A zeroth order Sobolev–Poincaré inequality on John domains

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joint works with B. Dyda, R. Hurri–Syrjänen, L. Ihnatsyeva, N. Marola

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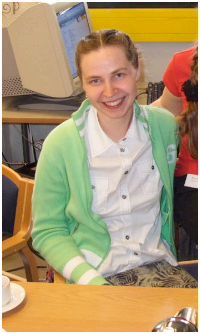
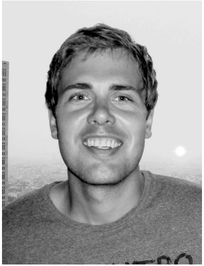
Our guideline is: *Sobolev–Poincaré equals John*

– Bojarski, Buckley, Hajłasz, Koskela, Martio, Reshetnyak...

Bartek, Ritva, Liza and Niko...



https://tuhat.halvi.helsinki.fi/portal/files/9726040/marola.jpg

Definition of John domains

A domain G in $\mathbf{R}^n$, $n \geq 2$ is a c-John domain, $c \geq 1$, if each pair of points $x_1, x_2 \in G$ can be joined by a rectifiable curve $\gamma : [0, \ell] \to G$ parametrized by its arc length such that

$$\operatorname{dist}(\gamma(t), \partial G) \geq \min\{t, \ell - t\}/c$$

for every $t \in [0, \ell]$.

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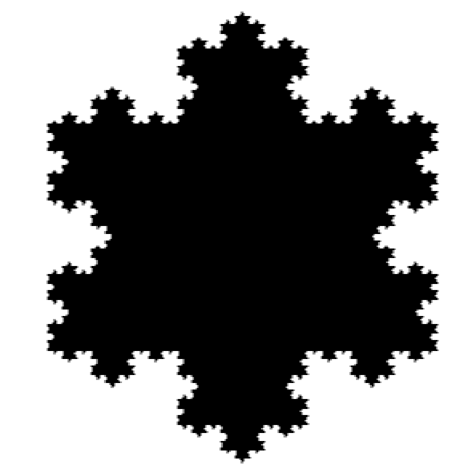
- Exhaustion property by Väisälä

Example: the Koch snowflake domain is John



Definition: Separation property

A domain G with a fixed point x_0 satisfies a separation property if there exists a constant $C_0 > 0$ as follows.

For each $x \in G$ there is a curve γ joining x to x_0 in G so that one of the following conditions is true for every t:

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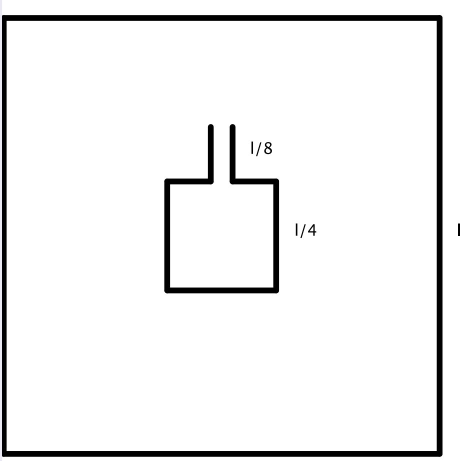
A domain with the separation property



Definition: Fractional Sobolev norm

Suppose that G is an open set in $\mathbf{R}^n$. For $0 < p < \infty$ and $0 < \kappa, \delta < 1$, and for appropriate measurable functions u on G we write

$$|u|_{W^{\delta,p}_{\kappa}(G)} = \left(\int_G \int_{B(x,\kappa \operatorname{dist}(x,\partial G))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\, dy\, dx \right)^{1/p}.$$

When $G = \mathbf{R}^n$ both of the integrals are taken over the whole space.

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Definition: Homogeneous Sobolev space

The homogeneous fractional Sobolev space $\dot{W}^{\delta,p}_{\kappa}(G)$ consists of the measurable functions $u : G \to \mathbb{R}$ with $|u|_{W^{\delta,p}_{\kappa}(G)} < \infty$.

