Martingale theory

Wednesday, 12th October, 2011
8.30 - 10.00 MaD 380

For questions 1-4:
Let $M_n := \varepsilon_1 + \cdots + \varepsilon_n$ and $M_0 := 0$, where $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ are iid with $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$. Let $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_n := \sigma(\varepsilon_1, \cdots, \varepsilon_n)$ for $n = 1, 2, \ldots$ and

$$\tau(\omega) := \inf\{n \geq 0 : M_n(\omega) = -10\}.$$

You can use the fact that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{M_n}{\sqrt{n}} = \infty, \liminf_{n\to\infty} \frac{M_n}{\sqrt{n}} = -\infty\right) = 1$$

(see e.g. page 14 in S. Geiss: Stochastic processes in discrete time).

1. Prove that $\mathbb{P}(\tau < \infty) = 1$.
2. Is there a constant $c > 0$ such that $\tau(\omega) \leq c$ a.s.?

   Hint: Assume YES, compute M_τ, and apply the OPTIONAL STOPPING THEOREM.
3. Is $\{\omega \in \Omega : \inf_n M_n(\omega) \leq -10\} \in \mathcal{F}_\tau$?
4. Is $\{\omega \in \Omega : \sup_n M_n(\omega) \geq 2\} \in \mathcal{F}_\tau$?

For questions 5-7:
Assume that a, b, c are integers with $-\infty < b < a < c < \infty$. Assume i.i.d. random variables $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ such that $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2$ and define $M_0 := a$ and $M_n := a + \sum_{k=1}^n \varepsilon_k$. Let

$$\tau(\omega) := \inf\{n \geq 0 : M_n \in \{b, c\}\}$$

with $\inf \emptyset := \infty$.

5. Prove that $\mathbb{P}(\tau < \infty) = 1$.
6. Prove that

   $$\mathbb{E} M_{\tau \wedge N} = a$$

   where $N \geq 1$ is a fixed integer.
7. (extra) Deduce that $\mathbb{E} M_\tau = a$ and compute the probability that the process M hits first b.