Fractional Hardy inequalities on domains and applications

Antti Vähäkangas

joint with D. Edmunds, R. Hurri–Syrjänen, L. Ihnatsyeva

University of Helsinki

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I. 'Fat'sets

Riesz capacities

If $1 < p < \infty$ and $0 < sp < n$, then the (s,p) outer capacity of a set E in $\mathbb{R}^n$ is

$$R_{s,p}(E) = \inf\{\|f\|_p^p \,:\, \mathcal{I}_s f \geq 1 \text{ on } E\}\,, \quad \mathcal{I}_s f = |x|^{s-n} * f\,.$$

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Uniform fatness

A set E is (s,p) uniformly fat, if there is a positive λ such that

$$R_{s,p}(B(x,r) \cap E) \geq \lambda r^{n-sp}$$

for every $x \in E$ and $r > 0$.

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Theorem (Self-improvement of uniform fatness; Lewis '88)

Suppose E is a closed (s,p) uniformly fat set, $1 < p < \infty$, $0 < sp < n$. Then, there is $\epsilon = \epsilon(n,s,p,\lambda)$ so that E is (β,q) uniformly fat if $sp - \epsilon < \beta q \leq sp$.

Definition of d-sets

A (closed) subset $S \subset \mathbb{R}^n$ is called a d-set, if

$$\mathcal{H}^d(B(x,r) \cap S) \simeq r^d$$

for every $x \in S$ and $0 < r < \operatorname{diam}(E)$.

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Proposition

Let $1 < p < \infty$ such that $0 < sp < n$. Fix

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and suppose E is an unbounded and closed d-set. Then E is (s,p) uniformly fat.

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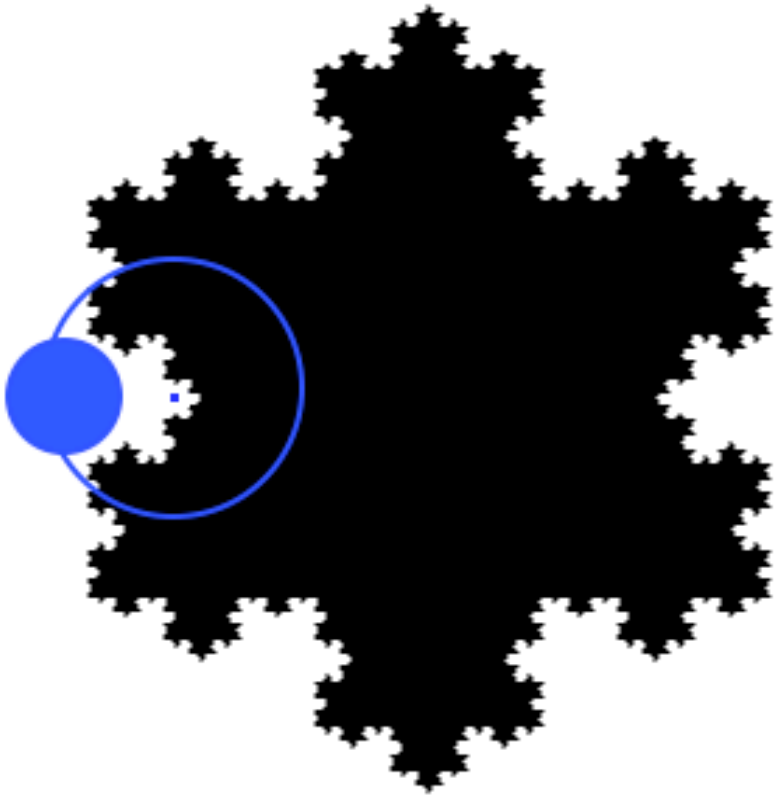
An example

The complement of the snowflake domain is an n-set, hence (s,p) uniformly fat ($1 < p < \infty$, $0 < sp < n$).



II. Hardy inequalities and uniform fatness

The classical Hardy inequality

We say that an open set G in $\mathbb{R}^n$ admits p-Hardy inequality, if

$$\int_G \frac{|f(x)|^p}{\operatorname{dist}(x,\partial G)^p}\,dx \lesssim \int_G |\nabla f(x)|^p\,dx\,, \quad f \in C_0^\infty(\mathrm{G})\,.$$

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Theorem (Lewis '88)

Suppose that the complement $\mathbb{R}^n \setminus G$ of an open set G is $(1,p)$ uniformly fat, $1 < p \le n$. Then G admits the p-Hardy inequality.