Theorem: An improved fractional Poincaré inequality

Let G be a bounded John domain in $\mathbf{R}^n$, $n \geq 2$, $1 \leq p, q < \infty$, and $\kappa, \delta \in (0,1)$. Suppose

$$1/p - 1/q \leq \delta/n.$$

Then there is $c > 0$ such that, for every $u \in L^q(G)$, we have

$$\int_G |u(x) - u_G|^q \, dx \leq c\biggl(\int_G \int_{B(x,\kappa \mathrm{dist}(x,\partial G))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}} \, dy \, dx \biggr)^{q/p}$$

$$= c|u|^q_{W^{\delta,p}_\kappa(G)}$$

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- Hurri-Syrjänen–V., J. Anal. Math. 120 (2013)

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- '$B(x, \kappa \operatorname{dist}(x, \partial G)) \Rightarrow G$' — on bounded domains that are n-sets
- Extending the classical case due to Martio, Reshetnyak, Bojarski ($p = 1$).

Ingredients of a proof I:
A fractional representation formula of a bounded John domain G

For every Lebesgue point $x \in G$ of $u \in L^1(G)$ and $\delta \in (0,1)$

$$|u(x) - u_{B_0}| \le C \int_G \frac{|\nabla^\delta| u(y)}{|x-y|^{n-\delta}}\, dy = C \cdot \mathcal{I}^\delta(\chi_G |\nabla^\delta| u)(x)\,,$$

where $B_0 \subset G$ is a (suitable) ball and

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- The proof as in the classical case (Reshetnyak)
- The Riesz potential $\mathcal{I}^\delta$ is bounded $L^p \to L^{np/(n-\delta p)}$, $1 < p < n/\delta$

Ingredients of proof II: *Weak Poincaré implies Poincaré*

Let $0 < p \leq q < \infty$; $\kappa, \delta \in (0,1)$.
The following two inequalities are equivalent if μ is a positive Borel measure on an open set $G \subset \mathbf{R}^n$ so that $\mu(G) < \infty$.

$$\forall u \in L^\infty(G;\mu) \, : \, \inf_{c\in\mathbb{R}} \sup_{\sigma>0} \mu\{x \in G : |u(x) - c| > \sigma\}\sigma^q \leq C_1 |u|^q_{W^{\delta,p}_\kappa(G;\mu)}$$

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- Dyda–Ihnatsyeva–V. 2013
- Based on a fractional Maz'ya-type truncation method: Dyda–V. 2013

Remark: Sobolev–Poincaré is scaling-invariant

Let Q be a cube in $\mathbf{R}^n$, $n \geq 2$, $1 \leq p, q < \infty$, and $\delta \in (0,1)$. Suppose

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Then, for every $u \in L^q(Q)$, we have

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Here the constant $c > 0$ is independent of Q and u.

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- If $1/p - 1/q = \delta/n$, then $1 + q\delta/n - q/p = 0$
- This is the 'scale-invariant' case of a *Sobolev–Poincaré inequality*

Sobolev–Poincaré inequality for general John domains

Suppose that G in $\mathbf{R}^n$ is a bounded or unbounded c-John domain. Let $\kappa, \delta \in (0,1)$ be given and $1 \le p, q < \infty$ so that $1/p - 1/q = \delta/n$.

Then inequality

$$\inf_{a \in \mathbb{R}} \int_G |u(x) - a|^q \, dx \le C(\delta, \kappa, p, n, c) |u|^q_{W^{\delta,p}_\kappa(G)}$$

holds for each $u \in \dot{W}^{\delta,p}_\kappa(G)$.

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- Ongoing work with R. Hurri-Syrjänen
- Based upon the exhaustion property of John domains

Theorem: Necessity of John condition

Let G be a domain in $\mathbf{R}^n$ of finite measure
that satisfies the separation property.

Suppose that $1 \le p, q < \infty$ and $\delta \in (0,1)$ are given such that

$$1/p - 1/q = \delta/n$$

and there is $\kappa \in (0,1)$ and $c > 0$ such that inequality

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- Dyda–Ihnatsyeva–V. 2013 (in preparation)

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- The classical case by Buckley–Koskela, Math. Res. Lett., 2 (1995)

An observation

The relation $1/p - 1/q = \delta/n$ in the case of $\delta = 0$ yields

$$p = q \qquad (0 < p < \infty).$$

$\therefore$ in a zeroth order Sobolev–Poincaré inequality, a *size condition* should be L^q-based, and a *smoothness condition* should be of order zero.

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Informal definition: An order zero Sobolev–Poincaré inequality

$$\inf_{c \in \mathbb{R}} \int_G |u(x) - c|^q \, dx \leq C \Lambda_u^q(G)$$

$u \mapsto \Lambda_u^q(G)$ is an 'L^q-type functional' which makes inequality scaling invariant, and measures the 'zeroth order smoothness' of u.

