i-1},t_i]}(t)dt \\ &= \mathbb{E}\sum_{i=1}^n \xi_{i-1}^2(t_i-t_{i-1}). \qquad (7.2)\end{aligned}$$

Comparing (7.1) and (7.2) implies the claim (a).

(b) From (a) the square integrability of $(I_t(L))_{t\geq 0}$ follows, because

$$\mathbb{E}(I_t(L))^2 = \mathbb{E}\int_0^t L_s^2 ds = \mathbb{E}\sum_{i=1}^n \xi_{i-1}^2(t_i \wedge t - t_{i-1}\wedge t) \leq c^2 t, \tag{7.3}$$

because $\xi_{i-1}^2 \leq c^2$ by the definition of simple processes. A martingale is said to be **continuous** if it has almost surely continuous paths. Hence, it needs to be verified that

$$\mathbb{P}(\{\omega \in \Omega : (I_t(L))_{t\leq 0}(\omega) \text{ is continuous in } t\}) = 1.$$

By the definition of the Brownian motion, $t \mapsto W_t(\omega)$ is a continuous function for all $\omega \in \Omega$. This implies

$$\begin{aligned} I_t(L)(\omega) &= \sum_{i=1}^n \xi_{i-1}(\omega)(W_{t_i\wedge t}(\omega) - W_{t_{i-1}\wedge t}(\omega)) \\ &\to \sum_{i=1}^n \xi_{i-1}(\omega)(W_{t_i\wedge s}(\omega) - W_{t_{i-1}\wedge s}(\omega)) = I_s(L)(\omega), \end{aligned}$$

as $t \to s$ and thus $I_t(L)(\omega)$ is continuous in t for all $\omega \in \Omega$.

Yet it needs to be shown that $(I_t(L))_{t\geq 0}$ is a martingale:

(1) If $t \in (t_{k-1}, t_k]$, then

$$I_t(L) = \sum_{i=1}^{k-1} \xi_{i-1}(W_{t_i} - W_{t_i-1}) + \xi_{k-1}(W_t - W_{t_{k-1}}).$$

The random variables ξ_{i-1} are $\mathcal{F}_{t_{i-1}}$-measurable, ξ_{k-1} is $\mathcal{F}_{t_{k-1}}$-measurable, the terms $(W_{t_i} - W_{t_{i-1}})$ are $\mathcal{F}_{t_i}$-measurable and the term $(W_t - W_{t_{k-1}})$ is $\mathcal{F}_t$-measurable. Since $\mathcal{F}_{t_{i-1}} \subseteq \mathcal{F}_t$, $I_t(L)$ is $\mathcal{F}_t$-measurable, and $(I_t(L))_{t\geq 0}$ is adapted.

(2) $\mathbb{E}|I_t(L)| \leq (\mathbb{E}|I_t(L)|^2)^{\frac{1}{2}} < \infty$, because inequality (7.3).

(3) For $0 \leq s < t$, assume $t_{k-1} < s < t \leq t_k$. Then

$$\begin{aligned} \mathbb{E}[I_t(L)|\mathcal{F}_s] &= \sum_{i=1}^{k-1} \mathbb{E}\left[\xi_{i-1}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_s\right] \\ &+ \mathbb{E}\left[\xi_{k-1}(W_t - W_{t_{k-1}})|\mathcal{F}_s\right] \end{aligned}$$

$$= \sum_{i=1}^{k-1} \xi_{i-1}(W_{t_i} - W_{t_{i-1}}) + \xi_{k-1}(W_s - W_{t_{k-1}})$$
$$= I_s(L) \text{ almost surely.}$$

(c) Clear from the definition.

□

4 Itô's integral for general integrands

Is it possible to define

$$\int_0^T W_t dW_t \ ?$$

The process (W_t) is not piecewise constant, so $(W_t) \notin \mathcal{L}_0$. In this section, the definition of $I_t(L)$ will be extended to a larger class of integrands L. The results are not proven here, but the proofs can be found in [3], [4] and [7].

Definition 7.10. Let $\mathcal{L}_2$ be the space of the processes $L = (L_t)_{t\in[0,T]}$ such that

1. L is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable,
2. L is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted,
3. $\mathbb{E}\int_0^T L_t^2 dt < \infty$.

Lemma 7.11. *Let $L \in \mathcal{L}_2$. Then there exists a sequence $(L^n)_{n\geq 0}$ of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |L_t - L_t^n|^2 dt = 0.$$

Definition 7.12. Let $L \in \mathcal{L}_2$. Then define

$$I_t(L) := \lim_{n\to\infty} \int_0^t L_s^n dW_s,$$

where the limit is in L_2-sense, i.e. $I_t(L)$ is the random variable such that

$$\lim_{n\to\infty} \mathbb{E}(I_t(L) - I_t(L^n))^2 = 0.$$

Notation:

$$I_t(L) := \int_0^t L_s dW_s.$$

Proposition 7.13 (Properties of $\int_0^t L_s dW_s, L \in \mathcal{L}_2$)**.**

(a) **Itô isometry:** $\mathbb{E}(I_t(L))^2 = \mathbb{E}\int_0^t L_s^2 ds$

(b) $(I_t(L))_{t\geq 0}$ *is square integrable and a martingale.*

(c) $I_t(\alpha L + \beta K) = \alpha I_t(L) + \beta I_t(K)$ *almost surely for all* $\alpha, \beta \in \mathbb{R}$, $L, K \in \mathcal{L}_2$.

