Martingale theory

Wednesday, 21st September, 2011
8.30 - 10.00 MaD 380

(1) Show that the conditional expectation is linear (Property (3)).

(2) Show that the conditional expectation becomes unconditional when the random variable is independent of the σ-algebra (Property (7)).

(3) Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 : \Omega \to \mathbb{R}$ be independent random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Define $f := \varepsilon_1(\varepsilon_2 + \varepsilon_3)(\varepsilon_1 + \varepsilon_3)$ and $\mathcal{G} := \sigma\big(\varepsilon_1, \varepsilon_2\big)$. Compute
$$\mathbb{E}\big(f|\mathcal{G}\big) : \Omega \to \mathbb{R}.$$

(4) Let $\Omega := [0,1]$, $\mathcal{F} := \mathcal{B}([0,1])$ and let λ be the Lebesgue measure on $[0,1]$. Define $f(x) := x^2$ and
$$\mathcal{G} := \sigma\left(\big[0, \frac{1}{2}\big], A \subseteq \big(\frac{1}{2}, 1\big], A \in \mathcal{F}\right).$$
Compute
$$\mathbb{E}\big(f|\mathcal{G}\big) : [0,1] \to \mathbb{R}.$$

(5) For $f \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ prove that
$$\big(\mathbb{E}\big(|f||\mathcal{G}\big)\big)^2 \leq \mathbb{E}\big(|f|^2|\mathcal{G}\big) \ a.s.$$
Hint: Use the proven properties of the conditional expectation to show that
$$\mathbb{E}\big(|f|^2|\mathcal{G}\big) = \mathbb{E}\big(\big[|f| - \mathbb{E}\big(|f| \ |\mathcal{G}\big)\big]^2 |\mathcal{G}\big) + \big[\mathbb{E}\big(|f| \ |\mathcal{G}\big)\big]^2.$$

(6) Show the monotone convergence property for the conditional expectation (Property (8)).