Martingale theory

Wednesday, 5th October, 2011
8.30 - 10.00 MaD 380

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Assume that $X : \Omega \to \mathbb{R}$ is a random variable and $f : \mathbb{R} \to \mathbb{R}$ a (Borel-measurable) function. Show that $f(X)$ is a random variable.

(b) Let $X, Y : \Omega \to \{0, 1\}$ be independent random variables. How many elements does the sub-σ-algebra $\mathcal{G} := \sigma(X, Y)$ contain?

(2) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\varepsilon_1, \varepsilon_2, ... : \Omega \to \mathbb{R}$ independent random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Let $v_0 \in \mathbb{R}$ and $v_n : \mathbb{R}^n \to \mathbb{R}$ be functions, $n = 1, 2, \ldots$ Define $M_0 := 0$, $M_1(\omega) := \varepsilon_1(\omega) v_0$,

$$M_n(\omega) := \varepsilon_1(\omega) v_0 + \sum_{k=2}^{n} \varepsilon_k(\omega) v_{k-1}\Big(\varepsilon_1(\omega), ..., \varepsilon_{k-1}(\omega)\Big) \text{ for } n = 2, 3, \ldots$$

and $\mathcal{F}_n := \sigma(\varepsilon_0, ..., \varepsilon_n)$ for $n = 0, 1, 2, \ldots$

(a) Is $(M_n)_{n=0}^{\infty}$ a martingale?

(b) Let $Z_0 := 1$ and

$$Z_n(\omega) := e^{M_n(\omega) - \frac{1}{2} \sum_{k=1}^{n} |v_{k-1}(\varepsilon_1(\omega), ..., \varepsilon_{k-1}(\omega))|^2}.$$

Show that $(Z_n)_{n=0}^{\infty}$ is a supermartingale.

Hint: in (b), notice that $e^x + e^{-x} \leq 2e^{\frac{1}{2}x^2}$

(3) Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration and $\sigma, \tau : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ stopping times.

(a) Show that $\sigma + \tau$ is a stopping time.

(b) Show that $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

For questions 4 and 5:
Let $\varepsilon_n : \Omega \to \mathbb{R}$, $n = 1, 2, \ldots$ independent random variables such that $\mathbb{P}(\varepsilon_n = -1) = \mathbb{P}(\varepsilon_n = 1) = \frac{1}{2}$ for all $i = 1, 2, \ldots$ Define $M_0 := 0$ and $M_n := \varepsilon_1 + \ldots + \varepsilon_n$, $n \geq 1$. Let $\mathcal{F}_0$ be the trivial σ-algebra and $\mathcal{F}_n := \sigma(\varepsilon_1, \ldots, \varepsilon_n)$. Are the following functions stopping times ($\inf \emptyset := \infty$)?

(4) (a) $\sigma(\omega) := \inf \{n \geq 0 : M_n = 0\}$

(b) $\sigma(\omega) := \inf \{n \geq 0 : M_n \in]10, 12[\}$

(c) $\sigma(\omega) := \inf\{n \geq 0 : M_n \in]10, 12[\} - 1$

(d) $\sigma(\omega) := \inf\{n \geq 0 : M_n \in]10, 12[\} + 1$

(5) (a) $\sigma(\omega) := \inf\{n \geq 0 : M_{n+1} \in]10, 12[\}$

(b) $\sigma(\omega) := \inf\{n \geq 0 : M_{n+1} \in]10, 11[\}$

(c) $\sigma(\omega) := \inf\{n \geq 0 : M_{n-1} = 10\}$

(d) $\sigma(\omega) := \inf\{n \geq 0 : M_{n-1} = 10\} - 1$

(6) Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n=0}^\infty)$ be a stochastic basis and $M = (M_n)_{n=0}^\infty$ be a martingale such that $M_0 = 0$ a.s. and $\mathbb{E}M_n^2 < \infty$ for all $n = 0, 1, 2, ...$ Define $\langle M \rangle_0 := 0$,

$$\langle M \rangle_n := \sum_{k=1}^{n} \mathbb{E}((M_k - M_{k-1})^2 | \mathcal{F}_{k-1}) \in [0, \infty).$$

The process $\langle M \rangle := (\langle M \rangle_n)_{n=0}^\infty$ is called *(predictable) bracket process.* Show that

(a) $(M_n^2 - \langle M \rangle_n)_{n=0}^\infty$ is a martingale,

(b) $\mathbb{E}M_n^2 = \mathbb{E}\langle M \rangle_n$ for all $n = 0, 1, ...$