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Localisation property of the gradient

A function $f \in C_0^\infty(G)$ satisfies

$$\int_{\mathbb{R}^n} |\nabla f|^p = \int_G |\nabla f|^p\,.$$

How to measure fractional smoothness for $s \in (0,1)$?

A consequence of Plancherel's identity: for every $f \in C_0^\infty(\mathbb{R}^n)$,

$$\left\|\Delta^{s/2} f\right\|^2_{L^2(\mathbb{R}^n)} := \left\|\xi \mapsto |\xi|^s \hat{f}(\xi)\right\|^2_{L^2(\mathbb{R}^n)} \simeq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\, dy\, dx\,.$$

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The Calderón operator $\Delta^{1/2}$

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Non-locality

The fractional powers $\Delta^{s/2}$ are non-local pseudodifferential operators.

Theorem (a 'local Hardy inequality'; Dyda, 2004)

Let G be a bounded Lipschitz domain, $s \in (0,1)$, and $1 < p < \infty$. Inequality $sp > 1$ (i.e. $n - sp < n - 1$) holds if, and only if,

$$\int_G \frac{|f(x)|^p}{\operatorname{dist}(x, \partial G)^{sp}}\, dx \lesssim \int_G \int_G \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx$$

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Theorem (a 'non-local Hardy inequality'; Edmunds, Hurri–Syrjänen, V., 2012)

Let $s \in (0,1)$ and $1 < p < n/s$. Let G be an open set, whose complement is (s,p) uniformly fat. Then,

$$\int_G \frac{|f(x)|^p}{\operatorname{dist}(x, \partial G)^{sp}}\, dx \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx$$

for every $f \in C_0^\infty(G)$.

A localisation property

Suppose G is an open set in $\mathbb{R}^n$, whose complement is (s,p) uniformly fat ($0 < s < 1$, $1 < p < n/s$). Then, the localisation property

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy\,dx \lesssim \int_G\int_G \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dy\,dx$$

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holds for all $f \in C_0^\infty(G)$ if, and only if, G admits the local Hardy inequality.

An example

Suppose G is a bounded Lipschitz domain in $\mathbb{R}^n$.
Then, the localisation property fails if $0 < sp \leq 1$. The reason is that the boundary is too thin, but in the general case there can be other obstructions.

III. Hardy inequalities on Triebel–Lizorkin spaces

A Triebel–Lizorkin space

Let $s \in (0,1)$ and $1 < p < \infty$.
A function f belongs to a Triebel–Lizorkin space $F^s_{pp}(\mathbb{R}^n)$, if

$$\|f\|_{F^s_{pp}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p} < \infty \, .$$

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Triebel–Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$

One can also define Triebel–Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$ for the full range of parameters $0 < p, q \leq \infty$ ($q = \infty$ if $p = \infty$) and $s \in \mathbb{R}$.

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- If $p \in (1, \infty) \setminus \{2\}$ and $k \in \mathbf{N}$, then $W^{k,p}(\mathbb{R}^n) = F^k_{p2}(\mathbb{R}^n) \neq F^k_{pp}(\mathbb{R}^n)$.

Theorem (a 'trace Hardy inequality'; Lizaveta Ihnatsyeva, A.V.; 2012)

Let S be a d-set in $\mathbb{R}^n$, $n-1<d<n$, $1<p<\infty$, and $1\le q\le\infty$. Then, if

$$n-sp<d\,,$$

we have

$$\int_{\mathbb{R}^n}\frac{|f(x)|^p}{\operatorname{dist}(x,S)^{sp}}dx\lesssim\|f\|^p_{F^s_{pq}(\mathbb{R}^n)}$$

for every $f\in F^s_{pq}(\mathbb{R}^n)$ such that $f|_S=0$.

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- Applications to pointwise multipliers $f\mapsto f\chi_G$;
- Applications to extension problems from the boundary trace.

IV. Pointwise multipliers

Theorem (Ihnatsyeva, V.; 2012)

Let G be a domain whose boundary is a d-set, $n-1<d<n$, and assume that $1<p<\infty$ and $1\le q<\infty$. If

$$n-sp<d\,,$$

then

$$f\mapsto \chi_G f$$

is a bounded linear operator in the subspace $\{f\in F^s_{pq}(\mathbb{R}^n)\,:\,f|_{\partial G}=0\}$.