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Informal definition: A weak order zero Sobolev–Poincaré inequality

$$\inf_{c\in\mathbb{R}} \sup_{\sigma>0} |\{x \in G : |u(x) - c| > \sigma\}|\sigma^q \le C\Lambda_u^q(G)$$

Definition: bounded mean oscillation (BMO)

For $1 \le p < \infty$ and $u \in L^1_{\mathrm{loc}}(G)$, define the $\mathrm{BMO}^p(G)$-seminorm by

$$|u|^p_{BMO^p(G)} = \sup_{Q \subset G \text{ cube}} \frac{1}{|Q|} \int_Q |u(x) - u_Q|^p \, dx \, .$$

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- F. John–L. Nirenberg (CPAM, 14 (1961), 415–426);

Theorem: L^q-averaging domains

Fix $1 \leq q < \infty$ and let $G \subset \mathbf{R}^n$ be an open set with finite measure. The scaling invariant inequality

$$\int_G |u(x) - u_G|^q \, dx \leq C \cdot |G| \cdot |u|^q_{\mathrm{BMO}^q(G)}$$

holds if, and only if, for a fixed $x_0 \in G$

$$\int_G k(x, x_0)^q \, dx < \infty \, ;$$

$k(x, x_0) = \inf_\gamma \int_\gamma \mathrm{dist}(z, \partial G)^{-1} \, dz$ is the quasihyperbolic distance

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$$\int_G |u(x) - u_G|^q \, dx \leq C \cdot |G| \cdot |u|^q_{\mathrm{BMO}^q(G)}$$

holds if, and only if, for a fixed $x_0 \in G$

$$\int_G k(x, x_0)^q \, dx < \infty \,;$$

$k(x, x_0) = \inf_\gamma \int_\gamma \mathrm{dist}(z, \partial G)^{-1} \, dz$ is the quasihyperbolic distance

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- Appeared earlier in her doctoral dissertation (supervisor F. Gehring)
- Not Sobolev–Poincaré, as $|G| \cdot \|u\|^q_{\mathrm{BMO}^q(G)}$ is not measuring L^q-size

Definition

For $u \in L^1(G)$, and $0 < p < \infty$ and $\tau \geq 1$, write

$$|u|^p_{A^{0,p}_\tau(G)} := \sup_{\mathcal{Q}_\tau(G)} \sum_{Q \in \mathcal{Q}_\tau(G)} |Q| \left(\frac{1}{|Q|} \int_Q |u(x) - u_Q| \, dx \right)^p .$$

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- Spaces $A^{\delta,p}_\rho(G)$ obtained from 'generalized Poincaré inequalities' by Heikkinen–Koskela–Tuominen, Studia Math. 181 (2007)
- What are the relations of these spaces to other function spaces?

Theorem: An order zero Sobolev–Poincaré inequality

Suppose that G is a bounded John domain in $\mathbf{R}^n$, $n \geq 2$.
Let $1 \leq q < \infty$ and $\tau \geq 1$. Then

$$\inf_{c \in \mathbb{R}} \sup_{\sigma > 0} |\{x \in G : |u(x) - c| > \sigma\}| \sigma^q \leq C(n, \tau, q, c) |u|^q_{A^{0,q}_\tau(G)} \qquad (\star)$$

for every $u \in L^1(G)$.

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- Hurri-Syrjänen–Marola–V. 2013 (John–Nirenberg for cubes, $\tau = 1$)

Theorem: Necessity of John condition

Let G be a domain in $\mathbf{R}^n$ $(n \geq 2)$ of finite measure and satisfying the separation property.

Let $\tau \geq 1$ and $1 < q < \infty$ be given such that inequality

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- Hurri-Syrjänen–Marola–V. 2013, $q \geq n/(n-1)$.
- Dyda–Ihnatsyeva-V. 2013, in preparation

Fix $1 < p < n/\delta$ and $0 < \delta < 1$ such that $1/p - 1/q = \delta/n$.

