Calderón problem

Lecture notes, Spring 2008

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CHAPTER 1

Introduction

Electrical Impedance Tomography (EIT) is an imaging method with potential applications in medical imaging and nondestructive testing. The method is based on the following important inverse problem.

> **Calderón problem:** Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

In this course we will prove a fundamental uniqueness result due to Sylvester and Uhlmann, which states that the conductivity is determined by the boundary measurements. We will also consider stable dependence of the conductivity on boundary measurements, and the case where measurements are only made on part of the boundary. In addition, we will discuss useful techniques in partial differential equations, Fourier analysis, and inverse problems.

Let us begin by recalling the mathematical model of EIT, see [**5**] for details. The purpose is to determine the electrical conductivity $\gamma(x)$ at each point $x \in \Omega$, where $\Omega \subseteq \mathbf{R}^n$ represents the body which is imaged (in practice $n = 3$). We assume that Ω is a bounded open subset of $\mathbf{R}^n$ with C^∞ boundary, and that γ is a positive C^2 function in $\overline{\Omega}$.

Under the assumption of no sources or sinks of current in Ω, a voltage potential f at the boundary $\partial\Omega$ induces a voltage potential u in Ω, which solves the Dirichlet problem for the conductivity equation,

$$\left\{ \begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \qquad \text{in } \Omega, \\ u &= f \qquad \text{on } \partial\Omega. \end{aligned} \right. \tag{1.1}$$

Since γ is positive, there is a unique weak solution $u \in H^1(\Omega)$ for any boundary value $f \in H^{1/2}(\partial\Omega)$. One can define the Dirichlet to Neumann map (DN map) formally as

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

This is the current flowing through the boundary. More precisely, the DN map is defined weakly as

$$(\Lambda_\gamma f, g)_{\partial\Omega} = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where u is the solution of (1.1), and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. The pairing on the boundary is integration with respect to the surface measure,

$$(f, g)_{\partial\Omega} = \int_{\partial\Omega} f g \, dS.$$

With this definition Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$.

The Calderón problem (also called the inverse conductivity problem) is to determine the conductivity function γ from the knowledge of the map Λ_γ. That is, if the measured current $\Lambda_\gamma f$ is known for all boundary voltages $f \in H^{1/2}(\partial\Omega)$, one would like to determine the conductivity γ. There are several aspects of this inverse problem which are interesting both for mathematical theory and practical applications.

1. **Uniqueness.** If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, show that $\gamma_1 = \gamma_2$.
2. **Reconstruction.** Given the boundary measurements Λ_γ, find a procedure to reconstruct the conductivity γ.
3. **Stability.** If Λ_{γ_1} is close to Λ_{γ_2}, show that γ_1 and γ_2 are close (in a suitable sense).
4. **Partial data.** If Γ is a subset of $\partial\Omega$ and if $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ for all boundary voltages f, show that $\gamma_1 = \gamma_2$.

Starting from the work of Calderón in 1980, the inverse conductivity problem has been studied intensively. In the case where $n \geq 3$ and all conductivities are in $C^2(\overline{\Omega})$, the following positive results are an example of what can be proved.

Theorem. *(Sylvester-Uhlmann 1987) If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

Theorem. *(Nachman 1988) There is a convergent algorithm for reconstructing γ from Λ_γ.*

Theorem. *(Alessandrini 1988) Let $\gamma_j \in H^s(\Omega)$ for $s > \frac{n}{2} + 2$, and assume that $\|\gamma_j\|_{H^s(\Omega)} \leq M$ and $1/M \leq \gamma_j \leq M$ $(j = 1, 2)$. Then*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)})$$

where $\omega(t) = C|\log t|^{-\sigma}$ *for small* $t > 0$*, with* $C = C(\Omega, M, n, s) > 0$*,* $\sigma = \sigma(n, s) \in (0, 1)$*.*

Theorem. *(Kenig-Sjöstrand-Uhlmann 2007) Assume that* Ω *is convex and* Γ *is any open subset of* $\partial\Omega$*. If* $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ *for all* $f \in H^{1/2}(\partial\Omega)$*, and if* $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$*, then* $\gamma_1 = \gamma_2$ *in* Ω*.*

During this course we will discuss the methods involved in these results. The main tool will be the construction of special solutions, called *complex geometrical optics* solutions, to the conductivity equation and related equations. This will involve Fourier analysis, and we will begin the course with a discussion of n-dimensional Fourier series.

References. This course a continuation of the class "Impedanssitomografian perusteet" (Principles of EIT) given by Petri Ola [**5**]. From [**5**] we will mainly use the solvability of the Dirichlet problem for elliptic equations, the definition of the DN map, and results which state that the boundary values of the conductivity can be recovered from the DN map. Chapter 2 on multiple Fourier series is classical, an excellent reference is Zygmund [**8**]. In Chapter 3 we prove the uniqueness result of Sylvester-Uhlmann [**6**]. The proof of the main estimate, Theorem 3.7, follows Hähner [**4**]. The stability question is taken up in Chapter 4, where the main results are due to Alessandrini [**1**]. The treatment also benefited from Feldman-Uhlmann [**3**]. In the final Chapter 5 we introduce Carleman estimates and prove the result in Bukhgeim-Uhlmann [**2**], which states that it is enough to measure currents on roughly half of the boundary to determine the conductivity.

CHAPTER 2

Multiple Fourier series

2.1. Fourier series in L^2

Joseph Fourier laid the foundations of the mathematical field now known as Fourier analysis in his 1822 treatise on heat flow. The basic question is to represent periodic functions as sums of elementary pieces. If the period of f is 2π and the elementary pieces are sine and cosine functions, then the desired representation would be

$$f(x) = \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx)).$$

Since $e^{ikx} = \cos(kx) + i\sin(kx)$, we may alternatively consider the series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}. \tag{2.1}$$

We will need to represent functions of n variables as Fourier series. If f is a function in $\mathbf{R}^n$ which is 2π-periodic in each variable, then a natural analog of (2.1) would be

$$f(x) = \sum_{k \in \mathbf{Z}^n} c_k e^{ik\cdot x}.$$

This is the form of Fourier series which we will study. Note that the terms on the right-hand side are 2π-periodic in each variable.

There are many subtle issues related to various modes of convergence for the series above. However, we will mostly just need the case of convergence in L^2 norm for Fourier series of L^2 functions, and in this case no problems arise. Consider the cube $Q = [-\pi, \pi]^n$, and define an inner product on $L^2(Q)$ by

$$(f, g) = (2\pi)^{-n} \int_Q f\bar{g}\, dx, \quad f, g \in L^2(Q).$$

With this inner product, $L^2(Q)$ is a separable infinite-dimensional Hilbert space. The space of functions which are locally square integrable and

2π-periodic in each variable may be identified with $L^2(Q)$. Therefore, we will consider Fourier series of functions in $L^2(Q)$.

LEMMA 2.1. *The set* $\{e^{ik\cdot x}\}$ *is an orthonormal subset of* $L^2(Q)$.

PROOF. A direct computation: if $k, l \in \mathbf{Z}^n$ then

$$\begin{aligned}(e^{ik\cdot x}, e^{il\cdot x}) &= (2\pi)^{-n} \int_Q e^{i(k-l)\cdot x}\,dx \\ &= (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i(k_1-l_1)x_1} \cdots e^{i(k_n-l_n)x_n}\,dx_n \cdots dx_1 \\ &= \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}\end{aligned}$$

□

We recall a Hilbert space fact: if $\{e_j\}_{j=1}^{\infty}$ is an orthonormal subset of a separable Hilbert space H, then the following are equivalent:

(1) $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis, in the sense that any $f \in H$ may be written as the series
$$f = \sum_{j=1}^{\infty} (f, e_j) e_j$$
with convergence in H,

(2) for any $f \in H$ one has
$$\|f\|^2 = \sum_{j=1}^{\infty} |(f, e_j)|^2,$$

(3) if $f \in H$ and $(f, e_j) = 0$ for all j, then $f \equiv 0$.

If the condition (3) is satisfied, the orthonormal system $\{e_j\}$ is called *complete*. The main point is that $\{e^{ik\cdot x}\}_{k\in\mathbf{Z}^n}$ is complete in $L^2(Q)$.

