Martingale theory

Wednesday, 26th October, 2011
8.30 - 10.00 MaD 380

1. (a) **Find an example of a sequence of random variables converging almost surely but not in L_1.**

   Take for example $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = n\chi_{[0,\frac{1}{n}[}(x)$ (with $\mathbb{P} = \lambda$). Then $\lim_{n\to\infty} f_n = 0$ for all $x \in]0,1]$ but $||f_n - 0||_{L_1} = ||f_n||_{L_1} = \int_0^{\frac{1}{n}} n d\lambda(x) = 1$ $(\not\to 0)$.

   Comment: we do not have an integrable majorant $g \in L_1 : g \geq |f_n|$ so that the dominated convergence theorem is not available.

   (b) **Find an example of a sequence of random variables converging in L_1 but not almost surely.**

   Take for example Haar functions as in Exercise 2.1. then for all $x \in [0,1]$ and all $m \in \mathbb{N}$ there exists an index $N_{x,m} \in \mathbb{N}$ such that $|h_{N_{x,m}}(x)| = 1$, i.e. $h_m \not\to 0$ a.s. as $m \to 0$.

   Comment: we do find a subsequence conbverging to 0 a.s.

   (c) **Prove that if $X_n \to X$ a.s. and $|X_n| \leq Y$ for some $Y \in L_1$, then $X \in L_1$ and $X_n \to X$ in L_1.**

   Since $|X_n(\omega)| \leq Y(\omega)$, we have $|X(\omega)| = |\lim X_n(\omega)| \leq Y(\omega)$ for all $\omega \in \Omega$ so that $X \in L_1$. By dominated convergence, $\lim ||X_n - X||_{L_1} = \lim \mathbb{E}|X_n - X| = \mathbb{E} \lim |X_n - X| = 0$, because $|X_n - X| \leq |X_n| + |X| \leq 2Y$.

2. (a) **Let $X : \Omega \to \mathbb{R}$ with $\mathbb{E}|X| < \infty$. Show that $\int_{\{|X|\geq c\}} |X| d\mathbb{P} \to 0$ as $c \to \infty$.**

   By dominated convergence, since $\chi_{\{|X|\geq c\}}|X| \leq |X|$,
   $\lim_{c\to\infty} \int_{|X|\geq c} |X| d\mathbb{P} = \lim_{c\to\infty} \int_\Omega \chi_{\{|X|\geq c\}}|X| d\mathbb{P} =$
   $\int_\Omega \lim_{c\to\infty} \chi_{\{|X|\geq c\}}|X| d\mathbb{P} = 0.$

   (b) **Prove the fact given as hint in Exercise 3.2 b): $\frac{1}{2}(e^\alpha + e^{-\alpha}) \leq e^{\frac{\alpha^2}{2}}$.**

   $\frac{1}{2}(e^\alpha + e^{-\alpha}) = \frac{1}{2}\sum_{k=0}^\infty \frac{\alpha^k + (-\alpha)^k}{k!} = \frac{1}{2}\sum_{k=0}^\infty \frac{2\alpha^{2k}}{(2k)!} = \sum_{k=0}^\infty \frac{\alpha^{2k}}{(2k)!} \leq \sum_{k=0}^\infty \frac{\alpha^{2k}}{2^k k!} = \sum_{k=0}^\infty \frac{\left(\frac{\alpha^2}{2}\right)^k}{(k!} = e^{\frac{\alpha^2}{2}}$

3. (a) **Let $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ be iid Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. Let $S_n := \varepsilon_1 + \cdots + \varepsilon_n$. Prove that**