By an improved fractional Sobolev–Poincaré inequality on cubes,

$$\begin{aligned}
|u|^q_{A^{0,q}_\tau(G)} &\le 2 \sum_{Q\in\mathcal{Q}_\tau(G)} |Q| \left(\frac{1}{|Q|} \int_Q |u(x) - u_Q| \, dx \right)^q \\
&\le 2 \sum_{Q\in\mathcal{Q}_\tau(G)} \int_Q |u(x) - u_Q|^q \, dx \\
&\le c \sum_{Q\in\mathcal{Q}_\tau(G)} \left(\int_Q \int_{B(x,\operatorname{dist}(x,\partial Q)/2)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} \, dy \, dx \right)^{q/p} \\
&\le c \left(\int_G \int_{B(x,\operatorname{dist}(x,\partial G)/2)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} \, dy \, dx \right)^{q/p} .
\end{aligned}$$

Thus, an improved fractional Sobolev–Poincaré inequality holds in G. This combined with the assumptions implies the John condition. $\square$

Theorem: An order zero Sobolev–Poincaré inequality

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for every $u \in L^1(G)$.

For the proof of a Sobolev–Poincaré, we need notation.

Notation: Whitney cubes

For a bounded John domain G, let $\mathcal{W}^\tau(G)$ be its Whitney decomposition such that

$$\tau Q^* = \tau \tfrac{9}{8} Q \subset G$$

for a cube $Q \in \mathcal{W}^\tau(G)$.
$\mathcal{W}_j^\tau(G)$ denotes the Whitney cubes whose side length is 2^{-j} ($j \in \mathbf{Z}$);

We need to show that, for each $u \in L^1(G)$,

$$\inf_{c\in\mathbb{R}} \sup_{\sigma>0} |\{x \in G : \ |u(x) - c| > \sigma\}|\sigma^q \le C(n, c, q)|u|^q_{A^{0,q}_\tau(G)} \, .$$

Fix $Q_0 \in \mathcal{W}^\tau(G)$. By the triangle inequality, for almost every $x \in G$,

$$\begin{aligned} |u(x) - u_{Q_0^*}| &\le \left| u(x) - \sum_{Q\in\mathcal{W}^\tau(G)} u_{Q^*}\chi_Q(x) \right| + \left| \sum_{Q\in\mathcal{W}^\tau(G)} u_{Q^*}\chi_Q(x) - u_{Q_0^*} \right| \\ &= g_1(x) + g_2(x) \, . \end{aligned}$$

Hence, for a fixed $\sigma > 0$, we have

$$\begin{aligned} &\sigma^q |\{x \in G : \ |u(x) - u_{Q_0^*}| > \sigma\}| \\ &\le \boxed{\sigma^q \, |\{x \in G : \ g_1(x) > \sigma/2\}|} + \boxed{\sigma^q \, |\{x \in G : \ g_2(x) > \sigma/2\}|} \end{aligned}$$

Recall that

$$g_1(x) = \left| u(x) - \sum_{Q \in \mathcal{W}^\tau(G)} u_{Q^*} \chi_Q(x) \right|$$

The 'local' term is estimated by the result of John and Nirenberg:

$$\begin{aligned}
\sigma^q \left|\{x \in G : \ g_1(x) > \sigma/2\}\right| &= \sum_{Q \in \mathcal{W}^\tau(G)} \sigma^q |\{x \in \operatorname{int}(Q) \ : \ g_1(x) > \sigma/2\}| \\
&\le \sum_{Q \in \mathcal{W}^\tau(G)} \sigma^q |\{x \in Q^* : \ |u(x) - u_{Q^*}| > \sigma/2\}| \\
&\le C 2^q \sum_{Q \in \mathcal{W}^\tau(G)} |u|^q_{A_1^{0,q}(Q^*)} \le C |u|^q_{A_\tau^{0,q}(G)}
\end{aligned}$$

Recall that

$$g_2(x) = \left| \sum_{Q \in \mathcal{W}^\tau(G)} u_{Q^*} \chi_Q(x) - u_{Q_0^*} \right|$$

The ‘chain’ term is estimated as follows:

$$\begin{aligned}
\sigma^q \left|\{x \in G : \ g_2(x) > \sigma/2\}\right| &= \sigma^q \sum_{Q \in \mathcal{W}^\tau(G)} \left|\{x \in \mathrm{int}(Q) \ : \ |u_{Q^*} - u_{Q_0^*}| > \sigma/2\}\right| \\
&= \sum_{\substack{Q \in \mathcal{W}^\tau(G) \\ |u_{Q^*} - u_{Q_0^*}| > \sigma/2}} \sigma^q |Q| \\
&\leq \boxed{2^q \sum_{Q \in \mathcal{W}^\tau(G)} |Q||u_{Q^*} - u_{Q_0^*}|^q}
\end{aligned}$$