(d) $\mathbb{E} I_t(L) = 0$.

Remark 7.14. By (b), $(I_t(L))_{t\geq 0}$ is square integrable and a martingale. What about the continuity of $t \mapsto I_t(L)(\omega)$? So far, for each $t \in [0,T]$ the random variable $I_t(L)$ has been defined as a limit in L_2-sense, i.e. $I_t(L)$ is $\mathbb{P}$-a.s. unique.
The stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are called **modifications** of each other, if $X_t = Y_t$ almost surely for all $t \geq 0$. It can be shown that $(I_t(L))_{t\geq 0}$ has a modification which has almost surely continuous paths $t \mapsto I_t(L)(\omega)$. From now on we will assume that $(I_t(L))_{t\geq 0}$ refers to the modification which has almost surely continuous paths. It can be shown that

$$\lim_{n,m\to\infty} \mathbb{E} \sup_{0\leq t\leq T} \left| \int_0^t L_s^n - L_s^m dW_s \right|^2 = 0.$$

5 Itô's formula

Assume a given stochastic process

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dW_s,$$

where the second term is a Riemann-integral with

- $b(s) = b(s,\omega)$ is jointly measurable, i.e. b is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable,
- $b(s)$ is $\mathcal{F}_s$-measurable, $\mathbb{E}\int_0^T |b(s)| ds < \infty$, and

- $\sigma \in \mathcal{L}_2$.

Then the Itô integral is defined.

Proposition 7.15 (Itô's formula)**.** *Let* $f \in C^{1,2}([0,T] \times \mathbb{R})$*. Then for* $t \in [0,T]$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, X_s)(b(s)ds + \sigma(s)dW_s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\sigma^2(s)ds. \end{aligned}$$

Remark 7.16. If

$$\left(\frac{\partial f}{\partial x}(s, X_s)\sigma(s)\right)_{s\in[0,T]} \notin \mathcal{L}_2,$$

then a more general definition is needed for

$$\int_0^t \frac{\partial f}{\partial x}(s, X_s)\sigma(s)dW_s.$$

(see, for example [3], section 3.1.)

Example 7.17. Let $f(t,x) = e^{x-\frac{t}{2}}$, $X_t = W_t$. Then

$$\begin{aligned} f(t, W_t) &= e^{W_t - \frac{t}{2}} = 1 + \int_0^t -\frac{1}{2} e^{W_s - \frac{s}{2}} ds + \int_0^t e^{W_s - \frac{s}{2}} dW_s + \frac{1}{2}\int_0^t e^{W_s - \frac{s}{2}} ds \\ &= 1 + \int_0^t e^{W_s - \frac{s}{2}} dW_s. \end{aligned}$$

8. Continuous time market models

1 The stock price process

A **continuous time market model** consists of

(1) a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

(2) a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, that

- satisfies the 'usual conditions'
- $\mathcal{F}_0$ is **trivial**, i.e. for $A \in \mathcal{F}_0$, $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.
- $\mathcal{F}_T = \mathcal{F}$.

(3) $d + 1$ traded assets:

- d stocks: $S_1(t), ..., S_d(t)$
- one bank account $S_0(t)$

Now assume, that $r : [0, T] \to [0, \infty)$ with $r(0) = 0$, for example $r(t) = r_0 t$, $r_0 \geq 0$, and $S_0(t) = e^{r(t)}$. We can interpret the stocks model as a 'generalized geometric Brownian motion'. For $d = 1$ let

$$S_1(t) = S_1(0) \exp\left(\int_0^t \sigma(s) dW_s + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds \right), \qquad (8.1)$$

where α, σ are bounded, measurable and adapted processes. Now Itô's formula implies that

$$(S_1(t)) \text{ is given by (8.1)} \iff S_1(t) = S_1(0) + \int_0^t \alpha(s) S_1(s) ds + \int_0^t \sigma(s) S_1(s) dW_s$$

for all $t \in [0, T]$.

Special case: geometric Brownian motion (with drift)

$$S_1(t) = S_1(0) e^{\sigma W_t + (\alpha - \frac{\sigma^2}{2})t}, \; t \in [0, T]$$
$$\iff \quad S_1(t) = S_1(0) + \alpha \int_0^t S_1(s) ds + \sigma \int_0^t S_1(s) dW_s, \; t \in [0, T]$$

For $d > 1$, the random influence is assumed to come from a d**-dimensional Brownian motion**

$$W = (W_t^1, ... W_t^d)_{t \in [0,T]},$$

where $(W_t^1)_{t \in [0,T]}, (W_t^2)_{t \in [0,T]}, ..., (W_t^d)_{t \in [0,T]}$ are **independent** Brownian motions. Then for $i = 1, ..., d$, $S_i(t)$ is defined as

$$S_i(t) := S_i(0) + \int_0^t \alpha_i(s) S_i(s) ds + \sum_{j=1}^{d} \int_0^t S_i(s) \sigma_{ij}(s) dW_s^j \tag{8.2}$$

for all $t \in [0, T]$, α_i, σ_{ij} bounded, measurable and adapted.

2 Trading strategies

Assume, that there are shares/stocks $S_1(t), ..., S_d(t), \; t \in [0, T]$ of the form 8.2, and a non-random bank account $S_0(t), \; t \in [0, T]$.