$$\mathbb{E}(\varepsilon_1 | \sigma(S_n)) = \frac{S_n}{n} \quad a.s.$$

$S_n = \mathbb{E}(S_n|\sigma(S_n)) = \mathbb{E}(\varepsilon_1 + \dots + \varepsilon_n|\sigma(S_n)) = \mathbb{E}(\varepsilon_1|\sigma(S_n)) + \dots + \mathbb{E}(\varepsilon_n|\sigma(S_n)) = n\mathbb{E}(\varepsilon_1|\sigma(S_n))$ a.s, since
$\mathbb{E}(\varepsilon_j|\sigma(S_n)) = \mathbb{E}(\varepsilon_1|\sigma(S_n))$ a.s. for all $j = 1, \dots, n$:
$\sigma(S_n) = \sigma(\{\omega \in \Omega : S_n = k\} : k = -n, \dots, n)$ because $S_n \in \{-n, \dots, n\}$. Thus $\int_{\{S_n=k\}} \varepsilon_j d\mathbb{P} =$
$\int_{\{S_n=k,\varepsilon_j=1\}} \varepsilon_j d\mathbb{P} + \int_{\{S_n=k,\varepsilon_j=-1\}} \varepsilon_j d\mathbb{P} = \int_{\{S_n=k,\varepsilon_j=1\}} 1 d\mathbb{P} + \int_{\{S_n=k,\varepsilon_j=-1\}} -1 d\mathbb{P} =$
$\mathbb{P}(S_n = k, \varepsilon_j = 1) - \mathbb{P}(S_n = k, \varepsilon_j = -1) = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{k-1} - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\binom{n-1}{k+1} =$
$\mathbb{P}(S_n = k, \varepsilon_1 = 1) - \mathbb{P}(S_n = k, \varepsilon_1 = -1) = \dots = \int_{\{S_n=k\}} \varepsilon_1 d\mathbb{P}$.

(b) **Is the process $S = (S_n)_{n=0}^\infty$ above uniformly integrable?**
Take $\mathcal{F}_n := \sigma(\varepsilon_1, \dots, \varepsilon_n)$. Then $S = (S_n)_{n=0}^\infty$ is a martingale. $||S_n - S_{n-1}||_{L_1} = ||\varepsilon_n||_{L_1} = 1$ for all $n = 1, 2, \dots$ so that $(S_n)_{n=0}^\infty$ is not a Cauchy sequence in L_1. By Theorem 8.1 it is not uniformly integrable.

4. **Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n=0}^\infty)$ be a stochastic basis, $\mathcal{F}_\infty := \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$, and $Z \in L_1$. What can we say about the almost sure and L_1-convergence of**

   (a) $\mathbb{E}(Z|\mathcal{F}_n) \to_n \mathbb{E}(Z|\mathcal{F}_\infty)$,
   By Proposition 2.6, $M = (M_n)_{n=0}^\infty$ with $M_n = \mathbb{E}(Z|\mathcal{F}_n)$ is a martingale. Moreover, $M_\infty := \mathbb{E}(Z|\mathcal{F}_\infty)$ is its closure meant in Theorem 8.1.(3). Theorem 8.1.(2) ensures the L_1 convergence, and Theorem 8.1.(b) the almost sure convergence of $\mathbb{E}(Z|\mathcal{F}_n) \to_n \mathbb{E}(Z|\mathcal{F}_\infty)$.

   (b) $\mathbb{E}(Z|\mathcal{F}_n) \to_n Z$?
   Nothing. Take for example $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1]), \mathbb{P} = \lambda$ and $\mathcal{F}_n = \sigma([0, 2^{-n}[, [2^{-n}, 2^{-n+1}[, \dots, [2^{-1}, 1])$. If $Z(\omega) = \omega$, then $Z \in L_1(\Omega, \mathcal{F}, \mathbb{P})$ but $\mathbb{E}(Z|\mathcal{F}_n) \not\to Z$ in any sence.

5. **Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function, i.e. $|f(x) - f(y)| \le L|x-y|$. Let**

$$\xi_n(t) := \sum_{k=1}^{2^n} \frac{k-1}{2^n} \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(t),$$

   $\Omega := [0,1)$, $\mathcal{F}_n := \sigma(\xi_n)$, and

$$M_n(t) := \frac{f(\xi_n(t) + 2^{-n}) - f(\xi_n(t))}{2^{-n}}.$$

   (a) **Prove that $(\mathcal{F}_n)_{n=0}^\infty$ is a filtration and that $\mathcal{B}([0,1)) = \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$.**
   Since $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right[= \left[\frac{m-2}{2^{n+1}}, \frac{m-1}{2^{n+1}}\right[\cup \left[\frac{m-1}{2^{n+1}}, \frac{m}{2^{n+1}}\right[$ for $m = 2k \in \{1, 2, \dots, 2^{n+1}\}$, we have $\mathcal{F}_n = \sigma(\xi_n) = \sigma(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right[, k = 1, 2, \dots, 2^n) \subset \sigma(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right[, k = 1, 2, \dots, 2^{n+1}) = \mathcal{F}_{n+1}$. Trivially $\sigma(\bigcup_{n=0}^\infty \mathcal{F}_n) \subset \mathcal{B}([0,1))$. For the other direction, we need to show that (for example) any open interval $]a,b[\in \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$ for all $0 \le a < b \le 1$. (This is sufficient