LEMMA 2.2. *If* $f \in L^2(Q)$ *satisfies* $(f, e^{ik\cdot x}) = 0$ *for all* $k \in \mathbf{Z}^n$, *then* $f \equiv 0$.

The proof is given below. The main result on Fourier series of L^2 functions is now immediate. Below we denote by $\ell^2(\mathbf{Z}^n)$ the space of complex sequences $c = (c_k)_{k\in\mathbf{Z}^n}$ with norm

$$\|c\|_{\ell^2(\mathbf{Z}^n)} = \Big(\sum_{k\in\mathbf{Z}^n} |c_k|^2 \Big)^{1/2}.$$

THEOREM 2.3. *If $f \in L^2(Q)$, then one has the Fourier series*

$$f(x) = \sum_{k \in \mathbf{Z}^n} \hat{f}(k) e^{ik\cdot x}$$

with convergence in $L^2(Q)$, where the Fourier coefficients are given by

$$\hat{f}(k) = (f, e^{ik\cdot x}) = (2\pi)^{-n} \int_Q f(x) e^{-ik\cdot x}\, dx.$$

One has the Plancherel formula

$$\|f\|^2_{L^2(Q)} = \sum_{k \in \mathbf{Z}^n} |\hat{f}(k)|^2.$$

Conversely, if $c = (c_k) \in \ell^2(\mathbf{Z}^n)$, then the series

$$f(x) = \sum_{k \in \mathbf{Z}^n} c_k e^{ik\cdot x}$$

converges in $L^2(Q)$ to a function f satisfying $\hat{f}(k) = c_k$.

PROOF. The facts on the Fourier series of $f \in L^2(Q)$ follow directly from the discussion above, since $\{e^{ik\cdot x}\}_{k \in \mathbf{Z}^n}$ is a complete orthonormal system in $L^2(Q)$. For the converse, if $(c_k) \in \ell^2(\mathbf{Z}^n)$, then

$$\| \sum_{\substack{k \in \mathbf{Z}^n \\ M \leq |k| \leq N}} c_k e^{ik\cdot x} \|^2_{L^2(Q)} = \sum_{\substack{k \in \mathbf{Z}^n \\ M \leq |k| \leq N}} |c_k|^2$$

by orthogonality. Since the right hand side can be made arbitrarily small by choosing M and N large, we see that $f_N = \sum_{k \in \mathbf{Z}^n, |k| \leq N} c_k e^{ik\cdot x}$ is a Cauchy sequence in $L^2(Q)$, and converges to $f \in L^2(Q)$. One obtains $\hat{f}(k) = (f, e^{ik\cdot x}) = c_k$ again by orthogonality. $\square$

It remains to prove Lemma 2.2. We begin with the most familiar case, $n = 1$. The partial sums of the Fourier series of a function $f \in L^2([-\pi, \pi])$, extended as a 2π-periodic function into $\mathbf{R}$, are given by

$$\begin{aligned} S_m f(x) = \sum_{k=-m}^{m} \hat{f}(k) e^{ikx} &= \sum_{k=-m}^{m} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\, dy \Big) e^{ikx} \\ &= \int_{-\pi}^{\pi} D_m(x-y) f(y)\, dy \end{aligned}$$

where $D_m(x)$ is the *Dirichlet kernel*

$$D_m(x) = \frac{1}{2\pi}\sum_{k=-m}^{m} e^{ikx} = \frac{1}{2\pi}e^{-imx}(1 + e^{ix} + \ldots + e^{i2mx})$$
$$= \frac{1}{2\pi}e^{-imx}\frac{e^{i(2m+1)x}-1}{e^{ix}-1} = \frac{1}{2\pi}\frac{e^{i(m+\frac{1}{2})x} - e^{-i(m+\frac{1}{2})x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} = \frac{1}{2\pi}\frac{\sin((m+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}.$$

Thus, definining the *convolution* of two 2π-periodic locally integrable functions by

$$f * g(x) = \int_{-\pi}^{\pi} f(x-y)g(y)\,dy,$$

we see that $S_m f(x) = (D_m * f)(x)$. The Dirichlet kernel acts in a similar way as an approximate identity, but is problematic because it takes both positive and negative values.

Definition. A sequence $(K_N(x))_{N=1}^{\infty}$ of 2π-periodic continuous functions on the real line is called an *approximate identity* if

(1) $K_N \geq 0$ for all N,
(2) $\int_{-\pi}^{\pi} K_N(x)\,dx = 1$ for all N, and
(3) for all $\delta > 0$ one has

$$\lim_{N\to\infty} \sup_{\delta\leq|x|\leq\pi} K_N(x) \to 0.$$

Thus, an approximate identity (K_N) for large N resembles a Dirac delta at 0, extended in a 2π-periodic way. It is possible to approximate L^p functions by convolving them against an approximate identity.

Lemma 2.4. *Let (K_N) be an approximate identity. If $f \in L^p([-\pi,\pi])$ where $1 \leq p < \infty$, or if f is a continuous 2π-periodic function and $p = \infty$, then*

$$\|K_N * f - f\|_{L^p([-\pi,\pi])} \to 0 \quad \text{as } N \to \infty.$$

Proof. Since K_N has integral 1, we have

$$(K_N * f)(x) - f(x) = \int_{-\pi}^{\pi} K_N(y)[f(x-y) - f(x)]\,dy$$

Let first f be continuous and $p = \infty$. To estimate the L^∞ norm of $K_N * f - f$, we fix $\varepsilon > 0$ and compute

$$|(K_N * f)(x) - f(x)| \le \int_{-\pi}^{\pi} K_N(y)|f(x-y) - f(x)|\,dy$$

$$\le \int_{|y|\le\delta} K_N(y)|f(x-y)-f(x)|\,dy + \int_{\delta\le|y|\le\pi} K_N(y)|f(x-y)-f(x)|\,dy.$$

Here $\delta > 0$ is chosen so that

$$|f(x-y) - f(x)| < \frac{\varepsilon}{2} \quad \text{whenever } x \in \mathbf{R} \text{ and } |y| \le \delta.$$

This is possible because f is uniformly continuous. Further, we use the definition of an approximate identity and choose N_0 so that

$$\sup_{\delta\le|x|\le\pi} K_N(x) < \frac{\varepsilon}{4\pi\|f\|_{L^\infty}}, \quad \text{for } N \ge N_0.$$

With these choices, we obtain

$$|(K_N * f)(x) - f(x)| \le \frac{\varepsilon}{2}\int_{|y|\le\delta} K_N(y)\,dy + 4\pi\|f\|_{L^\infty} \sup_{\delta\le|x|\le\pi} K_N(x) < \varepsilon$$

whenever $N \ge N_0$. The result is proved in the case $p = \infty$.

Let now $f \in L^p([-\pi,\pi])$ and $1 \le p < \infty$. We need the integral form of Minkowski's inequality,

$$\Big(\int_X \Big|\int_Y F(x,y)\,d\nu(y)\Big|^p d\mu(x)\Big)^{1/p} \le \int_Y \Big(\int_X |F(x,y)|^p\,d\mu(x)\Big)^{1/p} d\nu(y),$$

which is valid on σ-finite measure spaces (X,μ) and (Y,ν), cf. the usual Minkowski inequality $\|\sum_y F(\,\cdot\,,y)\|_{L^p} \le \sum_y \|F(\,\cdot\,,y)\|_{L^p}$. Using this, we obtain

$$\|K_N * f - f\|_{L^p([-\pi,\pi])} \le \int_{-\pi}^{\pi} K_N(y)\|f(\,\cdot\,-y) - f\|_{L^p([-\pi,\pi])}\,dy$$

$$= \int_{\delta\le|y|\le\pi} K_N(y)\|f(\,\cdot\,-y)-f\|_{L^p}\,dy + \int_{|y|\le\delta} K_N(y)\|f(\,\cdot\,-y)-f\|_{L^p}\,dy$$

$$\le 4\pi\|f\|_{L^p} \sup_{\delta\le|x|\le\pi} K_N(x) + \sup_{|y|\le\delta}\|f(\,\cdot\,-y) - f\|_{L^p}.$$

Now, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(\,\cdot\,-y) - f\|_{L^p([-\pi,\pi])} < \frac{\varepsilon}{2} \quad \text{whenever } |y| \le \delta.$$

Thus the second term can be made arbitrarily small by choosing δ sufficiently small, and then the first term is also small if N is large. This shows the result. $\square$

Now, if the Dirichlet kernels (D_m) were an approximate identity, by Lemma 2.4 one would have $S_m f \to f$ in L^2 for any $f \in L^2$. This would in particular imply that any $f \in L^2$ which satisfies $(f, e^{ikx}) = \hat{f}(k) = 0$ for all $k \in \mathbf{Z}^n$, would be the zero function.