Definition: Chain decomposition

Fix a cube $Q_0 \in \mathcal{W}^\tau(G)$. Suppose that, for each $Q \in \mathcal{W}^\tau(G)$, we are given a chain of cubes

$$\mathcal{C}(Q) = (Q_0, \dots, Q_k) \subset \mathcal{W}^\tau(G)$$

joining Q_0 to $Q_k = Q$ such that for each $f \in L^1_{\text{loc}}(G)$,

$$|f_{Q^*} - f_{Q_0^*}| \le C(n) \sum_{R \in \mathcal{C}(Q)} \frac{1}{|R^*|} \int_{R^*} |f(x) - f_{R^*}|\, dx\,.$$

Then the family $\{\mathcal{C}(Q) : Q \in \mathcal{W}^\tau(G)\}$ is a *chain decomposition* of G.

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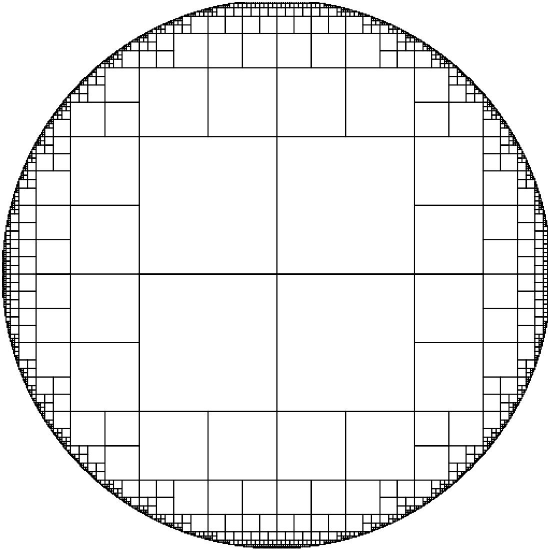
Then the family $\{\mathcal{C}(Q) \,:\, Q \in \mathcal{W}^\tau(G)\}$ is a *chain decomposition* of G.

Definition: Shadow of a Whitney cube

The *shadow* of a Whitney cube $Q \in \mathcal{W}^\tau(G)$ is the family

$$\mathcal{S}(R) = \{Q \in \mathcal{W}^\tau(G) \,:\, R \in \mathcal{C}(Q)\}\,.$$



Picture : (Almost) Whitney cubes

Lemma: Chain decomposition of a bounded John domain G

Fix $1 < q < \infty$. There is a chain decomposition of G satisfying:

The constants σ and ρ depend on τ, n, q, and the John constant c.

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(2) $\sharp\{R \in \mathcal{W}^{\tau}_{j}(G) : \ R \in \mathcal{C}(Q)\} \leq 2^{\rho}$ for each $Q \in \mathcal{W}^{\tau}(G)$ and $j \in \mathbf{Z}$;

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(3) The following inequality holds,

$$\sup_{j \in \mathbf{Z}} \sup_{R \in \mathcal{W}_j^\tau(G)} \frac{1}{|R|} \sum_{k=j-\rho}^{\infty} \sum_{\substack{Q \in \mathcal{W}_k^\tau(G) \\ Q \in \mathcal{S}(R)}} |Q|(\rho + 1 + k - j)^q < \sigma \,.$$

The constants σ and ρ depend on τ, n, q, and the John constant c.

We need to estimate

$$(\diamond) := \sum_{Q \in \mathcal{W}^\tau(G)} |Q||u_{Q^*} - u_{Q_0^*}|^q$$

By the property (1) of the chain decomposition,

$$(\diamond) \leq C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k^\tau(G)} |Q| \left(\sum_{j=-\infty}^{k+\rho} \underbrace{\sum_{\substack{R \in \mathcal{W}_j^\tau(G) \\ R \in \mathcal{C}(Q)}} \frac{1}{|R^*|} \int_{R^*} |u(x) - u_{R^*}|\, dx}_{=:\Sigma_j} \right)^q$$

$$= C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k^\tau(G)} |Q| \left(\sum_{j=-\infty}^{k+\rho} \underbrace{(\rho + 1 + k - j)^{-1} (\rho + 1 + k - j)}_{=1} \Sigma_j \right)^q$$

Using Hölder's inequality and that $\sum_{j=-\infty}^{k+\rho}(\rho+1+k-j)^{-q'} \lesssim 1$,

$$(\diamond) \le C \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{W}_k^\tau(G)} |Q| \sum_{j=-\infty}^{k+\rho} (\rho+1+k-j)^q \cdot \boxed{\Sigma_j^q}.$$

