Fractional Hardy inequalities on domains and applications

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I. ‘Fat’ sets
Riesz capacities

If $1 < p < \infty$ and $0 < sp < n$, then the $(s, p)$ outer capacity of a set $E$ in $\mathbb{R}^n$ is

$$R_{s,p}(E) = \inf \{ \| f \|_p^p : \mathcal{I}_s f \geq 1 \text{ on } E \}, \quad \mathcal{I}_s f = |x|^{s-n} * f.$$
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Uniform fatness

A set $E$ is $(s, p)$ uniformly fat, if there is a positive $\lambda$ such that

$$R_{s,p}(B(x, r) \cap E) \geq \lambda r^{n-sp}$$

for every $x \in E$ and $r > 0$. 
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**Theorem (Self-improvement of uniform fatness; Lewis '88)**
Suppose $E$ is a closed $(s, p)$ uniformly fat set, $1 < p < \infty$, $0 < sp < n$. Then, there is $\epsilon = \epsilon(n, s, p, \lambda)$ so that $E$ is $(\beta, q)$ uniformly fat if $sp - \epsilon < \beta q \leq sp$. 

Definition of $d$-sets

A (closed) subset $S \subseteq \mathbb{R}^n$ is called a $d$-set, if

$$\mathcal{H}^d(B(x, r) \cap S) \asymp r^d$$

for every $x \in S$ and $0 < r < \text{diam}(E)$. 

An example

The complement of the snowflake domain is an $n$-set, hence $(s, p)$ uniformly fat $(1 < p < 1, 0 < sp < n)$. 
Definition of $d$-sets

A (closed) subset $S \subset \mathbb{R}^n$ is called a $d$-set, if

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Proposition

Let $1 < p < \infty$ such that $0 < sp < n$. Fix

$$n - sp < d \leq n,$$

and suppose $E$ is an unbounded and closed $d$-set. Then $E$ is $(s, p)$ uniformly fat.
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The complement of the snowflake domain is an $n$-set, hence $(s, p)$ uniformly fat ($1 < p < \infty$, $0 < sp < n$).
'Fat' sets
II. Hardy inequalities and uniform fatness
The classical Hardy inequality

We say that an open set $G$ in $\mathbb{R}^n$ admits $p$-Hardy inequality, if

$$
\int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^p} \, dx \lesssim \int_G |\nabla f(x)|^p \, dx, \quad f \in C_0^\infty(G).
$$

Theorem (Lewis '88)

Suppose that the complement $\mathbb{R}^n \cap G$ of an open set $G$ is $(1, p)$-uniformly fat, then $G$ admits the $p$-Hardy inequality.

Localisation property of the gradient

A function $f \in C_1^0(G)$ satisfies

$$
\int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx = \int_G |\nabla f(x)|^p \, dx, \quad f \in C_0^\infty(G).
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Suppose that the complement $\mathbb{R}^n \setminus G$ of an open set $G$ is $(1, p)$ uniformly fat, $1 < p \leq n$. Then $G$ admits the $p$-Hardy inequality.
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How to measure fractional smoothness for $s \in (0, 1)$?

A consequence of Plancherel’s identity: for every $f \in C_0^\infty(\mathbb{R}^n)$,

$$\left\| \Delta^{s/2} f \right\|^2_{L^2(\mathbb{R}^n)} := \left\| \xi \mapsto |\xi|^s \hat{f}(\xi) \right\|^2_{L^2(\mathbb{R}^n)} \asymp \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} \, dy \, dx.$$
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The Calderón operator $\Delta^{1/2}$

For every $f \in C_0^\infty(\mathbb{R}^n)$,

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\]

The Calderón operator \( \Delta^{1/2} \)

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Non-locality

The fractional powers \( \Delta^{s/2} \) are non-local pseudodifferential operators.
Theorem (a ‘local Hardy inequality’; Dyda, 2004)

Let $G$ be a bounded Lipschitz domain, $s \in (0, 1)$, and $1 < p < \infty$. Inequality $sp > 1$ (i.e. $n - sp < n - 1$) holds if, and only if,

$$\int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \lesssim \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx$$

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for every \( f \in C_0^\infty (G) \).

