Local Hardy inequalities and self-improvement of uniform fatness

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1. Analysis on metric spaces (briefly)

We assume throughout the talk that $X = (X, d, \mu)$ is a metric measure space, where μ is a doubling Borel regular measure and

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In particular, there exists $C_D \geq 1$ such that

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We do not always assume that X is complete.

For simplicity, we assume that $\operatorname{diam}(X) = \infty$
(this is not, however, needed in the statements nor the arguments).

A measurable function $g \geq 0$ on X is a 'p-weak upper gradient' ($1 \leq p < \infty$) of a function $u : X \to \mathbb{R}$ if, for p-a.e. curve $\gamma : [0, 1] \to X$ joining two arbitrary points x and y in X, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds \, .$$

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Example 1.1

When u is a Lipschitz function on X, the *upper pointwise Lipschitz constant* of u at $x \in X$ is defined as

$$\operatorname{Lip}(u,x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r} \,.$$

The Borel function $g = \operatorname{Lip}(u, \cdot)$ is a p-weak upper gradient of u.

A metric measure space X supports a '(q,p)-Poincaré inequality', for exponents $1 \le q, p < \infty$, if there are $C > 0$ and $\lambda \ge 1$ such that

- for all balls $B(x,r) \subset X$,
- for all measurable functions u on X,
- and for all p-weak upper gradients g of u

we have:

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{1/q} \le C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}. \qquad (1)$$

The left-hand side of (1) is interpreted as ∞ if the integral-average

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu \,.$$

is not defined (i.e. if u is not integrable on B).

A typical standing assumption for analysis on metric spaces is a $(1,p)$-Poincaré inequality (a.k.a. p-Poincaré inequality).

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When $1 < p < \infty$, the p-Poincaré inequality is self-improving in various ways. For instance:

- If X is complete, then there is $1 < p_0 < p$ such that X supports a (p,p_0)-Poincaré inequality (with a larger λ).
- As a consequence, for a complete space supporting a $(1,p)$-Poincaré inequality, we obtain the (q,q)-Poincaré inequalities in the range

$$p_0 \leq q \leq p$$

and with uniform constants (use Hölder).

If X is not a complete space, an improved Poincaré inequalities hypothesis (PI) will be assumed in some of the statements:

(PI) Let $1 < p < \infty$ be given. We assume that there are $1 < p_0 < p$, $C_P > 0$ and $\tau \geq 1$ such that X supports the (q, q)-Poincaré inequality

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C_P \operatorname{diam}(B) \left(\fint_{\tau B} g^q \, d\mu \right)^{1/q}$$

for every $p_0 \leq q \leq p$.

Lemma 1.2

If X is complete and supports a $(1, p)$-Poincaré inequality for $1 < p < \infty$, then (PI) is valid for X.

There are books available on the topic of analysis on metric spaces:

- *Nonlinear Potential Theory on Metric Spaces*, A. Björn and J. Björn; 2011.
- *Sobolev Spaces on Metric Measure Spaces*, J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson; 2015.

2. Uniformly fat sets

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The '(Lipschitz) variational p-capacity' of K with respect to Ω is

$$\operatorname{cap}_p(K, \Omega) = \inf \int_\Omega g^p \, d\mu \,, \tag{2}$$

where the infimum is taken over all Lipschitz functions u on X, such that $\chi_K \leq u \leq \chi_\Omega$, and all p-weak upper gradients g of u.

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If there are no such functions u, we set $\operatorname{cap}_p(K,\Omega) = \infty$.

A closed set $E \subset X$ is 'uniformly p-fat', for $1 \leq p < \infty$, if there exists a constant $0 < c_0 \leq 1$ such that

$$\operatorname{cap}_p(E \cap \overline{B(x,r)}, B(x,2r)) \geq c_0 \operatorname{cap}_p(\overline{B(x,r)}, B(x,2r))$$

for all $x \in E$ and all $r > 0$.

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for all $x \in E$ and all $r > 0$.

Example 2.1

Recall that a closed set $E \subset \mathbb{R}^n$ is an 'Ahlfors d-regular set' with $0 < d \leq n$, if

$$\mathcal{H}^d(B(x,r) \cap E) \simeq r^d$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(E)$.

Suppose that $p > n - d$ and that $E \subset \mathbb{R}^n$ is a closed unbounded Ahlfors d-regular set, then E is p-uniformly fat.