By property (2) of the chain decomposition,

$$\Sigma_j^q = \left(\sum_{\substack{R\in\mathcal{W}_j^\tau(G)\\ R\in\mathcal{C}(Q)}} \frac{1}{|R^*|}\int_{R^*} |u(x)-u_{R^*}|\,dx \right)^q$$

$$\le C \sum_{\substack{R\in\mathcal{W}_j^\tau(G)\\ R\in\mathcal{C}(Q)}} \left(\frac{1}{|R^*|}\int_{R^*} |u(x)-u_{R^*}|\,dx\right)^q \le C \sum_{\substack{R\in\mathcal{W}_j^\tau(G)\\ R\in\mathcal{C}(Q)}} \frac{|u|^q_{A_1^{0,q}(R^*)}}{|R^*|}$$

Hence, we obtain

$$
\begin{aligned}
(\diamond) &\leq C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k^\tau(G)} |Q| \sum_{j=-\infty}^{k+\rho} (\rho + 1 + k - j)^q \sum_{\substack{R \in \mathcal{W}_j^\tau(G) \\ R \in \mathcal{C}(Q)}} \frac{|u|^q_{A_1^{0,q}(R^*)}}{|R^*|} \\
&= C \sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j^\tau(G)} |u|^q_{A_1^{0,q}(R^*)} \cdot \boxed{\frac{1}{|R|} \sum_{k=j-\rho}^{\infty} \sum_{\substack{Q \in \mathcal{W}_k^\tau(G) \\ Q \in \mathcal{S}(R)}} |Q| (\rho + 1 + k - j)^q} \\
&\leq C \sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j^\tau(G)} |u|^q_{A_1^{0,q}(R^*)} \leq C |u|^q_{A_\tau^{0,q}(G)}\,.
\end{aligned}
$$

This concludes the proof. □

Let us elaborate on the validity of inequality

$$\sup_{j\in\mathbf{Z}} \sup_{R\in\mathcal{W}_j^\tau(G)} \frac{1}{|R|} \sum_{k=j-\rho}^{\infty} \sum_{\substack{Q\in\mathcal{W}_k^\tau(G)\\ Q\in\mathcal{S}(R)}} |Q|(\rho+1+k-j)^q < \sigma$$

- $\ell(R)/\ell(Q) = 2^{k-j}$. Thus, for each $\varepsilon > 0$,

$$(\rho+1+k-j)^q \simeq \log_2^q(\ell(R)/\ell(Q)) \lesssim (\ell(R)/\ell(Q))^\varepsilon$$

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- Since $\cup_{Q\in\mathcal{S}(R)} Q \subset C(n,c)R$ if $R\in\mathcal{W}^\tau(G)$, there is $y_R\in\partial G$ s.t.

$$\frac{1}{|R|}\sum_{Q\in\mathcal{S}(R)} |Q|(\ell(R)/\ell(Q))^\varepsilon \simeq \ell(R)^{\varepsilon-n} \sum_{Q\in\mathcal{S}(R)} \int_Q \operatorname{dist}(x,\partial G)^{-\varepsilon}\,dx$$

$$\leq \boxed{\ell(R)^{\varepsilon-n} \int_{B(y_R,C\ell(R))} \operatorname{dist}(x,\partial G)^{-\varepsilon}\,dx}$$

Lemma: Aikawa dimension of $\partial G < n$ for a bounded John domain G

Let G be a bounded c-John domain in $\mathbb{R}^n$. Then there is $0 < \varepsilon = \varepsilon(n, c)$ such that

$$\int_{B(y,r)} \operatorname{dist}(x, \partial G)^{-\varepsilon}\, dx \leq C(n, c) r^{n-\varepsilon}$$

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for every $y \in \partial G$ and for every $0 < r < \operatorname{diam}(\partial G)$.

Sketch of the proof

(a) the boundary ∂G of a bounded John domain is porous in $\mathbf{R}^n$
(b) the Assouad dimension of a porous set in $\mathbf{R}^n$ is strictly less than n
(c) the Assouad dimension of ∂G coincides with its Aikawa dimension

- $B(y,r) \cap G$: Hajłasz–Koskela, J. London Math. Soc. 58 (1998)
- (b) : Luukkainen, J. Korean Math. Soc. 35 (1998)
- (c) : Lehrbäck–Tuominen, J. Math. Soc. Japan 65 (2013)