Martingale theory

Wednesday, 19th October, 2011
8.30 - 10.00 MaD 380

For questions 1. and 2.: Let $\Omega = [0,1)$, $\mathbb{P} = \lambda$ (the Lebesgue measure) and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Define $\mathcal{F}_n = \sigma([0,2^{-n}), [2^n, 2^{-n+1}), \ldots, [2^{-1}, 1))$ for all $n \geq 1$, and $M_0 = 1$, $M_n = 2^n \chi_{[0,2^{-n})}$.

1. Draw M_n for $n = 0,1,2,3$. Prove that the process $M = (M_n)_{n=0}^{\infty}$ is a martingale. Is it uniformly integrable? Is it closable?
   [Hint: see p. 68 in S. Geiss: Stochastic processes in discrete time]

2. Prove that

$$\lim_N \int_0^1 \sup_{n=1,\ldots,N} M_n(t)dt = \infty \quad \text{but} \quad \int_0^1 M_N(t)dt = 1.$$

   What is the connection to Doob's maximal inequalities?

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3. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1), \mathcal{B}([0,1)), \lambda)$ and

$$f_n(t) := \chi_{[0,1/n)}(t) n^{\frac{1}{p}}.$$

   For what $0 < p < \infty$ is the family $(f_n)_{n=1}^{\infty}$ uniformly integrable?

For questions 4.-5.: Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1), \mathcal{B}([0,1)), \lambda)$ and

$$M_n(t) := h_0(t) + \cdots + h_n(t),$$

where $h_n(t) := 2^n \chi_{[0,1/2^{n+1})} - 2^n \chi_{[1/2^{n+1},1/2^n)}$. Let $\mathcal{F}_n := \sigma(h_0, ..., h_n)$.

4. Show that $M = (M_n)_{n \geq 0}$ is a martingale.

5. Is there a constant $c > 0$ such that for all $N = 1, 2, ...$ one has

$$\int_0^1 \sup_{n=1,\ldots,N} |M_n(t)|dt \leq c \int_0^1 |M_N(t)|dt?$$

6. Is there a random variable $M_\infty : [0,1) \to \mathbb{R} \in L_1$ such that

$$M_n = \mathbb{E}(M_\infty | \mathcal{F}_n) \quad a.s.?$$