However, D_m is not an approximate identity because it takes negative values. One does get an approximate identity if a different summation method used: instead of the partial sums $S_m f$ consider the *Cesàro sums*

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{m=0}^{N} S_m f(x).$$

This can be written in convolution form as

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{m=0}^{N} (D_m * f)(x) = (F_N * f)(x)$$

where F_N is the *Fejér kernel*,

$$\begin{aligned}
F_N(x) &= \frac{1}{2\pi(N+1)} \sum_{m=0}^{N} \frac{e^{i(m+\frac{1}{2})x} - e^{-i(m+\frac{1}{2})x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{1}{2\pi(N+1)} \frac{e^{i\frac{1}{2}x} \frac{e^{i(N+1)x}-1}{e^{ix}-1} - e^{-i\frac{1}{2}x} \frac{e^{-i(N+1)x}-1}{e^{-ix}-1}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{1}{2\pi(N+1)} \frac{e^{i(N+1)x} - 1 + e^{-i(N+1)x} - 1}{(e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x})^2} \\
&= \frac{1}{2\pi(N+1)} \frac{\sin^2(\frac{N+1}{2}x)}{\sin^2(\frac{1}{2}x)}.
\end{aligned}$$

Clearly this is nonnegative, and in fact F_N is an approximate identity (exercise). It follows from Lemma 2.4 that Cesàro sums of the Fourier series an L^p function always converge in the L^p norm if $1 \le p < \infty$.

LEMMA 2.5. *If $f \in L^p([-\pi, \pi])$ where $1 \le p < \infty$, or if f is a continuous 2π-periodic function and $p = \infty$, then*

$$\|\sigma_N f - f\|_{L^p([-\pi,\pi])} \to 0 \quad \text{as } N \to \infty.$$

PROOF OF LEMMA 2.2. We begin with the case $n = 1$. If $f \in L^2([-\pi, \pi])$ and $(f, e^{ikx}) = 0$ for all $k \in \mathbf{Z}$, then $S_m f = 0$ and also $\sigma_N f = 0$ for all N. By Lemma 2.5 it follows that $f \equiv 0$.

Now let $n \geq 2$, and assume that $f \in L^2(Q)$ and $(f, e^{ik\cdot x}) = 0$ for all $k \in \mathbf{Z}^n$. Since $e^{ik\cdot x} = e^{ik_1x_1} \cdots e^{ik_nx_n}$, we have

$$\int_{-\pi}^{\pi} h(x_1; k_2, \ldots, k_n) e^{-ik_1x_1}\, dx_1 = 0$$

for all $k_1 \in \mathbf{Z}$, where

$$h(x_1; k_2, \ldots, k_n) = \int_{[-\pi,\pi]^{n-1}} f(x_1, x_2, \ldots, x_n) e^{-i(k_2x_2+\ldots+k_nx_n)}\, dx_2 \cdots dx_n.$$

Now $h(\,\cdot\,; k_2, \ldots, k_n)$ is in $L^2([-\pi, \pi])$ by the Cauchy-Schwarz inequality. By the completeness of the system $\{e^{ik_1x_1}\}$ in one dimension, we obtain that $h(\,\cdot\,; k_2, \ldots, k_n) = 0$ for all $k_2, \ldots, k_n \in \mathbf{Z}$. Applying the same argument in the other variables shows that $f \equiv 0$. $\square$

We have now proved the main facts on Fourier series of L^2 functions.

2.2. Sobolev spaces

In this section we wish to consider Sobolev spaces of periodic functions. In fact, most of the results given here will not be used in their present form, but they provide good motivation for the developments in Chapter 3.

Let $T^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$ be the n-dimensional torus. Note that $L^2(Q)$ above may be identified with $L^2(T^n)$. However, $C(Q)$ is different from $C(T^n)$; in fact $C(T^n)$ (resp. $C^\infty(T^n)$) can be identified with the continuous (resp. C^∞) 2π-periodic functions in $\mathbf{R}^n$.

First we need to define weak derivatives of periodic functions. This is similar to the nonperiodic case, except that here the test function space will be $C^\infty(T^n)$. We use the notation

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

Definition. Let $f \in L^2(T^n)$. We say that $D^\alpha f \in L^2(T^n)$ if there is a function $v \in L^2(T^n)$ which satisfies

$$\int_{T^n} f D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_{T^n} v\varphi\, dx$$

for all $\varphi \in C^\infty(T^n)$. In this case we define $D^\alpha f = v$.

As in the nonperiodic case, weak derivatives are unique, and for smooth functions the definition coincides with the usual derivative.

This uses the fact that there are no boundary terms arising from integration by parts, because of periodicity.

DEFINITION. If $m \geq 0$ is an integer, we denote by $H^m(T^n)$ the space of functions $f \in L^2(T^n)$ such that $D^\alpha f \in L^2(T^n)$ for all $\alpha \in \mathbf{N}^n$ satisfying $|\alpha| \leq m$.

We equip $H^m(T^n)$ with the inner product

$$(f,g)_{H^m(T^n)} = \sum_{|\alpha|\leq m} (D^\alpha f, D^\alpha g).$$

Then $H^m(T^n)$ is a Hilbert space.

If f and $D^\alpha f$ are in $L^2(T^n)$, one may compute the Fourier coefficients of $D^\alpha f$ to be

$$\begin{aligned}
(D^\alpha f)\hat{}(k) &= (2\pi)^{-n} \int_{T^n} D^\alpha f(x) e^{-ik\cdot x}\, dx \\
&= (2\pi)^{-n}(-1)^{|\alpha|} \int_{T^n} f(x) D^\alpha(e^{-ik\cdot x})\, dx \\
&= k^\alpha \hat{f}(k)
\end{aligned}$$

by the definition of weak derivative, since $e^{ik\cdot x} \in C^\infty(T^n)$. This motivates the following characterization of $H^m(T^n)$ in terms of Fourier coefficients.

LEMMA 2.6. *Let $f \in L^2(T^n)$. Then $f \in H^m(T^n)$ if and only if $\langle k\rangle^m \hat{f} \in \ell^2(\mathbf{Z}^n)$, where $\langle k\rangle = (1 + k_1^2 + \ldots + k_n)^{1/2}$.*

PROOF. One has

$$\begin{aligned}
f \in H^m(T^n) &\Leftrightarrow D^\alpha f \in L^2(T^n) \quad \text{for } |\alpha| \leq m \\
&\Leftrightarrow k^\alpha \hat{f}(k) \in \ell^2(\mathbf{Z}^n) \quad \text{for } |\alpha| \leq m \\
&\Leftrightarrow (k_1^2, \ldots, k_n^2)^\alpha |\hat{f}(k)|^2 \in \ell^1(\mathbf{Z}^n) \quad \text{for } |\alpha| \leq m.
\end{aligned}$$

If the last condition is satisfied, then

$$\langle k\rangle^{2m} |\hat{f}(k)|^2 = \sum_{|\beta|\leq m} c_\beta (k_1^2, \ldots, k_n^2)^\beta |\hat{f}(k)|^2 \in \ell^1(\mathbf{Z}^n),$$

consequently $\langle k\rangle^m \hat{f}(k) \in \ell^2(\mathbf{Z}^n)$. Conversely, if $\langle k\rangle^m \hat{f}(k) \in \ell^2(\mathbf{Z}^n)$, then $k^\alpha \hat{f}(k) \in \ell^2(\mathbf{Z}^n)$ for $|\alpha| \leq m$ because $|k_j| \leq \langle k\rangle$. $\square$

It is now easy to prove a version of the Sobolev embedding theorem.