Lemma 2.2 (A boundary p-Poincaré inequality)

Suppose that X supports a $(1,p)$-Poincaré inequality with an exponent $1 \le p < \infty$. Let $E \subset X$ be a uniformly p-fat set.

Then there exists a constant $C > 0$ such that

- *for all Lipschitz functions u on X such that $u = 0$ in E,*
- *for all p-weak upper gradients g of u,*
- *for all balls $B = B(w,r) \subset X$ with $w \in E$ and $r > 0$,*

we have

$$\left(\fint_B |u|^p \, d\mu \right) \le C \operatorname{diam}(B) \left(\fint_{\tau B} g^p \, d\mu \right)^{1/p},$$

where τ is the dilatation from the (p,p)-Poincaré inequality.

Proof of Lemma 2.2.

By testing the p-capacity and applying the (p,p)-Poincaré, we first obtain a 'capacitary (Maz'ya) Poincaré inequality'

$$\fint_B |u|^p \, d\mu \leq \frac{C}{\operatorname{cap}_p(\overline{2^{-1}B} \cap \{u = 0\}, B\}} \int_{\tau B} g^p \, d\mu \,, \qquad (3)$$

where by assumptions

$$E \cap \overline{2^{-1}B} \subset \overline{2^{-1}B} \cap \{u = 0\}$$

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and τ is the dilatation from the (p,p)-Poincaré inequality.

Next we apply the doubling condition of μ, a 'capacitary comparison' inequality and the uniform fatness-assumption as follows

$$\frac{\mu(B)}{\operatorname{diam}(B)^p} \simeq \frac{\mu(B)}{r^p} \lesssim \operatorname{cap}_p(\overline{2^{-1}B}, B) \lesssim \operatorname{cap}_p(E \cap \overline{2^{-1}B}, B) \,.$$

This concludes the proof. □

Theorem 2.3 (Self-improvement of uniform fatness)

Let $1 < p < \infty$ and let X be a complete metric space with a doubling measure and supporting a $(1, p)$-Poincaré inequality. Assume that $E \subset X$ is a uniformly p-fat closed set. Then there exists $1 < q < p$ such that E is uniformly q-fat (quantitatively).

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- There is a simple proof for the self-improvement of uniform Q-fatness in the setting of Ahlfors Q-regular metric measure spaces R. Korte and N. Shanmugalingam (Math. Z. '10) via global Hardy inequalities;

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- There is a simple proof for the self-improvement of uniform Q-fatness in the setting of Ahlfors Q-regular metric measure spaces R. Korte and N. Shanmugalingam (Math. Z. '10) via global Hardy inequalities;
- A new proof by Lehrbäck–Tuominen–V. 2015 via local Hardy inequalities.

The following new result is for non-complete spaces; the proof is quite elementary (but still not easy nor very short).

Theorem 2.4 (Lehrbäck–Tuominen–V. 2015)

Let $1 < p < \infty$ and suppose that X supports the improved (q, q)-Poincaré inequalities (PI) for $p_0 \leq q \leq p$. Assume that $E \subset X$ is a uniformly p-fat closed set and that

$$E \cap \overline{B(w, r)}$$

is compact for all $w \in E$ and all $r > 0$.

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Then there is $0 < \varepsilon < p - p_0$ such that E is uniformly $(p-\varepsilon)$-fat.

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- Hardy inequalities (we will return to this aspect);
- Higher integrability of the gradients of solutions to PDE's — T. Kilpeläinen and P. Koskela (Nonlinear Anal. '94);
- Higher integrability for the upper gradients of certain p-quasiminimizers on metric measure spaces — O. E. Maasalo [O. E. Kansanen] and A. Zatorska-Goldstein (J. Lond. Math. Soc. '08).

4. Pointwise and local Hardy inequalities

The following result has provided a starting point for the sudy of 'pointwise Hardy inequalities'.

Theorem 2.5 (Kinnunen–Martio, Math. Res. Lett. '97)

Let $1 < p \le n$ and $E \subset \mathbb{R}^n$ be a uniformly p-fat closed set. Then

$$|u(x)| \lesssim \operatorname{dist}(x, E) \left(\mathscr{M}|\nabla u|^p(x)\right)^{1/p}$$

whenever $x \in \mathbb{R}^n \setminus E$ and $u \in C_0^\infty(\mathbb{R}^n \setminus E)$.

Here $\mathscr{M}$ is the usual Hardy–Littlewood maximal operator in $\mathbb{R}^n$.