Definition 8.1. The stochastic processes

$$\varphi(t) := (\varphi_0(t), ..., \varphi_d(t)), \; t \in [0, T],$$

form a **trading strategy**, if

(a) the $\varphi_i : [0, T] \times \Omega \to \mathbb{R}$ are $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable and adapted ($\varphi_i(t)$ is $\mathcal{F}_t$-measurable for all t), $i = 0, ..., d$.

(b) $\sum_{i=0}^{d} \int_0^T \mathbb{E} \varphi_i(t)^2 S_i(t)^2 dt < \infty$.

In the definition above, $\varphi_i(t)$ denotes the amount of shares of asset i ($1 \leq i \leq d$) held in the portfolio at time t. $\varphi_i(t) < 0$ means **short sales**: selling a stock which is not owned, only borrowed.

9. Risk neutral pricing

We want to have a method to compute the fair price of an option (so that riskless profit = **arbitrage** is not possible). The method will be 'risk neutral pricing' using an equivalent martingale measure.

Definition 9.1. A probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is a **(strong) equivalent martingale measure** if

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, i.e. $\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$ for all $A \in \mathcal{F}$

(ii) The **discounted price processes**

$$\tilde{S} := \left(\frac{S(t)}{S_0(t)} \right)_{t \geq 0}$$

are a $\mathbb{Q}$-martingales.

It will be shown that (like in the discrete time case) for certain models an equivalent martingale measure (EMM) exists

- uniquely
- not uniquely (there are more than one EMM) or
- not at all.

Definition 9.2.

(a) The **value of the portfolio** $\varphi_t = (\varphi_0(t), \varphi_1(t), ..., \varphi_d(t))$ at time t is given by

$$V_t(\varphi) := \sum_{i=0}^{d} \varphi_i(t) S_i(t), \ t \in [0, T].$$

The process $(V_t(\varphi))_{t \in [0,T]}$ is called the **value process** of the trading strategy φ.

(b) The **gains process** is

$$G_t(\varphi) := \sum_{i=0}^{d} \int_0^t \varphi_i(u) dS_i(u),$$

if the stochastic integral is well-defined.

(c) A trading strategy is called **self-financing** if

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t), \quad t \in [0,T].$$

Remark 9.3. If

$$S_i(t) = S_i(0) + \int_0^t \alpha_i(s) S_i(s) ds + \sum_{j=1}^{d} \int_0^t S_i(s)\sigma_{ij}(s) dW_s^j,$$

then we define

$$\int_0^t \varphi_i(u) dS_i(u) := \int_0^t \varphi_i(u)\alpha_i(u) S_i(u) du + \sum_{j=1}^{d} \int_0^t \varphi_i(u) S_i(u) \sigma_{ij} dW_u^j.$$

By denoting the first term by A and the second term by B, then by Hölder's inequality

$$\mathbb{E}\bigg(\int_0^t \varphi_i(u) dS_i(u) \bigg)^2 = \mathbb{E}(A+B)^2 \le 2\mathbb{E}A^2 + 2\mathbb{E}B^2.$$

We can easily see that the gains process is square integrable:

$$\mathbb{E}A^2 = \mathbb{E}\bigg(\int_0^t \varphi_i(u)\alpha_i(u) S_i(u) du \bigg)^2 \le t\mathbb{E} \int_0^t \Big(\varphi_i(u) S_i(u) \alpha_i(u) \Big)^2 du < \infty$$

and for $\mathbb{E}B^2$,

$$\mathbb{E}\bigg(\int_0^t \varphi_i(u) S_i(u) \sigma_{ij}(u) dW_u^j \bigg)^2 = \sum_{j=1}^{d} \mathbb{E} \int_0^t \Big(\varphi_i(u) S_i(u) \sigma_{ij}(u) \Big)^2 du < \infty.$$

The above is true, because

$$\mathbb{E} \int_0^t a(u) dW_u^1 \int_0^t b(u) dW_u^2 = 0$$

if W^1 and W^2 are independent Brownian motions, and $a, b \in \mathcal{L}_2$. See the exercises for $(a(u))$ and $(b(u))$ simple processes.

Definition 9.4. Assume $\mathbb{Q}$ is an equivalent martingale measure. A strategy φ is **admissible**, if

(i) φ is self-financing,

(ii) $V_t(\varphi) \geq 0,\ t \in [0, T]$

(iii) $\mathbb{E}_{\mathbb{Q}} \sup\limits_{0 \leq t \leq T} \tilde{V}_t(\varphi)^2 < \infty$, where

$$\tilde{V}_t(\varphi) := \frac{V_t(\varphi)}{S_0(t)}.$$

Definition 9.5. A trading strategy φ is called an **arbitrage opportunity** if

(i) φ is admissible,

(ii) $V_0(\varphi) = 0$ and $\mathbb{P}(V_T(\varphi) > 0) > 0$.