since the Borel σ-algebra is generated by the open intervals.) Let $0 \le a < b \le 1$. Find sequences $(a_j)_{j=1}^{\infty}$ and $(b_j)_{j=1}^{\infty}$ such that $a_j = \frac{k_j}{2^{n_j}}$ for some $k_j, n_j \in \mathbb{N}$ and $(a_j)_{j=1}^{\infty}$ is non-increasing with $\lim_{j\to\infty} a_j = a$, and $b_j = \frac{k'_j}{2^{n'_j}}$ for some $k'_j, n'_j \in \mathbb{N}$ and $(b_j)_{j=1}^{\infty}$ is non-decreasing with $\lim_{j\to\infty} b_j = b$. Then $[a_j, b_j[\in \mathcal{F}_m$ for $m = \max(n_j, n'_j)$ and $]a, b[= \bigcup_{j=0}^{\infty} [a_j, b_j[\in \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$.

(b) **Prove that $(M_n)_{n=0}^{\infty}$ is a martingale with $|M_n(t)| \le L$.** *We assume $\mathbb{P} = \lambda$.*

Since f is Lipschitz, $|M_n(t)| = \left| \frac{f(\xi_n(t)+2^{-n}) - f(\xi_n(t))}{2^{-n}} \right| = 2^n \left| f(\xi_n(t) + 2^{-n}) - f(\xi_n(t)) \right| \le 2^n L 2^{-n} = L$ and thus $\mathbb{E}|M_n| < \infty$ for all n. Clearly, M_n is $\mathcal{F}_n$ measurable. Let $A \in \mathcal{F}_n$. Then $A = \bigcup_{k=1}^{2^n} I_k$, where $I_k = \emptyset$ or $I_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right[$ for each k. Then

$$\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} M_{n+1}(t)dt = \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} \frac{f(\xi_{n+1}(t) + 2^{-(n+1)}) - f(\xi_{n+1}(t))}{2^{-(n+1)}} dt$$

$$= 2^{n+1} \left[\int_{\frac{2k-2}{2^{n+1}}}^{\frac{2k-1}{2^{n+1}}} f(\xi_{n+1}(t) + 2^{-(n+1)}) - f(\xi_{n+1}(t))dt \right.$$

$$\left. + \int_{\frac{2k-1}{2^{n+1}}}^{\frac{2k}{2^{n+1}}} f(\xi_{n+1}(t) + 2^{-(n+1)}) - f(\xi_{n+1}(t))dt \right]$$

$$= 2^{n+1} \left[\frac{1}{2^{n+1}} \left(f(\frac{2k-1}{2^{n+1}}) - f(\frac{2k-2}{2^{n+1}}) + \frac{2k}{2^{n+1}}) - \frac{2k-1}{2^{n+1}}) \right) \right]$$

$$= f(\frac{2k}{2^{n+1}}) - f(\frac{2k-2}{2^{n+1}})$$

$$= 2^n \left[2^{-n} \left(f(\frac{k}{2^n}) - f(\frac{k-1}{2^n}) \right) \right]$$

$$= \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} \frac{f(\xi_n(t) + 2^{-n}) - f(\xi_n(t))}{2^{-n}} dt = \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} M_n(t)dt$$

so that $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$ a.s.

(c) **Prove that there is an integrable function $g : [0,1) \to \mathbb{R}$ such that $M_n = \mathbb{E}(g|\mathcal{F}_n)$ a.s.**

By Theorem 8.1, it is sufficient to prove that M is uniformly integrable. From part (b) above we know that $|M_n| \le L$ for all n. Therefore, $\int_{\{|M_n| \ge c\}} |M_n| d\mathbb{P} \le L\mathbb{P}(|M_n| \ge c) = 0$ for all $c > L$. Thus $\sup_n \int_{\{|M_n| \ge c\}} |M_n| d\mathbb{P} = 0$ for all $c > L$ and $\limsup_n \int_{\{|M_n| \ge c\}} |M_n| d\mathbb{P} = 0$.

