Martingale theory

Wednesday, 28th September, 2011
8.30 - 10.00 MaD 380

(1) Let $\Omega := [0,1)$ and let $(h_n)_{n=0}^{\infty}$, be the sequence of Haar-functions defined by $h_0 \equiv 1$ and

$$h_{2^{n-1}+k}(t) := \begin{cases} -1 & : \quad t \in \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \\ 1 & : \quad t \in \left[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right) \\ 0 & : \quad \text{else} \end{cases}$$

for $k = 0, ..., 2^{n-1}-1$ (draw pictures of the first functions) if $n \geq 1$. Define $\mathcal{F}_n := \sigma(h_0, ..., h_n)$ for $n = 0, 1, ...$ Describe $\mathcal{F}_n$ as easy as possible.

(2) In Problem 1, is $(M_n)_{n=0}^{\infty}$ with $M_n := h_0 + \cdots + h_n$ a martingale (with $\mathbb{P} = \lambda$, the Lebesgue measure)?

(3) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 : \Omega \to \{-1, 1\}$ be independent random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a (Borel measurable) function. Prove that

$$\mathbb{E}\left(f(\varepsilon_1, \varepsilon_2, \varepsilon_3)|\sigma(\varepsilon_1, \varepsilon_2)\right) = g(\varepsilon_1, \varepsilon_2)$$

where $g(\varepsilon_1, \varepsilon_2) := \frac{1}{2}\left[f(\varepsilon_1, \varepsilon_2, -1) + f(\varepsilon_1, \varepsilon_2, 1)\right]$.

(4) Let $0 < p < 1$ and $c \in \mathbb{R}$, and let $\varepsilon_1^{(p)}, \varepsilon_2^{(p)}, \ldots : \Omega \to \{-1, 1\}$ be independent random variables with $\mathbb{P}\left(\varepsilon_n^{(p)} = -1\right) = p$ for all $n = 1, 2, \ldots$ Define $M_0 := 1$ and $M_n := e^{\sum_{i=1}^{n} \varepsilon_n^{(p)} + cn}$ for $n = 1, 2, \ldots$. Find a condition on c so that $M = (M_n)_{n=0}^{\infty}$ is a martingale w.r.t. the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$, where $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(\varepsilon_1^{(p)}, \ldots, \varepsilon_n^{(p)})$.
(Exponential random walk; for help, see p. 48 of the lecture notes of S. Geiss)

(5) Assume that $\varepsilon_1, ..., \varepsilon_n : \Omega \to \mathbb{R}$ are independent random variables such that $\mathbb{P}(\varepsilon_i = 1) = p$ and $\mathbb{P}(\varepsilon_i = -1) = q$ for some $p, q \in (0, 1)$ with $p + q = 1$. Define the stochastic process $X_k := e^{a(\varepsilon_1 + ... + \varepsilon_k) + bk}$ for $k = 1, ..., n$ and $X_0 := 1$ with $a > 0$ and $b \in \mathbb{R}$ and the filtration $(\mathcal{F}_k)_{k=0}^{n}$ with $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_k := \sigma(\varepsilon_1, ..., \varepsilon_k)$. Assume that $-a + b > 0$. Why there cannot exist random variables $\varepsilon_1, ..., \varepsilon_n : \Omega \to \{-1, 1\}$ such that $(X_k)_{k=0}^{n}$ is a martingale? (i.e. we cannot find p, q as above)

(6) Let $0 < p < \infty$, $\Omega = [0, 1)$, and $\mathcal{F}_n := \sigma\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) : k = 1, ..., 2^n\right)$ and λ the Lebesgue measure. Define $M_n(t) := 2^{\frac{n}{p}}$ for $t \in [0, 2^{-n})$ and $M_n(t) := 0$ for $t \in [2^{-n}, 1)$ for $n = 0, 1, ...$

(a) Show that $(M_n)_{n=0}^{\infty}$ is a martingale for $p = 1$.
(b) Is $(M_n)_{n=0}^{\infty}$ a super- or sub-martingale for $p \neq 1$?