Any non-negative random variable H we will call **an option**. Often H is a function of $S_i(T)$, the terminal value of the i-th share price process. For example,

$$H = f(S_i(T)) = (S_i(T) - K)^+,$$

the European call option. But there also exist **"basket options"** like

$$H = \big(S_1(T) + ... + S_d(T) - K\big)^+$$

or **"Asian options"**, depending on the whole process S, for example,

$$H = \left(\frac{1}{T}\int_0^T S_1(t)dt - K\right)^+.$$

Proposition 9.6. *If the equivalent martingale measure $\mathbb{Q}$ exists uniquely and $\mathbb{E}_{\mathbb{Q}} H^2 < \infty$, then there exists an admissible trading strategy*

$$\varphi = \big(\varphi_0(s), ..., \varphi_d(s)\big)_{s \in [0,T]}$$

such that

$$\tilde{H} = V_0(\varphi) + \sum_{i=1}^d \int_0^T \varphi_i(s) d\tilde{S}_i(s).$$

The theorem says that if there is exactly one $\mathbb{Q} \sim \mathbb{P}$ such that $(\tilde{S}_i(t))_{t\in[0,T]}$, $i = 1, ..., d$ are $\mathbb{Q}$-martingales, then any option $H \geq 0$ with $\mathbb{E}_{\mathbb{Q}} H^2 < \infty$ can be hedged. It can be shown, that $\mathbb{E}_{\mathbb{Q}} \int_0^T \varphi_i(t) d\tilde{S}_i(t) = 0$. Hence

$$\mathbb{E}_{\mathbb{Q}} \tilde{H} = V_0(\varphi),$$

which is the **fair price** of the option H.

Since arbitrage opportunities appear in reality only temporarily, the aim is to construct market models which do not admit arbitrage opportunities.

Theorem 9.7 (Fundamental Theorem of Asset pricing)**.** *If a market model has an equivalent martingale measure* $\mathbb{Q}$*, then it does not admit arbitrage.*

Remark 9.8. The other implication is not always true in this general setting. But there exists an "if and only if"-relation, if "no arbitrage opportunities" is replaced by "no free lunch with vanishing risk" (see [2], page 235).

10. The Black-Scholes model

Assume a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and a Brownian motion W with respect to $\mathbb{F}$. We will consider the model suggested by Black and Scholes. The model consists of

- one riskless asset $S_0(t) = e^{rt},\ t \geq 0$, where $r > 0$ is the (instantaneous) interest rate, and
- one risky asset $S(t) = s_0 e^{\sigma W_t - \frac{\sigma^2}{2}t + \mu t}$.

It can be easily checked, that $S_0(t)$ solves the differential equation

$$\begin{cases} dS_0(t) &= rS_0(t)dt \\ S_0(0) &= 1, \end{cases} \tag{10.1}$$

and that $S_0(t)$ is the only solution of (10.1). It is also true, that

$$\begin{cases} dS(t) &= \sigma S(t)dW_t + \mu S(t)dt \\ S(0) &= s_0 \end{cases} \tag{10.2}$$

is solved by

$$S(t) = s_0 e^{\sigma W_t - \frac{\sigma^2}{2}t + \mu t}.$$

1 The equivalent martingale measure $\mathbb{Q}$

We want to determine the equivalent martingale measure $\mathbb{Q}$. It has the properties:

1. $\mathbb{Q} \sim \mathbb{P}$
2. $\tilde{S}(t) = \frac{S(t)}{S_0(t)} = s_0 e^{\sigma W_t - \frac{\sigma^2}{2}t + (\mu - r)t}$, $t \in [0, T]$ is a $\mathbb{Q}$-martingale.

The measure $\mathbb{Q}$ can be found using Girsanov's theorem:

Proposition 10.1 (Girsanov's theorem)**.**
Let $(\theta_t)_{t\in[0,T]}$ be an adapted process satisfying $\int_0^T \theta_s^2 ds < \infty$ almost surely and such that the process

$$H_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \ t \in [0,T]$$

is a martingale. Then with respect to $\mathbb{Q}$,

$$\mathbb{Q}(A) := \int_A H_T d\mathbb{P}(\omega),$$

the process

$$B_t := W_t + \int_0^t \theta_s ds, \ 0 \le t \le T$$

is a **standard Brownian motion**.

Remark 10.2. It is often difficult in applications to check whether $(H_t)_t$ is a martingale. The **Novikov condition**

$$\mathbb{E}e^{\frac{1}{2}\int_0^T \theta_t^2 dt} < \infty$$

is a sufficient condition for (H_t) being a martingale.
Now by Itô's formula,

$$\begin{aligned}\tilde{S}(t) &= s_0 + \sigma\int_0^t \tilde{S}(s)dW_s + \int_0^t \tilde{S}(s)(-\frac{\sigma^2}{2} + (\mu - r))ds \\ &+ \frac{\sigma^2}{2}\int_0^t \tilde{S}(s)ds \\ &= s_0 + \sigma\int_0^t \tilde{S}(s)dW_s + \int_0^t \tilde{S}(s)(\mu - r)ds \\ &= s_0 + \sigma\int_0^t \tilde{S}(s)dB_s,\end{aligned}$$

with $B_s = W_s + \frac{\mu - r}{\sigma}s$. Hence $\mathbb{Q}$ is given by $\mathbb{Q}(A) := \int_A H_t d\mathbb{P}$, where

$$H_t := \exp\left(-\frac{\mu - r}{\sigma}W_t + \frac{(\mu - r)^2}{2\sigma^2}t\right).$$

2 Pricing: The Black-Scholes formula

Assume we have an European call-option with strike price $K > 0$ at time $T > 0$,

$$f(x) = (x - K)^+$$

What is the fair price of the option $f(S_T) = (S_T - K)^+$? This question will be answered using the so-called **martingale representation** (see [7]).