Corollary 2.6 (Global Hardy inequality — Wannebo and Lewis)

Let $1 < p \leq n$ (actually $1 < p < \infty$) and suppose that $E \subset \mathbb{R}^n$ is a uniformly p-fat closed set. Then

$$\int_{\mathbb{R}^n \setminus E} \frac{|u(x)|^p}{\operatorname{dist}(x,E)^p}\, dx \lesssim \int_{\mathbb{R}^n \setminus E} |\nabla u(x)|^p\, dx$$

for every $u \in C_0^\infty(\mathbb{R}^n \setminus E)$.

Proof.

By the self-improvement of uniform fatness, there is $1 < q < p$ such that E is uniformly q-fat. By Theorem 2.5, if $x \in \mathbb{R}^n \setminus E$,

$$\frac{|u(x)|}{\operatorname{dist}(x,E)} \lesssim \left(\mathscr{M}|\nabla u|^q(x)\right)^{1/q}$$

Raising this to power p and applying the boundedness of maximal operator in $L^{p/q}$ gives the result (observe that $p/q > 1$). □

For a limiting upper radius $0 < R \leq \infty$, we let

$$\mathscr{M}_R u(x) = \sup_{\substack{0<r\leq R \\ r<\infty}} \fint_{B(x,r)} |u(y)|\, d\mu(y)$$

be an 'R-restricted Hardy–Littlewood maximal function' of a locally integrable function u on X.

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Definition 2.7

Let $1 \leq p < \infty$ and $E \subset X$ be closed. The open set $X \setminus E$ admits a 'pointwise p-Hardy inequality' if there exist constants $c_H > 0$ and $L \geq 1$ such that

$$|u(x)| \leq c_H \operatorname{dist}(x, E) \big(\mathscr{M}_{L \operatorname{dist}(x,E)} g^p(x)\big)^{1/p}$$

for a.e. $x \in X \setminus E$, for every Lipschitz function u on X satisfying $u = 0$ in E, and for every p-weak upper gradient g of u.

Theorem 2.8 (Korte–Lehrbäck–Tuominen, Math. Ann. '11)

Let $1 \leq p < \infty$ and let X be a complete, doubling metric measure space supporting a $(1, p)$-Poincaré inequality.

Then, for a closed set $E \subset X$, the following are quantitatively equivalent:

(a) *E is uniformly p-fat;*

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(b) *E admits a boundary p-Poincaré inequality;*

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(c) *$X \setminus E$ admits a pointwise p-Hardy inequality.*

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(a) *E is uniformly p-fat;*

(b) *E admits a boundary p-Poincaré inequality;*

(c) *$X \setminus E$ admits a pointwise p-Hardy inequality.*

With a new proof for the self-improvement of uniform fatness in mind, we would like to add the following item to the list:

(d) $X \setminus E$ admits a local p-Hardy inequality.

Theorem 2.9 (A local p-Hardy inequality)

Let $1 < p < \infty$ and let X be a complete, doubling metric measure space supporting a $(1,p)$-Poincaré inequality. Assume that $E \subset X$ is a closed uniformly p-fat set.

Then there exists $L \geq 1$ such that

$$\int_{B(w,r)\setminus E} \frac{|u(x)|^p}{\operatorname{dist}(x,E)^p}\, d\mu(x) \lesssim \int_{B(w,2Lr)} g(x)^p\, d\mu(x)$$

whenever u is a Lipschitz function on X such that $u = 0$ in E, g is a p-weak upper gradient of u, and $B(w,r)$ is a ball with $w \in E$.

Proof.

By self-improvement of both Poincaré and uniform fatness, there exists $1 < q < p$ such that X supports $(1, q)$-Poincaré inequality and E is uniformly q-fat.

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Recall that $w \in E$. Hence, by the pointwise q-Hardy inequality,

$$\frac{|u(x)|}{\operatorname{dist}(x, E)} \lesssim \left(\mathscr{M}_{L \operatorname{dist}(x,E)}\, g^q(x)\right)^{1/q} \leq \left(\mathscr{M}_\infty\left(\chi_{B(w,2Lr)} g^q\right)(x)\right)^{1/q}$$

for almost every $x \in B(w, r) \setminus E$.

Proof.

By self-improvement of both Poincaré and uniform fatness, there exists $1 < q < p$ such that X supports $(1, q)$-Poincaré inequality and E is uniformly q-fat.

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for almost every $x \in B(w, r) \setminus E$.