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The proof is a consequence of Hardy inequality and this Lemma;

Let G be a domain whose boundary is porous in $\mathbb{R}^n$. Then, if $1<p<\infty$, $1\le q<\infty$, and $s>0$, we have

$$\|f\chi_G\|_{F^s_{pq}(\mathbb{R}^n)}\lesssim \|f\|_{F^s_{pq}(\mathbb{R}^n)}+\left(\int_G \frac{|f(y)|^p}{\operatorname{dist}(y,\partial G)^{sp}}\,dy\right)^{1/p}$$

for every $f\in F^s_{pq}(\mathbb{R}^n)$.

V. Calderón-type extension

Traces and extension; a general framework

Suppose that there is a bounded and surjective trace operator

$$f \mapsto f|_S \colon A'(\mathbb{R}^n) \to A(S)\,.$$

Here $A'(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and $A(S) \subset L^p(S)$ are Banach function spaces, and $S \subset \mathbb{R}^n$ is a d-set with $0 < d \leq n$.

Then, we say that a bounded linear operator

$$\mathrm{Ext}_S : A(S) \to A'(\mathbb{R}^n)$$

is an *extension operator*, if $(\mathrm{Ext}_S f)|_S = f$ ($\mathcal{H}^d$ a.e.) for every $f \in A(S)$.

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An example

The trace operator is bounded and surjective

$$F^s_{pq}(\mathbb{R}^n) \to B^{s-(n-d)/p}_{pp}(S)$$

if S is a d-set with $0 < d < n$ and $n - sp < d$ ($1 < p < \infty$, $1 \leq q \leq \infty$).

Theorem (Calderón, 1961)

Let G be a Lipschitz domain in $\mathbb{R}^n$ and $k \in \{1, 2, \ldots\}$.
Then, the trace operator $W^{k,p}(\mathbb{R}^n) \to W^{k,p}(G)$ is bounded and surjective, and there is an extension operator

$$\mathrm{Ext}_G : W^{k,p}(G) \to W^{k,p}(\mathbb{R}^n)$$

such that $\mathrm{Ext}_G f(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus G$ if $f \in W_0^{k,p}(G)$.

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Theorem (Shvartsman, 2006; Ihnatsyeva, V., 2012)

Suppose G is a domain in $\mathbb{R}^n$, whose closure is an n-set, and whose boundary is a d-set, with $n - 1 < d < n$. Let $1 < p < \infty$, $1 \leq q < \infty$, and $n - sp < d$.

Then, the trace operator $F^s_{pq}(\mathbb{R}^n) \to F^s_{pq}(G)$ is bounded and surjective, and there is an extension operator

$$\mathrm{Ext_G} : F^s_{pq}(G) \to F^s_{pq}(\mathbb{R}^n)$$

such that $\mathrm{Ext_G} f(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus G$ assuming that $f|_{\partial G} = 0$.

Step I

We identify a given $f \in F^s_{pq}(G)$ with its Shvartsman extension $\in F^s_{pq}(\mathbb{R}^n)$. Define $g := f|_{\partial G}$; by a trace theorem, Ihnatsyeva, V. 2011,

$$\|g\|_{B^{s-(n-d)/p}_{pp}(\partial G)} \lesssim \|f\|_{F^s_{pq}(G)} \cdot$$

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Step II

Define $h := \mathrm{Ext}_{\partial G}(g)$; here

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Step III

Define $\mathrm{Ext}_G(f) := (f-h)\chi_G + h$. This has the 'zero extension property' for functions with zero trace on ∂G. Moreover,

$$\|h\|_{F^s_{pq}(\mathbb{R}^n)} = \|\mathrm{Ext}_{\partial G} g\|_{F^s_{pq}(\mathbb{R}^n)} \lesssim \|g\|_{B^{s-(n-d)/p}_{pp}(\partial G)} \lesssim \|f\|_{F^s_{pq}(G)} \,.$$

Since $(f-h)|_{\partial G} = 0$, $\|(f-h)\chi_G\|_{F^s_{pq}(\mathbb{R}^n)} \lesssim \|f-h\|_{F^s_{pq}(\mathbb{R}^n)} \lesssim \|f\|_{F^s_{pq}(G)}$.