Definition 10.3. Let $(X_t)_{0 \le t \le T}$ be a stochastic process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Set

$$\begin{aligned} \mathcal{F}_t &:= \sigma(X_s, 0 \le s \le t) \\ &= \text{the smallest } \sigma\text{-algebra, such that } X_s \text{ is} \\ &\quad\ \mathcal{F}_t\text{-measurable for all } s \in [0,t]. \end{aligned}$$

Then $(\mathcal{F}_t)_{t\in[0,T]}$ is the **filtration generated by** (X_t).

(ii) Now all $\mathbb{P}$-null sets of $\mathcal{F}$ are added to each $\mathcal{F}_t$:

$$\mathcal{F}_t^X := \sigma(\mathcal{F}_t \cup \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}).$$

Then $(\mathcal{F}_t^X)_{t\in[0,T]}$ is again a filtration, and it is called the **augmented natural filtration of** $(X_t)_{t\in[0,T]}$.

Proposition 10.4 (Brownian martingale representation)**.**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t)_{t\in[0,T]}$ a Brownian motion, $(\mathcal{F}_t^W)_{t\in[0,T]}$ its augmented natural filtration. Let $(M_t)_{t\in[0,T]}$ be a martingale with respect to $(\mathcal{F}_t^W)_{t\in[0,T]}$ such that $\mathbb{E}M_T^2 < \infty$. Then there exists an $(\mathcal{F}_t^W)_{t\in[0,T]}$-adapted process $(L_t)_{t\in[0,T]}$, such that $\mathbb{E}\int_0^T L_t^2 dt < \infty$, and

$$M_t = M_0 + \int_0^t L_s dW_s \text{ almost surely for all } t \in [0,T].$$

The above Proposition 10.4 is applied in the following way:

- $\widetilde{(S(T) - K)}^+$ is square integrable:

$$\mathbb{E}_{\mathbb{Q}}((\widetilde{S(T) - K})^+)^2 \le \mathbb{E}_{\mathbb{Q}}(\tilde{S}(T))^2 = 1 < \infty$$

- $M_t := \mathbb{E}_{\mathbb{Q}}\big[(\widetilde{S(T) - K})^+ | \mathcal{F}_t^B\big]$ is a square integrable martingale with respect to $(\mathcal{F}_t^B)_{t\in[0,T]}$, where $(B_s)_{s\in[0,T]}$ is a Brownian motion with respect to $\mathbb{Q}$.

- Proposition 10.4 implies that there exists a process $(L_s)_{s\in[0,T]}$ which is $(\mathcal{F}_t^B)$-adapted and $\mathbb{E}_{\mathbb{Q}} \int_0^T L_s^2 ds < \infty$, such that

$$\big(\widetilde{S(T) - K}\big)^+ = M_T = M_0 + \int_0^T L_s dB_s. \tag{10.3}$$

On the other hand, if the equation

$$\big(\widetilde{S(T) - K}\big)^+ = \tilde{V}_T(\varphi) = V_0(\varphi) + \int_0^T \varphi(u) d\tilde{S}(u) \tag{10.4}$$

would be true with $(\varphi(u))$ being adapted and $\mathbb{E}_{\mathbb{Q}}\big(\int_0^T \varphi(u) d\tilde{S}(u)\big)^2 < \infty$, then $(\varphi(u))_{u\in[0,T]}$ would be a self-financing trading strategy. Because of

$$\tilde{S}(t) = s_0 + \sigma \int_0^t \tilde{S}(s) dB_s,$$

where $(B_s)_{s\in[0,T]}$ is the Brownian motion with respect to $\mathbb{Q}$, (10.4) implies

$$(\widetilde{S(T) - K})^+ = V_0(\varphi) + \int_0^T \varphi(u)\tilde{S}(u)\sigma dB_u. \tag{10.5}$$

By comparing (10.3) and (10.5),

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbb{Q}} \left| (\widetilde{S(T) - K})^+ - (\widetilde{S(T) - K})^+ \right|^2 \\
&= \mathbb{E}_{\mathbb{Q}} \left(M_0 + \int_0^T L_s dB_s - V_0(\varphi) - \int_0^T \varphi(u)\tilde{S}(u)\sigma dB_u \right)^2 \\
&= \mathbb{E}_{\mathbb{Q}} \Big(M_0 - V_0(\varphi) \Big)^2
\end{aligned}$$

$$\begin{aligned}
&+ \; 2\mathbb{E}_{\mathbb{Q}}\Big(M_0 - V_0(\varphi)\Big)\Big(\int_0^T L_s dB_s - \int_0^T \varphi(u)\tilde{S}(u)\sigma dB_u\Big) \\
&+ \; \mathbb{E}_{\mathbb{Q}}\Big(\int_0^T L_s dB_s - \int_0^T \varphi(u)\tilde{S}(u)\sigma dB_u\Big)^2 \\
&= \; \mathbb{E}_{\mathbb{Q}}\Big(M_0 - V_0(\varphi)\Big)^2 + \mathbb{E}_{\mathbb{Q}}\int_0^T \Big|L_u - \varphi(u)\tilde{S}(u)\sigma\Big|^2 du,
\end{aligned}$$

where the last equality follows from the fact that $\mathbb{E}_{\mathbb{Q}}\int_0^T L_u - \varphi(u)\tilde{S}(u)\sigma dB_u = 0$ and Itô 's isometry. This implies, that

$$\begin{aligned}
V_0(\varphi) &= M_0 \\
\varphi(u) &= \frac{L_u}{\tilde{S}(u)\sigma} \text{ and} \\
\tilde{V}_t(\varphi) &= M_t
\end{aligned}$$