Raising this to power p and using the boundedness of the maximal operator $\mathscr{M}_\infty$ on $L^{p/q}$ gives the result. □

5. Self-improvement of uniform fatness revisited [arXiv:1512.06514]

Let us now rewind and go back to our main-result.

Theorem 2.10 (Lehrbäck–Tuominen–V. 2015)

Let $1 < p < \infty$ and suppose that X supports the improved (q, q)-Poincaré inequalities (PI) for $p_0 \leq q \leq p$. Assume that $E \subset X$ is a uniformly p-fat closed set and that

$$E \cap \overline{B(w, r)}$$

is compact for all $w \in E$ and all $r > 0$.

Then there is $0 < \varepsilon < p - p_0$ such that E is uniformly $(p - \varepsilon)$-fat.

(Both ε and the constant appearing in the uniform $(p - \varepsilon)$-fatness condition are quantitative.)

Next we will sketch the proof of Theorem 2.10

Step I: The absorption principle

The proof is based on the 'absorption principle', for $0 < \delta < 1$,

$$A^* \leq (1-\delta)A^* + \alpha \quad \Rightarrow \quad A^* \leq \frac{\alpha}{\delta}\,.$$

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It is *critical* to choose the appropriate quantities A^* and α

In particular (and contrary to the intuition) it can be better to first increase A^* in order to 'balance the inequality', for $0 < \delta < 1$,

$$A^* \leq A \quad \text{and} \quad A \leq (1-\delta)A + \alpha \quad \Rightarrow \quad A^* \leq \frac{\alpha}{\delta}\,.$$

Example 2.11

For instance, in order to obtain an estimate

$$A^*(x) = |u(x)| \lesssim \alpha(x)$$

via absorption, one can first try to replace A^* with $A(x) = \mathscr{M}\,u(x)$ (this is often beneficial but quite vaguely described above).

In our setting, a 'balanced inequality' will be gained by choosing

$$A = \int_{B(w,r)} \frac{|u(x)|^q}{\operatorname{dist}(x,E)^q}\, d\mu(x)\,, \qquad w \in E\,,$$

with a suitable exponent q, depending on the situation at hand.

We establish and improve local Hardy inequalities via absorption

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with a suitable exponent q, depending on the situation at hand.

We establish and improve local Hardy inequalities via absorption

There is a natural competitor (which is difficult to work with)

$$A^* = r^{-q} \int_{B(w,r)} |u(x)|^q\, d\mu(x) \leq \int_{B(w,r)} \frac{|u(x)|^q}{\operatorname{dist}(x,E)^q}\, d\mu(x) = A\,,$$

since $\operatorname{dist}(x,E) \leq d(x,w) < r$ for every $x \in B(w,r)$. Observe that A^* would correspond to the boundary p-Poincaré inequality.

Step II: Truncation

Fix, once and for all, a ball $B = B(w, r)$ with $w \in E$ and $r > 0$.

Lemma 2.12 (A truncation lemma)

Assume that $E \subset X$ is a uniformly p-fat closed set. Let

$$E_B^0 = E \cap \overline{\frac{1}{2}B}$$

and then define inductively, for every $j \in \mathbb{N}$, that

$$E_B^j = \bigcup_{x \in E_B^{j-1}} E \cap \overline{B(x, 2^{-j-1}r)}\,, \quad \text{and set} \quad E_B = \overline{\bigcup_{j \in \mathbb{N}_0} E_B^j}.$$

Then E_B is a 'locally uniformly p-fat set' (up to scale r).

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- The truncation of E to the ball B somehow appears in *each known proof* for the self-improvement of uniform fatness;

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- In fact, we came up with this truncation independently;
- The truncation is needed for a 'local absorption lemma'.

Lemma 2.13 (Local absorption lemma)

Fix $\sigma \geq 1$, $\varsigma \geq 2$, and $p_0 \leq q \leq p$.
Assume that u is a Lipschitz function on X such that $u = 0$ on E_B and g is a q-weak upper gradient of u such that inequality

$$\int_{\varsigma B \setminus E_B} \frac{|u(x)|^q}{\operatorname{dist}(x, E_B)^q}\, d\mu(x)$$
$$\leq C_1 \int_{\sigma\varsigma B \setminus E_B} \frac{|u(x)|^q}{\operatorname{dist}(x, E_B)^q}\, d\mu(x) + C_2 \int_{\sigma\varsigma B} g(x)^q\, d\mu(x)$$

holds with some constants $C_1, C_2 > 0$. Then

$$C_3 \int_{\sigma\varsigma B \setminus E_B} \frac{|u(x)|^q}{\operatorname{dist}(x, E_B)^q}\, d\mu(x) \leq C_4 \int_{\tau\sigma\varsigma B} g(x)^q\, d\mu(x)\,,$$

where $C_3 = 1 - C_1(1 + C_{C_D,\sigma,\varsigma,p})$ and $C_4 = (1 + C_2)C_{X,\sigma,\varsigma,q}$.