THEOREM 2.7. *If $m > n/2$ then $H^m(T^n) \subseteq C(T^n)$.*

PROOF. Let $f \in H^m(T^n)$, so that $\langle k\rangle^m \hat{f} \in \ell^2(\mathbf{Z}^n)$ and

$$f(x) = \sum_{k\in\mathbf{Z}^n} \hat{f}(k)e^{ik\cdot x}.$$

Let $M_k = |\hat{f}(k)e^{ik\cdot x}| = \langle k\rangle^{-m}(\langle k\rangle^m \hat{f}(k))$. We have

$$\sum_{k\in\mathbf{Z}^n} M_k \leq \|\langle k\rangle^{-m}\|_{\ell^2(\mathbf{Z}^n)}\|\langle k\rangle^m \hat{f}(k)\|_{\ell^2(\mathbf{Z}^n)} < \infty,$$

by Lemma 2.6 and since $m > n/2$. Since the terms in the Fourier series of f are continuous functions, this Fourier series converges absolutely and uniformly into a continuous function in T^n by the Weierstrass M-test. □

The final result in this section will be elliptic regularity in the periodic case. Consider a second order differential operator $P(D)$ acting on 2π-periodic functions in $\mathbf{R}^n$,

$$P(D) = \sum_{|\alpha|\leq 2} a_\alpha D^\alpha,$$

where a_α are complex constants. The principal part of $P(D)$ is

$$P_2(D) = \sum_{|\alpha|=2} a_\alpha D^\alpha.$$

We say that $P(D)$ is *elliptic* if $P_2(D)$ has real coefficients, and

$$P_2(k) > 0$$

whenever $k \in \mathbf{Z}^n \smallsetminus \{0\}$. The following proof also indicates how Fourier series are used in the solution of partial differential equations.

THEOREM 2.8. *Let $P(D)$ be an elliptic second order differential operator with constant coefficients, and assume that $u \in L^2(T^n)$ solves the equation*

$$P(D)u = f \quad \text{in } T^n,$$

for some $f \in L^2(T^n)$. Then $u \in H^2(T^n)$.

PROOF. Taking the Fourier coefficients on both sides of $P(D)u = f$ gives

$$P(k)\hat{u}(k) = \hat{f}(k), \quad k \in \mathbf{Z}^n. \tag{2.2}$$

We have

$$P_2(k) = |k|^2 P_2(k/|k|) \geq c|k|^2$$

for some $c > 0$, by ellipticity. Then for $k \neq 0$,

$$\begin{aligned}|P(k)| &= |P_2(k) + \sum_{j=1}^{n} a_j k_j + a_0| \\ &\geq |P_2(k)| - (\sum_{j=1}^{n} |a_j|)|k| - |a_0| \\ &\geq c|k|^2 - C|k|.\end{aligned}$$

If $C' > 0$ is sufficiently large, it follows that

$$|P(k)| \geq \frac{1}{2} c|k|^2, \qquad \text{for } |k| \geq C'.$$

From (2.2) we obtain

$$|\hat{u}(k)| = \left|\frac{\hat{f}(k)}{P(k)}\right| \leq \frac{2}{c|k|^2} |\hat{f}(k)|, \qquad |k| \geq C'.$$

Since $\hat{f}(k) \in \ell^2(\mathbf{Z}^n)$ this shows that $\langle k \rangle^2 \hat{u}(k) \in \ell^2(\mathbf{Z}^n)$, which implies $u \in H^2(T^n)$ as required. $\square$

CHAPTER 3

Uniqueness

In this chapter, we will discuss the proof of the following uniqueness result of Sylvester and Uhlmann.

THEOREM 3.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

In fact, this theorem will be reduced to a uniqueness result for the Schrödinger equation (also due to Sylvester and Uhlmann).

THEOREM 3.2. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ in Ω are well posed. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$ in Ω.*

The reduction of Theorem 3.1 to Theorem 3.2 is presented in Section 3.1 along with the relevant definitions. The proof of the uniqueness results relies on complex geometrical optics (CGO) solutions, which are constructed in Section 3.2 for the Schrödinger equation by Fourier analysis and perturbation arguments. Section 3.3 includes an integral identity relating boundary measurements to interior information about the coefficients, and this identity is used to prove Theorem 3.2 by taking an asymptotic limit with suitable CGO solutions.

3.1. Reduction to Schrödinger equation

The first step in the proof is the reduction of the conductivity equation to a Schrödinger equation,

$$(-\Delta + q)u = 0 \quad \text{in } \Omega,$$

where q is a function in $L^\infty(\Omega)$. This equation turns out to be easier to handle, since the principal part is the Laplacian $-\Delta$.

LEMMA 3.3. *If $\gamma \in C^2(\overline{\Omega})$ and $u \in H^1(\Omega)$ then*

$$-\nabla \cdot \gamma \nabla(\gamma^{-1/2} u) = \gamma^{1/2}(-\Delta + q)u \tag{3.1}$$

in the weak sense, where

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}.$$

PROOF. This is a direct computation: if $u \in C^2(\overline{\Omega})$ then

$$\begin{aligned} -\partial_j(\gamma \partial_j(\frac{1}{\sqrt{\gamma}} u)) &= -\partial_j(\sqrt{\gamma}\partial_j u) + \partial_j(\partial_j(\sqrt{\gamma})u) \\ &= -\sqrt{\gamma}\partial_j^2 u - \partial_j(\sqrt{\gamma})\partial_j u + \partial_j(\sqrt{\gamma})\partial_j u + \partial_j^2(\sqrt{\gamma})u. \end{aligned}$$

The claim follows by taking the sum over j. The case where $u \in H^1(\Omega)$ can be proved by approximation[1]. $\square$

If $q \in L^\infty(\Omega)$, we consider the Dirichlet problem for the Schrödinger equation,

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $f \in H^{1/2}(\partial\Omega)$. We say that $u \in H^1(\Omega)$ is a weak solution of (3.2) if $u|_{\partial\Omega} = f$, and if for all $\varphi \in H_0^1(\Omega)$ one has

$$\int_\Omega (\nabla u \cdot \nabla\varphi + qu\varphi)\,dx = 0.$$

---

[1]Let $(u_k) \subseteq C^2(\overline{\Omega})$ be a sequence with $u_k \to u$ in $H^1(\Omega)$. We have

$$-\nabla \cdot \gamma\nabla(\gamma^{-1/2}u_k) = \gamma^{1/2}(-\Delta + q)u_k.$$

If $a \in C^1(\overline{\Omega})$ then $u \mapsto au$ is a continuous map on $H^1(\Omega)$, since

$$\|au\|_{H^1(\Omega)} = \|au\|_{L^2(\Omega)} + \|(\nabla a)u + a\nabla u\|_{L^2(\Omega)} \leq 2(\|a\|_{L^\infty(\Omega)} + \|\nabla a\|_{L^\infty(\Omega)})\|u\|_{H^1(\Omega)}.$$

Using that γ is in $C^2(\overline{\Omega})$ and that $\nabla : H^m(\Omega) \to H^{m-1}(\Omega)$ is a continuous map, we have

$$\begin{array}{rcl} & u_k \to u & \text{in } H^1(\Omega) \\ \Longrightarrow & \gamma^{-1/2}u_k \to \gamma^{-1/2}u & \text{in } H^1(\Omega) \\ \Longrightarrow & \nabla(\gamma^{-1/2}u_k) \to \nabla(\gamma^{-1/2}u) & \text{in } L^2(\Omega) \\ \Longrightarrow & \gamma\nabla(\gamma^{-1/2}u_k) \to \gamma\nabla(\gamma^{-1/2}u) & \text{in } L^2(\Omega) \\ \Longrightarrow & -\nabla\cdot\gamma\nabla(\gamma^{-1/2}u_k) \to -\nabla\cdot\gamma\nabla(\gamma^{-1/2}u) & \text{in } H^{-1}(\Omega). \end{array}$$

Similarly, $\gamma^{1/2}(-\Delta + q)u_k \to \gamma^{1/2}(-\Delta + q)u$ in $H^{-1}(\Omega)$, which shows that (3.1) holds in the sense of $H^{-1}(\Omega)$.