$\mathbb{Q} \otimes dt$-almost everywhere. Since (B_u) is a Brownian motion with respect to $\mathbb{Q}$, by Proposition 7.13

$$\tilde{V}_t(\varphi) = V_0(\varphi) + \int_0^t \varphi(u)\tilde{S}(u)\sigma dB_u, \; t \in [0,T]$$

is a square integrable martingale.
Now, if

$$\begin{aligned}
V_0(\varphi) &= \text{the price of the option at time } 0. \\
\tilde{V}_t(\varphi) &= \text{discounted price of the option at time } t.
\end{aligned}$$

Then the price is

$$\begin{aligned}
V_t(\varphi) &= e^{rt}\mathbb{E}_{\mathbb{Q}}\left[\frac{(S(T)-K)^+}{e^{rT}}\middle|\mathcal{F}_t^B\right] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}_t^B] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(S(t)\frac{S(T)}{S(t)} - K\right)^+\middle|\mathcal{F}_t^B\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(x\frac{S(T)}{S(t)} - K\right)^+\middle|\mathcal{F}_t^B\right]\Bigg|_{x=S(t)},
\end{aligned}$$

where $S(t)$ is independent from $\mathcal{F}_t^B$ and $\mathcal{F}_t^B$-measurable. For the last equality, see [6], proposition A.25.
Now, set

$$F(t,x) := \mathbb{E}_{\mathbb{Q}} e^{-r(T-t)} \left(x \frac{S(T)}{S(t)} - K \right)^+,$$

then $V_t(\varphi) = F(t, S(t))$. By $\tilde{S}(u) = s_0 e^{\sigma B_u - \frac{\sigma^2 u}{2}}$

$$\begin{aligned} F(t,x) &= \mathbb{E}_{\mathbb{Q}} e^{-r(T-t)} \left(x \frac{e^{rT}\tilde{S}(T)}{e^{rt}\tilde{S}(t)} - K \right)^+ \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left(x e^{r(T-t)} e^{\sigma(B_T - B_t) - \frac{\sigma^2(T-t)}{2}} - K \right)^+ \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left(x e^{r(T-t)} e^{\sigma(B_{T-t}) - \frac{\sigma^2(T-t)}{2}} - K \right)^+, \end{aligned}$$

since $B_T - B_t \stackrel{d}{=} B_{T-t}$. By substituting $T - t =: a$ and denoting $X = \frac{B_{T-t}}{\sqrt{T-t}}$, making X standard Gaussian, i.e. $\mathbb{P}(X \leq z) = \mathcal{N}_{0,1}(z)$, $F(t,x)$ can be expressed as

$$\begin{aligned} F(t,x) &= e^{-ra} \mathbb{E}_{\mathbb{Q}} \left(x e^{ra} e^{\sigma\sqrt{a}X - \frac{\sigma^2 a}{2}} - K \right)^+ \\ &= e^{-ra} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x e^{ra} e^{\sigma\sqrt{a}z - \frac{\sigma^2 a}{2}} - K \right)^+ e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x e^{\sigma\sqrt{a}z - \frac{\sigma^2 a}{2}} - K e^{-ra} \right) \mathbb{1}_{\{z + d_2 \geq 0\}} e^{-\frac{z^2}{2}} dz, \end{aligned}$$

where

$$d_1 = \frac{\log \frac{x}{K} + (r + \frac{\sigma^2}{2})a}{\sigma\sqrt{a}} \quad \text{and } d_2 = d_1 - \sigma\sqrt{a},$$

with $a = T - t$. Then $\{z + d_2 \geq 0\} = \{-z \leq d_2\}$, and

$$\begin{aligned} F(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \left(x e^{-\sigma\sqrt{a}z - \frac{\sigma^2 a}{2}} - K e^{-ra} \right) e^{-\frac{z^2}{2}} dz \\ &= \ldots = x\mathcal{N}_{0,1}(d_1) - K e^{-ra} \mathcal{N}_{0,1}(d_2). \end{aligned}$$

This is the Black-Scholes formula

$$F(t,x) = x\mathcal{N}_{0,1}(d_1) - K e^{-r(T-t)} \mathcal{N}_{0,1}(d_2).$$

3 Example of infinitely many EMM's

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and (W_t^1) and (W_t^2) are independent $\mathbb{F}-$Brownian motions. Let $(S_t)_{t\in[0,T]}$, $S_t := e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ be the share price process. Then (S_t) is a $\mathbb{P}$-martingale with respect to $(\mathcal{F}_t^{W^1})_{t\in[0,T]}$, the natural filtration of (W_t^1). Set $H_t := e^{\theta W_t^2 - \frac{T\theta^2}{2}}$, $\theta > 0$. Then

$$\mathbb{Q}^\theta(A) := \mathbb{E}_{\mathbb{P}}\left(\mathbb{I}_A H_T\right)$$

is an equivalent martingale measure for any $\theta > 0$:

- $0 < H_T < \infty$ $\mathbb{P}$-a.s.
- Since $H_T\frac{S_t}{S_s}$ and $\mathcal{F}_s^{W_1}$ are independent, as are H_T and $\frac{S_t}{S_s}$, and H_T and S_T are square integrable, the martingale property is satisfied as

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^\theta}[S_t|\mathcal{F}_s^{W_1}] &= \mathbb{E}\left[H_T S_t|\mathcal{F}_s^{W_1}\right] \\
&= S_s\mathbb{E}\left[H_T\frac{S_t}{S_s}\middle|\mathcal{F}_s^{W_1}\right] \\
&= S_s\mathbb{E}\left(H_T\frac{S_t}{S_s}\right) \\
&= S_s\mathbb{E}H_T\mathbb{E}\frac{S_t}{S_s} = S_s,
\end{aligned}$$

  because $\mathbb{E}H_T = 1 = \mathbb{E}\frac{S_t}{S_s}$.