(d) **Prove that $f(\frac{k}{2^n}) = f(0) + \int_0^{\frac{k}{2^n}} g(t)dt$ for $k = 0, ..., .2^n - 1$.**

Since $f(\frac{k}{2^n}) = f(0) + f(\frac{k}{2^n}) - f(0) = f(0) + \sum_{j=1}^{k} \left[f(\frac{j}{2^n}) - f(\frac{j-1}{2^n}) \right]$ and $f(\frac{j}{2^n}) - f(\frac{j-1}{2^n}) = 2^n \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} f(\xi_n(t) + 2^{-n}) - f(\xi_n(t))dt = \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} M_n(t)dt$, we obtain $f(\frac{k}{2^n}) = f(0) + \int_0^{\frac{k}{2^n}} M_n(t)dt$. As $[0, \frac{k}{2^n}[\in \mathcal{F}_n$ and $M_n = \mathbb{E}(g|\mathcal{F}_n)$ a.s, we are done: $\int_0^{\frac{k}{2^n}} M_n(t)dt = \int_0^{\frac{k}{2^n}} g(t)dt$.

(e) **Prove that $f(x) = f(0) + \int_0^x g(t)dt$ for $x \in [0,1]$, i.e. g is the generalized derivative of f.**

Since g is integrable, the function $x \mapsto \int_0^x g(t)dt$ exists (i.e. $\int_0^x g(t)dt \in \mathbb{R}$ for all $x \in [0,1]$). Define $G : [0,1] \to \mathbb{R}$ by setting $G(x) = f(0) + \int_0^x g(t)dt$. By construction, G is continuous. By (d), we know that $G(s) = f(s)$ for all $s = \frac{k}{2^n}$, $n \in \mathbb{N}$, $k = 1, \ldots, 2^n$. Since the set $\{\frac{k}{2^n} : n \in \mathbb{N}, k = 1, \ldots, 2^n\}$ is dense in $[0,1]$, we obtain $G(t) = f(t)$ for all $t \in [0,1]$.

6. **Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_k)_{k=0}^n)$ with $\Omega = \{\omega_1, ..., \omega_N\}$, $\mathbb{P}(\{\omega_i\}) > 0$, and a process $(Z_k)_{k=0}^n$ such that Z_k is $\mathcal{F}_k$-measurable. Define**

$$U_n := Z_n$$

and, backwards,

$$U_k := \max\{Z_k, \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}$$

for $k = 0, ..., n-1$.

- **Show that $(U_k)_{k=0}^n$ is a supermartingale.**

  Adaptivity: $U_k := \max\{Z_k, \mathbb{E}(U_{k+1}|\mathcal{F}_k)\} \in \mathcal{F}_n$ since both Z_k and $\mathbb{E}(U_{k+1}|\mathcal{F}_k)$ are. Integrability: $\mathbb{E}|U_k| = \sum_{i=1}^N U_k(\omega_i)\mathbb{P}(\omega_i) < \infty$. Supermartingale property: $\mathbb{E}(U_k|\mathcal{F}_{k-1}) \le \max\{Z_{k-1}, \mathbb{E}(U_k|\mathcal{F}_{k-1})\} = U_{k-1}$.

- **Show that $(U_k)_{k=0}^n$ is the smallest supermartingale which dominates $(Z_k)_{k=0}^n$: if $(V_k)_{k=0}^n$ is a supermartingale with $Z_k \le V_k$, then $U_k \le V_k$ a.s.**

  Let $(V_k)_{k=0}^n$ be a supermartingale with $Z_k \le V_k$. Then $V_{n-1} \ge \mathbb{E}(V_n|\mathcal{F}_{n-1}) \ge \mathbb{E}(Z_n|\mathcal{F}_{n-1}) = \mathbb{E}(U_n|\mathcal{F}_{n-1}) \ge U_{n-1}$. Proof is complete by induction down from n: $V_{k-1} \ge \mathbb{E}(V_k|\mathcal{F}_{k-1}) \ge \mathbb{E}(U_k|\mathcal{F}_{k-1})$ and $V_{k-1} \ge Z_{k-1}$ together imply that $V_{k-1} \ge \max\{Z_{k-1}, \mathbb{E}(U_k|\mathcal{F}_{k-1})\} = U_{k-1}$.

- **Show that $\tau(\omega) := \inf\{k = 0, ..., n : Z_k(\omega) = U_k(\omega)\}$ ($\inf \emptyset := n$) is a stopping time.**

  Both Z and U are adapted; see Examole 3.4 for details.

The process $(U_k)_{k=0}^n$ is called SNELL-envelop of $(Z_k)_{k=0}^n$.