The problem (3.2) is said to be *well-posed* if the following three conditions hold:

1. (existence) there is a weak solution u in $H^1(\Omega)$ for any boundary value f in $H^{1/2}(\partial\Omega)$,
2. (uniqueness) the solution u is unique,
3. (stability) the solution operator $f \mapsto u$ is continuous from $H^{1/2}(\partial\Omega) \to H^1(\Omega)$, that is,
$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}.$$

If the problem is well-posed, we can define a DN map formally by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),\ f \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

This is analogous to the conductivity equation, and also here the precise definition of the DN map is given by the weak formulation

$$(\Lambda_q f, g)_{\partial\Omega} = \int_\Omega (\nabla u \cdot \nabla v + quv)\, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where u solves (3.2) and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. The next proof shows that this is a valid definition and Λ_q is a bounded linear map.

LEMMA 3.4. *If $q \in L^\infty(\Omega)$ is such that the problem (3.2) is well-posed, then Λ_q is a bounded linear map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, and satisfies*

$$(\Lambda_q f, g)_{\partial\Omega} = (f, \Lambda_q g)_{\partial\Omega}, \qquad f, g \in H^{1/2}(\partial\Omega).$$

PROOF. Fix $f \in H^{1/2}(\partial\Omega)$, and define a map $T : H^{1/2}(\partial\Omega) \to \mathbf{C}$ by

$$T(g) = \int_\Omega (\nabla u \cdot \nabla v + quv)\, dx,$$

where u solves (3.2) and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Since u is a solution, we have

$$\int_\Omega (\nabla u \cdot \nabla \varphi + qu\varphi)\, dx$$

for any $\varphi \in H^1_0(\Omega)$. Therefore we may replace v by $v + \varphi$ in the definition of $T(g)$. If $v, \tilde{v} \in H^1(\Omega)$ and $v|_{\partial\Omega} = \tilde{v}_{\partial\Omega} = g$, then $v - \tilde{v} \in H^1_0(\Omega)$, so indeed the definition does not depend on the particular choice of v (as long as $v \in H^1(\Omega)$ and $v|_{\partial\Omega} = g$).

Now, for $g \in H^{1/2}(\partial\Omega)$, use the one-sided inverse to the trace operator to obtain $v_g \in H^1(\Omega)$ with $\|v_g\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}$. Then by Cauchy-Schwarz

$$\begin{aligned} |T(g)| &\leq \int_\Omega |\nabla u \cdot \nabla v_g + quv_g| \, dx \leq C'\|u\|_{H^1(\Omega)}\|v_g\|_{H^1(\Omega)} \\ &\leq C''\|f\|_{H^{1/2}(\partial\Omega)}\|g\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

Thus $T : H^{1/2}(\partial\Omega) \to \mathbf{C}$ is a continuous map, and there is an element $\Lambda_q f \in H^{-1/2}(\partial\Omega)$ satisfying $(\Lambda_q f, g) = T(g)$. Consequently, the map $\Lambda_q : f \mapsto \Lambda_q f$ is a bounded linear map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

To show the last identity, let $f, g \in H^{1/2}(\Omega)$, and let u, v be solutions of (3.2) with boundary values $u|_{\partial\Omega} = f$, $v|_{\partial\Omega} = g$. Then

$$(\Lambda_q f, g)_{\partial\Omega} = \int_\Omega (\nabla u \cdot \nabla v + quv) \, dx = \int_\Omega (\nabla v \cdot \nabla u + qvu) \, dx = (\Lambda_q g, f)$$

by the definition of Λ_q. □

The problem (3.2) is not always well-posed, consider for instance the case $q = -\lambda$ where $\lambda > 0$ is an eigenvalue of the Laplacian in Ω (then there exists a nonzero $u \in H^1(\Omega)$ with $-\Delta u = \lambda u$ in Ω and $u|_{\partial\Omega} = 0$). However, for potentials q coming from conductivities, the Dirichlet problem is always well-posed, and there is a relation between the DN maps Λ_γ and Λ_q.

Lemma 3.5. *If $\gamma \in C^2(\overline{\Omega})$ and $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$, then the Dirichlet problem* (3.2) *is well-posed and*

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2} f) + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu} f\Big|_{\partial\Omega}, \quad f \in H^{1/2}(\Omega).$$

Proof. Fix $f \in H^{1/2}(\partial\Omega)$. We need to show that there is u in $H^1(\Omega)$ which solves (3.2). Motivated by Lemma 3.3, we take v be the solution of $\nabla \cdot \gamma\nabla v = 0$ in Ω with $v|_{\partial\Omega} = \gamma^{-1/2} f$. Then $u = \gamma^{1/2} v$ is a function in $H^1(\Omega)$ which solves $(-\Delta + q)u = 0$ with the right boundary values. The same argument with $f = 0$ shows that the solution is unique. Further, since $\gamma \in C^2(\overline{\Omega})$, we have the estimate

$$\|u\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega)} \leq C'\|\gamma^{-1/2} f\|_{H^{1/2}(\partial\Omega)} \leq C''\|f\|_{H^{1/2}(\partial\Omega)}.$$

If u solves (3.2), then $v = \gamma^{-1/2} u$ solves the conductivity equation and

$$\Lambda_\gamma(\gamma^{-1/2} f) = \gamma\frac{\partial v}{\partial\nu} = \gamma^{1/2}\frac{\partial u}{\partial\nu} - \frac{1}{2}\gamma^{-1/2}\frac{\partial\gamma}{\partial\nu} f.$$

Since $\Lambda_q f = \partial u/\partial \nu$, we obtain the relation between Λ_γ and Λ_q. $\square$

We may now show that uniqueness in the inverse conductivity problem can be deduced from the corresponding result for the Schrödinger equation.

Proof that Theorem 3.2 implies Theorem 3.1. Let γ_1, γ_2 be positive functions in $C^2(\overline{\Omega})$, and assume that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$, then $q_j \in L^\infty(\Omega)$ and by Theorem 3.5 the Dirichlet problems for $-\Delta + q_j$ are well posed. Also, the boundary reconstruction result ([**5**], Theorem 6.6.1) implies that $\gamma_1 = \gamma_2$ and $\partial\gamma_1/\partial\nu = \partial\gamma_2/\partial\nu$ on $\partial\Omega$. Thus, for any f in $H^{1/2}(\partial\Omega)$ one has

$$\begin{aligned}\Lambda_{q_1} f &= \gamma_1^{-1/2}\Lambda_{\gamma_1}(\gamma_1^{-1/2} f) + \frac{1}{2}\gamma_1^{-1}\frac{\partial\gamma_1}{\partial\nu} f\Big|_{\partial\Omega} \\ &= \gamma_2^{-1/2}\Lambda_{\gamma_2}(\gamma_2^{-1/2} f) + \frac{1}{2}\gamma_2^{-1}\frac{\partial\gamma_2}{\partial\nu} f\Big|_{\partial\Omega} \\ &= \Lambda_{q_2} f.\end{aligned}$$

Theorem 3.2 gives that $q_1 = q_2$ in Ω. Therefore,

$$\frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \quad \text{in } \Omega. \tag{3.3}$$

We would like to conclude from (3.3) that $\gamma_1 = \gamma_2$. Let $q = \gamma_1^{-1/2}\Delta\gamma_1 = \gamma_2^{-1/2}\Delta\gamma_2$, and consider the equation

$$(-\Delta + q)u = 0 \qquad \text{in } \Omega.$$

Both $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ solve this equation, and $\sqrt{\gamma_1}|_{\partial\Omega} = \sqrt{\gamma_2}|_{\partial\Omega}$ by boundary determination. Since the Schrödinger equation with potential q coming from a conductivity is well-posed, we obtain $\gamma_1 = \gamma_2$ by uniqueness of solutions. $\square$

Remark 3.6. We record for later use another argument showing that (3.3) implies $\gamma_1 = \gamma_2$. The equation (3.3) looks like a nonlinear PDE involving γ_1 and γ_2. To simplify the equation, we note that

$$\Delta(\log\sqrt{\gamma}) = \sum_{j=1}^n \partial_j\big(\frac{1}{\sqrt{\gamma}}\partial_j\sqrt{\gamma}\big) = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} - |\nabla(\log\sqrt{\gamma})|^2.$$

The equation becomes

$$\Delta(\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}) + |\nabla(\log\sqrt{\gamma_1})|^2 - |\nabla(\log\sqrt{\gamma_2})|^2 = 0.$$

Rewriting slightly, we get

$$\Delta(\log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}) + \nabla a \cdot \nabla(\log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}) = 0,$$

where $a = \log \sqrt{\gamma_1 \gamma_2}$. This is a linear equation for the C^2 function $v = \log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}$. Further, noting the identity

$$\nabla \cdot (e^a \nabla v) = e^a(\Delta v + \nabla a \cdot \nabla v),$$

and using the fact that $\gamma_1 = \gamma_2$ on $\partial\Omega$, we see that v solves the Dirichlet problem

$$\begin{cases} \nabla \cdot ((\gamma_1\gamma_2)^{1/2} \nabla v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem is well-posed, and we get $v \equiv 0$ and $\gamma_1 \equiv \gamma_2$.