4 Example of no EMM's

Assume $r = 0$ and price processes are chosen as

$$S_i(t) = S_i(0)e^{W_t + (\mu_i - \frac{1}{2})t}, \ i = 1, 2 \ \text{ and } \mu_1 \neq \mu_2.$$

Then $H_i^1 = e^{-\mu_i W_T + \frac{\mu_i^2}{2}T}$ defines according to Proposition 10.1 the equivalent martingale measure $\mathbb{Q}_i$ for S_i. For $\mathbb{Q}_1 = \mathbb{Q}_2$ the condition $\mu_1 = \mu_2$ would be needed.

11. Bonds

A **bond** is a debt security. The issuer (government, credit institutes, companies) has to pay interest (coupon) once or twice a year and to repay the amount (principal) and the interest at the maturity time. Zero-coupon bonds do not pay interest until maturity time. Modeling of a zero-coupon bond:

- $p(t,T)$ = price of a zero-coupon bond at time $t \leq T$ that pays 1 € at time T.
- $p(t,t) = 1$.

If the 'instantaneous' interest rate is a constant r, then

$$p(t,T) = e^{-r(T-t)}$$

because for 1 € one would get $e^{r(T-t)}$ interest for the time amount $T-t$. (The relation between annual interest rate r_a and *instantaneous interest rate* r is

$$r_a = \lim_{n\to\infty}\left(1+\frac{rT}{n}\right)^n - 1 = e^{rT} - 1 \quad \text{for } T = 1 \text{ year.)}$$

Now we assume that the interest rate is not fixed, but changes randomly.

Similar to Proposition 9.7, it holds that if the market model $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, S)$ has an equivalent martingale measure, then it does not admit arbitrage.
If $\mathbb{Q}$ is an equivalent martingale measure, then $\mathbb{Q} \sim \mathbb{P}$ and the discounted **price processes of the basic securities** have to be $\mathbb{Q}$-martingales. If we consider the bond market, then the discounted zero-coupon bonds have to be $\mathbb{Q}$-martingales.

Definition 11.1. The **fair price for a zero-coupon bond** with maturity T at time t is

$$\frac{p(t,T)}{S_0(t)} := \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{S_0(T)}\middle|\mathcal{F}_t\right] \text{ almost surely,}$$

under the assumption $S_0(t) = \exp(\int_0^t r(s)ds)$ with a random adapted process $(r(s))_{s\in[0,T]}$. So $p(t,T)$, the price of a bond at time t which matures at T is

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right].$$

Remark 11.2. There are other approaches to describe bonds and bond prices. See [2] for the Heath-Jarrow-Morton-model.
The "short rate" r(s) can be modeled by several processes:
Vasicek model, see [3]:

$$r(s) = r_0 + \int_0^s \alpha - \beta r(u)du + \int_0^s \gamma dW_u,$$

Cox-Ingersoll-Ross, see [3]:

$$r(s) = r_0 + \int_0^s \alpha - \beta r(u)du + \int_0^s \sigma(u)\sqrt{r(u)}dW_u,$$

and other more complicated ones.

12. Currency markets

Sometimes assets are needed in several countries simultaneously. We consider two countries: a domestic country with interest rate r_d and a foreign country with interest rate r_f, and bank accounts in both countries, denoted by $B_d(t) = e^{r_d t}$ and $B_f(t) = e^{r_f t}$. We introduce an *exchange rate process* $(Q(t))_{t\geq 0}$ to pass denomination in foreign to domestic currency. $(Q(t))_{t\geq 0}$ depends on the two economies, the policies of the governments, etc, so $(Q(t))$ is influenced by a 'multidimensional noise', modelled with independent Brownian motions $(W_t^1)_t, ..., (W_t^d)_t$. Then

$$\begin{cases} dQ(t) &= Q(t)\mu dt + Q(t)(\sigma_1 dW_t^1 + ... + \sigma_d dW_t^d) \\ Q(0) &= q_0 > 0 \end{cases}$$

if and only if

$$Q(t) = q_0 \exp\left(\sum_{i=1}^{d} \sigma_i W_t^i + \left(\mu - \frac{1}{2}\sum_{i=1}^{d}\sigma_i^2\right) t\right), \tag{12.1}$$

where μ and σ_i, $i = 1, ...d$ are positive constants. We compute the discounted value of the foreign savings account (discounted by the domestic interest rate) in domestic currency,

$$\tilde{Q}(t) := \frac{B_f(t)Q(t)}{B_d(t)}, \tag{12.2}$$

and get from (12.1) and (12.2) that

$$\tilde{Q}(t) = q_0 \exp\left(\sum_{i=1}^{d} \sigma_i W_t^i + \left(\mu - \frac{1}{2}\sum_{i=1}^{d}\sigma_i^2\right) t + (r_f - r_d)t\right).$$

