Martingale theory

Wednesday, 26th October, 2011
8.30 - 10.00 MaD 380

1. (a) Find an example of a sequence of random variables converging almost surely but not in L_1.

   (b) Find an example of a sequence of random variables converging in L_1 but not almost surely.

   (c) Prove that if $X_n \to X$ a.s. and $|X_n| \leq Y$ for some $Y \in L_1$, then $X \in L_1$ and $X_n \to X$ in L_1.

2. (a) Let $X : \Omega \to \mathbb{R}$ with $\mathbb{E}|X| < \infty$. Show that $\int_{\{|X| \geq c\}} |X| d\mathbb{P} \to 0$ as $c \to \infty$.

   (b) Prove the fact given as hint in Exercise 3.2 b): $\frac{1}{2}\left(e^{\alpha} + e^{-\alpha}\right) \leq e^{\frac{\alpha^2}{2}}$.

3. (a) Let $\varepsilon_1, \varepsilon_2, \ldots : \Omega \to \mathbb{R}$ be iid Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. Let $S_n := \varepsilon_1 + \cdots + \varepsilon_n$. Prove that

$$\mathbb{E}(\varepsilon_1 | \sigma(S_n)) = \frac{S_n}{n} \quad a.s.$$

   (b) Is the process $S = (S_n)_{n=0}^{\infty}$ above uniformly integrable?

4. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n=0}^{\infty})$ be a stochastic basis, $\mathcal{F}_\infty := \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$, and $Z \in L_1$. What can we say about the almost sure and L_1-convergence of

   (a) $\mathbb{E}(Z|\mathcal{F}_n) \to_n \mathbb{E}(Z|\mathcal{F}_\infty)$,

   (b) $\mathbb{E}(Z|\mathcal{F}_n) \to_n Z$?

   (Proofs/counterexamples)

5. Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function, i.e. $|f(x) - f(y)| \leq L|x-y|$. Let

$$\xi_n(t) := \sum_{k=1}^{2^n} \frac{k-1}{2^n} \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(t),$$

   $\Omega := [0,1)$, $\mathcal{F}_n := \sigma(\xi_n)$, and

$$M_n(t) := \frac{f(\xi_n(t) + 2^{-n}) - f(\xi_n(t))}{2^{-n}}.$$

   (a) Prove that $(\mathcal{F}_n)_{n=0}^{\infty}$ is a filtration and that $\mathcal{B}([0,1)) = \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$.

   (b) Prove that $(M_n)_{n=0}^{\infty}$ is a martingale with $|M_n(t)| \leq L$.

   (c) Prove that there is an integrable function $g : [0,1) \to \mathbb{R}$ such that $M_n = \mathbb{E}(g|\mathcal{F}_n)$ a.s.

(d) Prove that $f(\frac{k}{2^n}) = f(0) + \int_0^{\frac{k}{2^n}} g(t)dt$ for $k = 0, ..., .2^n - 1$.

(e) Prove that $f(x) = f(0) + \int_0^x g(t)dt$ for $x \in [0,1]$, i.e. g is the generalized derivative of f.

6. Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_k)_{k=0}^n)$ with $\Omega = \{\omega_1, ..., \omega_N\}$, $\mathbb{P}(\{\omega_i\}) > 0$, and a process $(Z_k)_{k=0}^n$ such that Z_k is $\mathcal{F}_k$-measurable. Define

$$U_n := Z_n$$

and, backwards,

$$U_k := \max\{Z_k, \mathbb{E}(U_{k+1}|\mathcal{F}_k)\}$$

for $k = 0, ..., n-1$.

- Show that $(U_k)_{k=0}^n$ is a super-martingale.
- Show that $(U_k)_{k=0}^n$ is the smallest super-martingale which dominates $(Z_k)_{k=0}^n$: if $(V_k)_{k=0}^n$ is a super-martingale with $Z_k \leq V_k$, then $U_k \leq V_k$ a.s.
- Show that $\tau(\omega) := \inf\{k = 0, ..., n : Z_k(\omega) = U_k(\omega)\}$ $(\inf \emptyset := n)$ is a stopping time.

The process $(U_k)_{k=0}^n$ is called SNELL-envelop of $(Z_k)_{k=0}^n$.

7. (Extra) What have we learnt about the (symmetric) random walk $S = (S_n)_{n=0}^\infty$,

$$S_n := \varepsilon_1 + \cdots + \varepsilon_n,$$

where $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ are independent Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$?
(No proofs; recall questions and possible answers.)