3.2. Complex geometrical optics

In this section, we will construct *complex geometrical optics* (CGO) solutions to the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } \Omega.$$

The potential q is assumed to be in $L^\infty(\Omega)$.

MOTIVATION. First let $q = 0$. We try as a solution to the equation $-\Delta u = 0$ the complex exponential at frequency $\zeta \in \mathbf{C}^n$,

$$u(x) = e^{i\zeta\cdot x}.$$

This satisfies

$$Du(x) = \zeta e^{i\zeta\cdot x}, \quad D^2u(x) = (\zeta \cdot \zeta)e^{i\zeta\cdot x}.$$

Thus, if $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = 0$, then $u(x) = e^{i\zeta\cdot x}$ solves $-\Delta u = 0$. Writing $\zeta = \mathrm{Re}\,\zeta + i\,\mathrm{Im}\,\zeta$, we see that

$$\zeta \cdot \zeta = 0 \iff |\mathrm{Re}\,\zeta| = |\mathrm{Im}\,\zeta|, \; \mathrm{Re}\,\zeta \cdot \mathrm{Im}\,\zeta = 0.$$

Now suppose q is nonzero. The function $u = e^{i\zeta\cdot x}$ is not an exact solution of $(-\Delta + q)u = 0$ anymore, but we can find solutions which resemble complex exponentials. These are the CGO solutions, which have the form

$$u(x) = e^{i\zeta\cdot x}(1 + r(x)). \tag{3.4}$$

Here r is a *correction term* which is needed to convert the approximate solution $e^{i\zeta\cdot x}$ to an exact solution.

In fact, we are interested in solutions in the *asymptotic limit* as $|\zeta| \to \infty$. This follows the principle that it is usually not possible to obtain explicit formulas for solutions to general equations, but in suitable asymptotic limits explicit expressions for solutions may exist.

We note that (3.4) is a solution of $(-\Delta + q)u = 0$ iff

$$e^{-i\zeta\cdot x}(-\Delta + q)e^{i\zeta\cdot x}(1 + r) = 0. \tag{3.5}$$

It will be convenient to conjugate the exponentials $e^{i\zeta\cdot x}$ into the Laplacian. By this we mean that

$$e^{-i\zeta\cdot x}D_j(e^{i\zeta\cdot x}v) = (D_j + \zeta_j)v,$$
$$e^{-i\zeta\cdot x}D^2(e^{i\zeta\cdot x}v) = (D + \zeta)^2 v = (D^2 + 2\zeta \cdot D)v.$$

We can rewrite (3.5) as

$$(D^2 + 2\zeta \cdot D + q)(1 + r) = 0.$$

This implies the following equation for r:

$$(D^2 + 2\zeta \cdot D + q)r = -q. \tag{3.6}$$

The solvability of (3.6) is the most important step in the construction of CGO solutions. We proceed in several steps.

3.2.1. Basic estimate. We first consider the free case in which there is no potential on the left hand side of (3.6).

THEOREM 3.7. *There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, and for any $f \in L^2(\Omega)$, the equation*

$$(D^2 + 2\zeta \cdot D)r = f \quad \text{in } \Omega \tag{3.7}$$

has a solution $r \in H^1(\Omega)$ satisfying

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|f\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq C_0\|f\|_{L^2(\Omega)}.$$

The idea of the proof is that (3.7) is a linear equation with constant coefficients, so one can try to solve it by the Fourier transform. Since $(D_j u)\hat{}(\xi) = \xi_j \hat{u}(\xi)$, the Fourier transformed equation is

$$(\xi^2 + 2\zeta \cdot \xi)\hat{r}(\xi) = \hat{f}(\xi).$$

We would like to divide by $\xi^2 + 2\zeta \cdot \xi$ and use the inverse Fourier transform to get a solution r. However, the symbol $\xi^2 + 2\zeta \cdot \xi$ vanishes for some $\xi \in \mathbf{R}^n$, and the division cannot be done directly.

It turns out that we can divide by the symbol if we use Fourier series in a large cube instead of the Fourier transform, and moreover take the Fourier coefficients in a shifted lattice instead of the usual integer coordinate lattice.

Proof of Theorem 3.7. Write $\zeta = s(\omega_1 + i\omega_2)$ where $s = |\zeta|/\sqrt{2}$ and ω_1 and ω_2 are orthogonal unit vectors in $\mathbf{R}^n$. By rotating coordinates in a suitable way, we can assume that $\omega_1 = e_1$ and $\omega_2 = e_2$ (the first and second coordinate vectors). Thus we need to solve the equation

$$(D^2 + 2s(D_1 + iD_2))r = f.$$

We assume for simplicity that Ω is contained in the cube $Q = [-\pi, \pi]^n$. Extend f by zero outside Ω into Q, which gives a function in $L^2(Q)$ also denoted by f. We need to solve

$$(D^2 + 2s(D_1 + iD_2))r = f \quad \text{in } Q. \tag{3.8}$$

Let $w_k(x) = e^{i(k+\frac{1}{2}e_2)\cdot x}$ for $k \in \mathbf{Z}^n$. That is, we consider Fourier series in the lattice $\mathbf{Z}^n + \frac{1}{2}e_2$. Writing

$$(u, v) = (2\pi)^{-n} \int_Q u\bar{v}\, dx, \quad u, v \in L^2(Q),$$

we see that $(w_k, w_l) = 0$ if $k \neq l$ and $(w_k, w_k) = 1$, so $\{w_k\}$ is an orthonormal set in $L^2(Q)$. It is also complete: if $v \in L^2(Q)$ and $(v, w_k) = 0$ for all $k \in \mathbf{Z}^n$ then $(ve^{-\frac{1}{2}ix_2}, e^{ik\cdot x}) = 0$ for all $k \in \mathbf{Z}^n$, which implies $v = 0$.

Hilbert space theory gives that f can be written as the series $f = \sum_{k\in\mathbf{Z}^n} f_k w_k$, where $f_k = (f, w_k)$ and $\|f\|^2_{L^2(Q)} = \sum_{k\in\mathbf{Z}^n} |f_k|^2$. Seeking also r in the form $r = \sum_{k\in\mathbf{Z}^n} r_k w_k$, and using that

$$Dw_k = (k + \frac{1}{2}e_2)w_k,$$

the equation (3.8) results in

$$p_k r_k = f_k, \quad k \in \mathbf{Z}^n,$$

where

$$p_k := (k + \frac{1}{2}e_2)^2 + 2s(k_1 + i(k_2 + \frac{1}{2})).$$

Note that $\operatorname{Im} p_k = 2s(k_2 + \frac{1}{2})$ is never zero, which was the reason for considering the shifted lattice. We define

$$r_k := \frac{1}{p_k} f_k$$

and

$$r := \sum_{k \in \mathbf{Z}^n} r_k w_k.$$

The last series converges in $L^2(Q)$ to a function $r \in L^2(Q)$ since

$$|r_k| \leq \frac{1}{|p_k|}|f_k| \leq \frac{1}{|2s(k_2 + \frac{1}{2})|}|f_k| \leq \frac{1}{s}|f_k|,$$

and then

$$\|r\|_{L^2(Q)} = \Big(\sum_k |r_k|^2\Big)^{1/2} \leq \frac{1}{s}\Big(\sum_k |f_k|^2\Big)^{1/2} = \frac{1}{s}\|f\|_{L^2(Q)}.$$

This shows the desired estimate in $L^2(Q)$.