To avoid arbitrage between domestic and foreign bond markets, such an equivalent martingale measure $\tilde{Q}(t)$ is needed that

$$\tilde{Q}(t) = q_0 + \int_0^t \tilde{Q}(s)(\mu + r_f - r_d)ds + \sum_{i=1}^{d} \sigma_i \int_0^t \tilde{Q}(s)dW_s^i$$

does not have the drift term $\int_0^t \tilde{Q}(s)(\mu + r_f - r_d)ds$. In case there exist d independent trading assets the equivalent martingale measure $\mathbb{Q}$ is unique. It is called the **domestic martingale measure**. By considering **currency options**, for example the currency European call-option:

$$f(Q(t)) := (Q(T) - K)^+$$

Then the fair price at time t is given by

$$C(t) = \mathbb{E}_{\mathbb{Q}}[e^{-r_d(T-t)}(Q(T) - K)^+|\mathcal{F}_t].$$

13. Credit risk

Credit risk is the risk of loss if a loan or its interest is not paid back. There exists two methods to model credit risk:

- **reduced form models**
  The modeler has the same information as the market. A point process is used to describe the default event.

- **structural models**
  The modeler has the same information as the firms manager. The traded assets itself are used to describe the default event.

Default event is the event where the debtor does not make a scheduled payment - 'the debtor defaults'.

1 A simple model for credit risk

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space, and $T^* > 0$ a time horizon. We assume a firm's value process

$$V(t) = V(0) + r\int_0^t V(s)ds - \delta\int_0^t V(s)ds + \sigma\int_0^t V(s)dW_s,$$

where r is the interest rate and δ is either the constant dividend rate if $\delta > 0$ or the constant 'pay-in' rate, if $\delta < 0$. The process can be equivalently written (see page 82) as

$$V(t) = V(0)\exp\left(\sigma W_t - \frac{\sigma^2 t}{2} + (r-\delta)t\right).$$

We assume that there exists an equivalent martingale measure $\mathbb{Q}$ to have an arbitrage-free market. Then the discounted tradable securities which pay no dividends follow $\mathbb{Q}$-martingales. By assuming that the firm's value process **is** a tradable security, we get for $\sigma \geq 0$ that

$$\widehat{V}(t) := V(0)e^{\sigma W_t - \frac{\sigma^2 t}{2}} = e^{-(r-\delta)t}V(t)$$

is a $\mathbb{Q}$-martingale if (W_t) is a $\mathbb{Q}$-Brownian motion. Now we consider a zero-coupon bond with notational amount F (=face value) and maturity $T \leq T^*$. So the cash received by the 'owner of the defaultable claim' (= the one who gives the credit to the firm in form of buying the bond) is

$$D_T = \begin{cases} F, & \text{if } V_T \geq F \\ V_T, & \text{if } V_T < F. \end{cases}$$

Now, define a 'default time'

$$\tau := T\mathbb{1}_{\{V_T<F\}} + \infty\mathbb{1}_{\{V_T\geq F\}}.$$

Then the payoff for the 'equity owners' (=owners of the firm) is

$$C_T = (V_T - F)^+,$$

which can be interpreted as a **call option on the value of the firm**. By Black-Scholes formula (see page 86),

$$C_t = \mathbb{E}_{\mathbb{Q}}(\widetilde{V_T - F})^+ = V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(d_1) - Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2) \qquad (13.1)$$

with

$$d_1 = \frac{\log\frac{V_t}{F} + (r - \delta + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_2 + \sigma\sqrt{T-t}.$$

The bond owners' (= the one who gave the credit) payoff is

$$D_T = F - (F - V_T)^+.$$

What is the **fair price of this bond**, or in other words, how much **credit** can this firm get?

Proposition 13.1. *Assume the above assumptions. Then*

Bonds = a risk-free payment − put-option on the value of the firm,

so we have

$$\begin{aligned} p(t,T) &= Fe^{-r(T-t)} - \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(F - V_T)^+ \middle| \mathcal{F}_T\right] \\ &= V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(-d_1) + Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2). \end{aligned}$$

Proof
The proof is done by using the call-put-parity

$$C_T - P_T = (V_T - F)^+ - (F - V_T)^+ = V_T - F$$

and knowing that

$$e^{-(r-\delta)t}V_t = M_t$$

is a martingale. We have

$$C_t = e^{rt}\mathbb{E}_{\mathbb{Q}}\big[\widetilde{C_T}\big|\mathcal{F}_t\big]$$

and

$$P_t = e^{rt}\mathbb{E}_{\mathbb{Q}}\big[\widetilde{P_T}\big|\mathcal{F}_t\big],$$

which imply that

$$\begin{aligned} C_t - P_t &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[V_T\big|\mathcal{F}_t\big] - e^{-r(T-t)}F \\ &= e^{-r(T-t)+(r-\delta)T-(r-\delta)t}V_t - e^{-r(T-t)}F \\ &= V_t e^{-\delta(T-t)} - e^{-r(T-t)}F. \end{aligned}$$

This yields

$$Fe^{-r(T-t)} - P_t = V_t e^{-\sigma(T-t)} - C_t$$

This implies

$$\begin{aligned} p(t,T) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[D_T|\mathcal{F}_t\big] \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[F - P_T|\mathcal{F}_t\big] \\ &= Fe^{-r(T-t)} - P_t \\ &= V_t e^{-\delta(T-t)} - V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(d_1) + Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2) \\ &= V_t e^{-\delta(T-t)}\mathcal{N}_{0,1}(-d_1) + Fe^{-r(T-t)}\mathcal{N}_{0,1}(d_2) \end{aligned}$$

□

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