Step III: Localized Wannebo argument

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The *last proof* is well written and easy to grasp. Actually, by applying the local absorption lemma, it allows a modification to provide a local p-Hardy inequality, as well.

Let us take a brief look at the details.

A 'β-regularization' is needed to make certain sums converge.

For this purpose, one introduces a free 'absorption parameter' $0 < \beta < 1$ by considering the function

$$v(x) = u(x)\,\mathrm{dist}(x, E_B)^{\beta/p}\,.$$

The exact value of β is to be determined later. Recall that E_B is the truncation of the set E to the fixed ball $B = B(w, r)$.

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Now v has a p-weak upper gradient g_v such that

$$g_v(x) \leq \frac{\beta}{p}|u(x)|\,\mathrm{dist}(x, E_B)^{\beta/p-1} + g(x)\,\mathrm{dist}(x, E_B)^{\beta/p}\,;$$

This fact follows from the Leibniz and chain rules.

Another natural ingredient is to consider the annuli

$$A_m = \{x \in 8\tau B \,:\, 2^{-m} r \leq \operatorname{dist}(x, E_B) < 2^{-m+1} r\}\,,$$

The following expansion in terms in the annuli is motivated by the observation that $\operatorname{dist}(x, E_B) \simeq 2^{-m} r$ if $x \in A_m$.

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Since E_B is locally p-fat, via capacitary p-Poincaré inequalities and the fact that $\beta > 0$, we obtain inequalities

$$\begin{aligned}\int_{2B\setminus E_B} \frac{|v(x)|^p}{\operatorname{dist}(x, E_B)^{p+\beta}}\, d\mu(x) &\leq \sum_{m=0}^{\infty} \int_{A_m} \frac{|v(x)|^p}{\operatorname{dist}(x, E_B)^{p+\beta}}\, d\mu(x) \\ &\leq \frac{C_{X,p}}{c_0 \beta} \int_{8\tau B\setminus E_B} \frac{g_v(x)^p}{\operatorname{dist}(x, E_B)^{\beta}}\, d\mu(x)\,.\end{aligned}$$

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Observe how the size of the ball grows! This is inevitable

Substituting u to the inequality above gives us

$$\int_{2B\setminus E_B} \frac{|u(x)|^p}{\operatorname{dist}(x,E_B)^p}\,d\mu = \int_{2B\setminus E_B} \frac{|v(x)|^p}{\operatorname{dist}(x,E_B)^{p+\beta}}\,d\mu(x)$$
$$\leq \frac{C_{X,p}\beta^{p-1}}{c_0}\int_{8\tau B\setminus E_B} \frac{|u(x)|^p}{\operatorname{dist}(x,E_B)^p}\,d\mu + \frac{C_{X,p}}{c_0\beta}\int_{8\tau B} g(x)^p\,d\mu\,.$$

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By resorting to local absorption lemma, and therein choosing β sufficiently small, we obtain a local p-Hardy inequality (however with E replaced by E_B, but this is not essential).

Step IV: Self-improvement of local Hardy inequality

We now have establised a local p-Hardy inequality. The Step IV is to 'ε-improve' this inequality (quantitatively).

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We are now in the heart of the self-improvement

By applying the local absorption lemma once again, we are able to localize the Koskela–Zhong argument. This step provides us with an 'ε-improved inequality'

$$\int_{12\tau^2 B\setminus E_B} \frac{|u(x)|^{p-\varepsilon}}{\operatorname{dist}(x,E_B)^{p-\varepsilon}}\,d\mu(x) \lesssim \int_{12\tau^3 B} \operatorname{Lip}(u,x)^{p-\varepsilon}\,d\mu(x)$$

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Observe that we have the pointwise upper Lipschitz constant on the right-hand side, instead of g. This is related to the delicate properties of q-weak upper gradients: we vary the exponent!

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Therefore an additional (but again standard) argument, based on connection between the Hausdorff content and uniform fatness, is needed to finish the proof in the non-complete case.

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The end