It remains to show that $Dr \in L^2(Q)$ with correct bounds. We have

$$Dr = \sum_{k \in \mathbf{Z}^n} (k + \frac{1}{2}e_2) r_k w_k.$$

The derivative is justified since this is a convergent series in $L^2(Q)$: we claim

$$|(k + \frac{1}{2}e_2) r_k| \leq 4|f_k|, \quad k \in \mathbf{Z}^n, \tag{3.9}$$

which implies that $\|Dr\|_{L^2(Q)} \leq 4\|f\|_{L^2(Q)}$. To show (3.9) we consider two cases: if $|k + \frac{1}{2}e_2| \leq 4s$ we have

$$|(k + \frac{1}{2}e_2) r_k| \leq \frac{4s}{2s|k_2 + 1/2|}|f_k| \leq 4|f_k|,$$

and if $|k + \frac{1}{2}e_2| \geq 4s$ then

$$||k + \frac{1}{2}e_2|^2 + 2sk_1| \geq |k + \frac{1}{2}e_2|^2 - 2s|k + \frac{1}{2}e_2| \geq \frac{1}{2}|k + \frac{1}{2}e_2|^2$$

which implies

$$|(k + \frac{1}{2}e_2) r_k| \leq \frac{|k + \frac{1}{2}e_2|}{\frac{1}{2}|k + \frac{1}{2}e_2|^2}|f_k| \leq \frac{1}{2s}|f_k|.$$

The statement is proved. $\square$

3.2.2. Basic estimate with potential. Now we consider the solution of (3.6) in the presence of a nonzero potential. It will be convenient to give a name to the solution operator in the free case.

NOTATION. Let $\zeta \in \mathbf{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta|$ sufficiently large. We denote by G_ζ the solution operator

$$G_\zeta : L^2(\Omega) \to H^1(\Omega),\ f \mapsto r,$$

where r is the solution to $(D^2 + 2\zeta \cdot D)r = f$ provided by Theorem 3.7.

THEOREM 3.8. *Let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C_0\|q\|_{L^\infty(\Omega)}, 1)$, and for any $f \in L^2(\Omega)$, the equation*

$$(D^2 + 2\zeta \cdot D + q)r = f \quad \text{in } \Omega \tag{3.10}$$

has a solution $r \in H^1(\Omega)$ satisfying

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|f\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq C_0\|f\|_{L^2(\Omega)}.$$

PROOF. If one has $q = 0$, a solution would be given by $r = G_\zeta f$. Here q may be nonzero, so we try a solution of the form

$$r := G_\zeta \tilde{f}, \tag{3.11}$$

where $\tilde{f} \in L^2(\Omega)$ is a function to be determined. Inserting (3.11) in the equation (3.10), and using that $(D^2 + 2\zeta \cdot D)G_\zeta = I$, we see that $\tilde{f}$ should satisfy

$$(I + qG_\zeta)\tilde{f} = f \quad \text{in } \Omega. \tag{3.12}$$

We have the norm estimate

$$\|qG_\zeta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{C_0\|q\|_{L^\infty(\Omega)}}{|\zeta|}.$$

If $|\zeta| \geq \max(2C_0\|q\|_{L^\infty(\Omega)}, 1)$ then

$$\|qG_\zeta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{1}{2}.$$

It follows that $I + qG_\zeta$ is an invertible operator on $L^2(\Omega)$, and the equation (3.12) has a solution

$$\tilde{f} := (I + qG_\zeta)^{-1}f.$$

The definition (3.11) for r implies

$$(D^2 + 2\zeta \cdot D + q)r = \tilde{f} + qG_\zeta \tilde{f} = (I + qG_\zeta)\tilde{f} = f,$$

and r indeed solves the equation (3.10). Since $\|(I+qG_\zeta)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \leq 2$, we have $\|\tilde{f}\|_{L^2(\Omega)} \leq 2\|f\|_{L^2(\Omega)}$. The norm estimates in Theorem 3.7 imply the desired estimates for r, if we replace C_0 by $2C_0$. □

3.2.3. Construction of CGO solutions. It is now easy to give the main result on the existence of CGO solutions. Note that the constant function $a \equiv 1$ always satisfies $\zeta \cdot \nabla a = 0$, so as a special case one obtains the solutions $u = e^{i\zeta\cdot x}(1 + r)$ in (3.4).

THEOREM 3.9. *Let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C_0\|q\|_{L^\infty(\Omega)}, 1)$, and for any function $a \in H^2(\Omega)$ satisfying*

$$\zeta \cdot \nabla a = 0 \quad \textit{in } \Omega,$$

the equation $(-\Delta + q)u = 0$ in Ω has a solution

$$u(x) = e^{i\zeta\cdot x}(a + r), \tag{3.13}$$

where $r \in H^1(\Omega)$ satisfies

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|(-\Delta + q)a\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq C_0\|(-\Delta + q)a\|_{L^2(\Omega)}.$$

PROOF. The function (3.13) is a solution of $(-\Delta + q)u = 0$ iff

$$e^{-i\zeta\cdot x}(-\Delta + q)e^{i\zeta\cdot x}(a + r) = 0. \tag{3.14}$$

As in the beginning of this section, we conjugate the exponentials into the derivatives and rewrite (3.5) as

$$(D^2 + 2\zeta \cdot D + q)(a + r) = 0.$$

Since $\zeta \cdot Da = 0$, this implies the following equation for r:

$$(D^2 + 2\zeta \cdot D + q)r = -(D^2 + q)a.$$

Theorem 3.8 guarantees the existence of a solution r satisfying the norm estimates above. Then (3.13) is the required solution to $(-\Delta+q)u = 0$ in Ω. □

3.3. Uniqueness proof

In this section we prove the Sylvester-Uhlmann uniqueness results. As shown in Section 3.1, uniqueness in the inverse conductivity problem (Theorem 3.1) follows from the uniqueness result for the Schrödinger equation, which we now recall.

THEOREM 3.2. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ in Ω are well posed. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$ in Ω.*

The starting point is an integral identity which relates the difference of the boundary measurements $\Lambda_{q_1} - \Lambda_{q_2}$ to the difference of the potentials.

LEMMA 3.8. *Let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ in Ω are well-posed. Then for any $f_1, f_2 \in H^{1/2}(\partial\Omega)$ one has*

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx,$$

where $u_j \in H^1(\Omega)$ is the solution of $(-\Delta + q_j) u_j = 0$ in Ω with boundary values $u_j|_{\partial\Omega} = f_j$, $j = 1, 2$.

PROOF. By the weak definition of the DN map, we have

$$\langle \Lambda_{q_1} f_1, f_2 \rangle = \int_\Omega (\nabla u_1 \cdot \nabla u_2 + q_1 u_1 u_2) \, dx$$

since u_1 is a solution with boundary values f_1, and u_2 has boundary values f_2. Also, since the DN map is self-adjoint,

$$\langle \Lambda_{q_2} f_1, f_2 \rangle = \langle f_1, \Lambda_{q_2} f_2 \rangle = \langle \Lambda_{q_2} f_2, f_1 \rangle = \int_\Omega (\nabla u_2 \cdot \nabla u_1 + q_2 u_2 u_1) \, dx.$$

The claim follows. □

PROOF OF THEOREM 3.2. Since $\Lambda_{q_1} = \Lambda_{q_2}$, we know from Lemma 3.8 that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0 \tag{3.15}$$

for *any* H^1 solutions u_j to the equations $(-\Delta + q_j) u_j = 0$, $j = 1, 2$. Thus, to prove that $q_1 = q_2$, it is enough to establish that products $u_1 u_2$ of such solutions are dense in $L^1(\Omega)$.

Fix $\xi \in \mathbf{R}^n$. We would like to choose the solutions in such a way that $u_1 u_2$ is close to $e^{ix\cdot\xi}$, since the functions $e^{ix\cdot\xi}$ form a dense set. We begin by taking unit vectors ω_1 and ω_2 in $\mathbf{R}^n$ such that $\{\omega_1, \omega_2, \xi\}$ is an orthogonal set (here we need that $n \geq 3$). Let

$$\zeta = s(\omega_1 + i\omega_2),$$

so that $\zeta \cdot \zeta = 0$. By Theorem 3.9, if s is sufficiently large there exist H^1 solutions u_1 and u_2 which satisfy $(-\Delta + q_j)u_j = 0$, and which are of the form

$$u_1 = e^{i\zeta\cdot x}(e^{ix\cdot\xi} + r_1),$$
$$u_2 = e^{-i\zeta\cdot x}(1 + r_2),$$

where $\|r_j\|_{L^2(\Omega)} \leq C/s$ for $j = 1, 2$. For the first solution we chose $a = e^{ix\cdot\xi}$ which satisfies $\zeta \cdot \nabla a = (\zeta \cdot \xi)e^{ix\cdot\xi} = 0$ by orthogonality, and for the second solution we chose a to be constant.