Theorem (a ‘non-local Hardy inequality’; Edmunds, Hurri–Syrjänen, V., 2012)

Let \( s \in (0, 1) \) and \( 1 < p < n/s \).
Let \( G \) be an open set, whose complement is \((s, p)\) uniformly fat. Then,

\[
\int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx
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for every \( f \in C_0^\infty (G) \).
A localisation property

Suppose $G$ is an open set in $\mathbb{R}^n$, whose complement is $(s, p)$ uniformly fat ($0 < s < 1$, $1 < p < n/s$). Then, the localisation property

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \lesssim \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx$$

holds for all $f \in C_0^\infty(G)$ if, and only if, $G$ admits the local Hardy inequality.
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holds for all $f \in C_0^\infty(G)$ if, and only if, $G$ admits the local Hardy inequality.

An example

Suppose $G$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Then, the localisation property fails if $0 < sp \leq 1$. The reason is that the boundary is too thin, but in the general case there can be other obstructions.
III. Hardy inequalities on Triebel–Lizorkin spaces
A Triebel–Lizorkin space

Let $s \in (0, 1)$ and $1 < p < \infty$. A function $f$ belongs to a Triebel–Lizorkin space $F_{pp}^s(\mathbb{R}^n)$, if

$$
\|f\|_{F_{pp}^s(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p} < \infty.
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Often one denotes $W^{s,p}(\mathbb{R}^n) = F_{pp}^s(\mathbb{R}^n)$. 

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Let \( s \in (0, 1) \) and \( 1 < p < \infty \).

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- Often one denotes \( W^{s,p}(\mathbb{R}^n) = F_{sp}^{pp}(\mathbb{R}^n) \).
- This space is also known as the fractional Sobolev space of order \( s \).
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Triebel–Lizorkin spaces \( F_{pq}^s(\mathbb{R}^n) \)

One can also define Triebel–Lizorkin spaces \( F_{pq}^s(\mathbb{R}^n) \) for the full range of parameters \( 0 < p, q \leq \infty \) (\( q = \infty \) if \( p = \infty \)) and \( s \in \mathbb{R} \).
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Triebel–Lizorkin spaces $F_{pq}^s(\mathbb{R}^n)$

One can also define Triebel–Lizorkin spaces $F_{pq}^s(\mathbb{R}^n)$ for the full range of parameters $0 < p, q \leq \infty$ ($q = \infty$ if $p = \infty$) and $s \in \mathbb{R}$.
- If $p \in (1, \infty) \setminus \{2\}$ and $k \in \mathbb{N}$, then $W^{k,p}(\mathbb{R}^n) = F_{p2}^k(\mathbb{R}^n) \neq F_{pp}^k(\mathbb{R}^n)$. 
Theorem (a ‘trace Hardy inequality’; Lizaveta Ihnatsyeva, A.V.; 2012)

Let $S$ be a $d$-set in $\mathbb{R}^n$, $n - 1 < d < n$, $1 < p < \infty$, and $1 \leq q \leq \infty$. Then, if

$$n - sp < d,$$

we have

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{\text{dist}(x, S)^{sp}} \, dx \lesssim \|f\|^p_{F_{pq}^s(\mathbb{R}^n)}$$

for every $f \in F_{pq}^s(\mathbb{R}^n)$ such that $f|_S = 0$. 
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- Applications to pointwise multipliers $f \mapsto f \chi_G$;
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- Applications to pointwise multipliers $f \mapsto f \chi_G$;
- Applications to extension problems from the boundary trace.
IV. Pointwise multipliers
Theorem (Ihnatsyeva, V.; 2012)

Let $G$ be a domain whose boundary is a $d$-set, $n - 1 < d < n$, and assume that $1 < p < \infty$ and $1 \leq q < \infty$. If

$$n - sp < d,$$

then

$$f \mapsto \chi_G f$$

is a bounded linear operator in the subspace $\{ f \in F_{pq}^s(\mathbb{R}^n) : f|_{\partial G} = 0 \}$. 

Theorem (Ihnatsyeva, V.; 2012)

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The proof is a consequence of Hardy inequality and this Lemma;

Let $G$ be a domain whose boundary is porous in $\mathbb{R}^n$. Then, if $1 < p < \infty$, $1 \leq q < \infty$, and $s > 0$, we have

$$\|f \chi_G\|_{F_{pq}^s(\mathbb{R}^n)} \lesssim \|f\|_{F_{pq}^s(\mathbb{R}^n)} + \left(\int_G \frac{|f(y)|^p}{\text{dist}(y, \partial G)^{sp}} \, dy\right)^{1/p}$$

for every $f \in F_{pq}^s(\mathbb{R}^n)$. 

V. Calderón-type extension
Traces and extension; a general framework

Suppose that there is a bounded and surjective trace operator

\[ f \mapsto f|_S : A'(\mathbb{R}^n) \to A(S). \]

Here \( A'(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \) and \( A(S) \subset L^p(S) \) are Banach function spaces, and \( S \subset \mathbb{R}^n \) is a \( d \)-set with \( 0 < d < n \).

Then, we say that a bounded linear operator

\[ \text{Ext}_S : A(S) \to A'(\mathbb{R}^n) \]

is an \textit{extension operator}, if \( (\text{Ext}_S f)|_S = f (\mathcal{H}^d \text{ a.e.}) \) for every \( f \in A(S) \).
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An example

The trace operator is bounded and surjective

\[ F_{pq}^s(\mathbb{R}^n) \to B_{pp}^{s-(n-d)/p}(S) \]

if \( S \) is a \( d \)-set with \( 0 < d < n \) and \( n - sp < d \) \((1 < p < \infty, 1 \leq q \leq \infty)\).
Theorem (Calderón, 1961)

Let $G$ be a Lipschitz domain in $\mathbb{R}^n$ and $k \in \{1, 2, \ldots \}$. Then, the trace operator $W^{k,p}(\mathbb{R}^n) \to W^{k,p}(G)$ is bounded and surjective, and there is an extension operator

$$\text{Ext}_G : W^{k,p}(G) \to W^{k,p}(\mathbb{R}^n)$$

such that $\text{Ext}_G f(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus G$ if $f \in W^{k,p}_0(G)$. 

Theorem (Shvartsman, 2006; Ihnatsyeva, V., 2012)

Suppose $G$ is a domain in $\mathbb{R}^n$, whose closure is an $n$-set, and whose boundary is a $d$-set, with $n + 1 < d < n$. Let $1 < p < \infty$, $1 \leq q < \infty$, and $n \text{sp} < d$. Then, the trace operator $F_{pq}^s(\mathbb{R}^n) \to F_{pq}^s(G)$ is bounded and surjective, and there is an extension operator

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Calderón-type extension

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Theorem (Shvartsman, 2006; Ihnatsyeva, V., 2012)

Suppose $G$ is a domain in $\mathbb{R}^n$, whose closure is an $n$-set, and whose boundary is a $d$-set, with $n - 1 < d < n$. Let $1 < p < \infty$, $1 \leq q < \infty$, and $n - sp < d$.

Then, the trace operator $F^{s}_{pq}(\mathbb{R}^n) \to F^{s}_{pq}(G)$ is bounded and surjective, and there is an extension operator

$$\text{Ext}_G : F^{s}_{pq}(G) \to F^{s}_{pq}(\mathbb{R}^n)$$

such that $\text{Ext}_G f(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus G$ assuming that $f|_{\partial G} = 0$. 
Step I

We identify a given $f \in F_{pq}^s(G)$ with its Shvartsman extension $\in F_{pq}^s(\mathbb{R}^n)$. Define $g := f|_{\partial G}$; by a trace theorem, Ihnatsyeva, V. 2011,

$$\|g\|_{B_{pp}^{s-(n-d)/p}(\partial G)} \lesssim \|f\|_{F_{pq}^s(G)}.$$
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Step II
Define $h := \text{Ext}_{\partial G}(g)$; here

$$\text{Ext}_{\partial G} : B_{pp}^{s-(n-d)/p}(\partial G) \to F_{pq}^s(\mathbb{R}^n)$$

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Calderón-type extension

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Step III

Define $\text{Ext}_G(f) := (f - h) \chi_G + h$. This has the ‘zero extension property’ for functions with zero trace on $\partial G$. Moreover,

$$\|h\|_{F_{pq}^s(\mathbb{R}^n)} = \|\text{Ext}_{\partial G}g\|_{F_{pq}^s(\mathbb{R}^n)} \lesssim \|g\|_{B_{pp}^{s-(n-d)/p}(\partial G)} \lesssim \|f\|_{F_{pq}^s(G)}.$$ 

Since $(f - h)|_{\partial G} = 0$, 

$$||(f - h)\chi_G\|_{F_{pq}^s(\mathbb{R}^n)} \lesssim \|f - h\|_{F_{pq}^s(\mathbb{R}^n)} \lesssim \|f\|_{F_{pq}^s(G)}.$$ 