Inserting these solutions in (3.15), we obtain

$$\int_\Omega (q_1 - q_2)(e^{ix\cdot\xi} + r_1)(1 + r_2)\,dx = 0. \tag{3.16}$$

In this identity, only r_1 and r_2 depend on s. Since the L^2 norms of r_1 and r_2 are bounded by C/s, it is possible to take the limit as $s \to \infty$ in (3.15), and then the terms involving r_1 and r_2 will vanish. Taking this limit in (3.16), we get

$$\int_\Omega (q_1 - q_2)e^{ix\cdot\xi}\,dx = 0.$$

This holds for every $\xi \in \mathbf{R}^n$. If $\tilde{q}$ is the function in $L^1(\mathbf{R}^n)$ which is equal to $q_1 - q_2$ in Ω and vanishes outside Ω, the last identity implies that the Fourier transform of $\tilde{q}$ vanishes for every frequency $\xi \in \mathbf{R}^n$. Consequently $\tilde{q} = 0$, and $q_1 = q_2$ in Ω. $\square$

CHAPTER 4

Stability

In the preceding chapter, we proved that if γ_1, γ_2 are two conductivities in $C^2(\overline{\Omega})$ such that Λ_{γ_1} is equal to Λ_{γ_2}, then $\gamma_1 \equiv \gamma_2$. Here we address the stability question: if γ_1, γ_2 are two conductivities such that Λ_{γ_1} is close to Λ_{γ_2}, does this imply that γ_1 is close to γ_2?

More precisely, we are looking for an estimate of the form

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*), \tag{4.1}$$

where $\|\cdot\|_* = \|\cdot\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ is the natural operator norm for the DN maps, and $\omega : [0,\infty) \to [0,\infty)$ is a *modulus of continuity*, that is, a continuous nondecreasing function satisfying $\omega(t) \to 0$ as $t \to 0^+$.

We begin with an example due to Alessandrini.

Example. Let $\mathbb{D}$ be the unit disc in $\mathbf{R}^2$, and let $\gamma_1, \gamma_2 \in L^\infty(\mathbb{D})$ be two conductivities such that $\gamma_1 \equiv 1$ in $\mathbb{D}$, and

$$\gamma_2(x) = \begin{cases} 1 + A, & |x| < r_0, \\ 1, & r_0 < |x| < 1, \end{cases}$$

where A is a positive constant and $r_0 \in (0,1)$. If $f \in H^{1/2}(\partial\mathbb{D})$, then f may be written as Fourier series

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}.$$

It can be shown (exercise) that

$$\Lambda_{\gamma_1} f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} |k| \hat{f}(k) e^{ik\theta},$$

$$\Lambda_{\gamma_2} f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} |k| \frac{2 + A(1 + r_0^{2|k|})}{2 + A(1 - r_0^{2|k|})} \hat{f}(k) e^{ik\theta}.$$

Then

$$\|(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})f\|^2_{H^{-1/2}} = \sum_{k=-\infty}^{\infty}(1+k^2)^{-1/2}k^2\left|1-\frac{2+A(1+r_0^{2|k|})}{2+A(1-r_0^{2|k|})}^2\right|^2|\hat{f}(k)|^2$$
$$= \sum_{k=-\infty}^{\infty}\frac{k^2}{1+k^2}\left(\frac{2Ar_0^{2|k|}}{2+A(1-r_0^{2|k|})}\right)^2(1+k^2)^{1/2}|\hat{f}(k)|^2.$$

The sum is actually over $k \neq 0$, and then the expression in parentheses is $\leq Ar_0^2$ since it attains its maximum value when $|k| = 1$. It follows that

$$\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_* = \sup_{\|f\|_{H^{1/2}}=1}\|(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})f\|_{H^{-1/2}} \leq Ar_0^2.$$

Now $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_* \to 0$ as $r_0 \to 0$, but

$$\|\gamma_1-\gamma_2\|_{L^\infty(\mathbb{D})} = A.$$

Thus, an estimate of the form (4.1) can not be valid if one only assumes that $\gamma_1, \gamma_2 \in L^\infty$.

It turns out that under certain *a priori* assumptions on γ_1 and γ_2, it is possible to prove a stability estimate with logarithmic modulus of continuity.

THEOREM 4.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_j, $j = 1, 2$, be two positive functions in $H^{s+2}(\Omega)$ with $s > n/2$, satisfying*

$$\tfrac{1}{M} \leq \gamma_j \leq M, \tag{4.2}$$

$$\|\gamma_j\|_{H^{s+2}(\Omega)} \leq M. \tag{4.3}$$

There are constants $C = C(\Omega, n, M, s) > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_*)$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\log t|^{-\sigma}, \quad 0 < t < 1/e.$$

Note that $\gamma_j \in H^{s+2}(\Omega)$ with $s > n/2$ implies that $\gamma_j \in C^2(\overline{\Omega})$ by Sobolev embedding. The logarithmic modulus of continuity is rather weak, in the sense that even small changes in γ can result in large changes in Λ_γ. However, it has been proved that the logarithmic modulus is optimal, and for instance Hölder type stability can not hold for conductivities of general form.

4.1. Schrödinger equation

As in the uniqueness proof, we will use the inverse problem for the Schrödinger equation to study the stability question. In the following, $\Omega \subseteq \mathbf{R}^n$ is a bounded open set with C^∞ boundary, and $n \geq 3$.

THEOREM 4.2. *Let $q_j \in L^\infty(\Omega)$ be two potentials such that the Dirichlet problems for $-\Delta + q_j$ are well-posed. Further, assume that*

$$\|q_j\|_{L^\infty(\Omega)} \leq M.$$

There is a constant $C = C(\Omega, n, M)$ such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq \omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*)$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\log t|^{-\frac{2}{n+2}}, \quad 0 < t < 1/e.$$

PROOF. Let $\xi \in \mathbf{R}^n$. We start from the identity in Lemma 3.8, which states that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = ((\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega}), u_2|_{\partial\Omega})_{\partial\Omega}, \tag{4.4}$$

for any $u_j \in H^1(\Omega)$ which solve $(-\Delta + q_j)u_j = 0$ in Ω. As in the proof of Theorem 3.2, let ω_1 and ω_2 be unit vectors such that $\{\omega_1, \omega_2, \xi\}$ is an orthogonal set. The choice of complex vectors is slightly different (we make this choice to obtain better constants), we take

$$\zeta_1 = \frac{s}{\sqrt{2}}\Big(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\omega_1 + \frac{1}{\sqrt{2}s}\xi + i\omega_2\Big),$$
$$\zeta_2 = -\frac{s}{\sqrt{2}}\Big(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\omega_1 - \frac{1}{\sqrt{2}s}\xi + i\omega_2\Big).$$

These satisfy $\zeta_j \cdot \zeta_j = 0$ and $|\zeta_1| = |\zeta_2| = s$. Theorem 3.9 ensures the existence of solutions u_j to $(-\Delta + q_j)u_j = 0$, provided that $s \geq \max(C_0 M, 1)$, of the form

$$u_1 = e^{i\zeta_1 \cdot x}(1 + r_1),$$
$$u_2 = e^{i\zeta_2 \cdot x}(1 + r_2),$$

with $\|r_j\|_{L^2(\Omega)} \leq \frac{C_0\|q_j\|_{L^\infty}}{s}$, and $C_0 = C_0(\Omega, n)$.

Inserting u_1 and u_2 in (4.4) and using that $e^{i(\zeta_1+\zeta_2)\cdot x} = e^{ix\cdot\xi}$, we obtain

$$\begin{aligned}\Big|\int_\Omega (q_1-q_2)e^{ix\cdot\xi}\,dx\Big| &\leq \|\Lambda_{q_1}-\Lambda_{q_2}\|_*\|u_1\|_{H^{1/2}(\partial\Omega)}\|u_2\|_{H^{1/2}(\partial\Omega)} \\ &\quad + \Big|\int_\Omega (q_1-q_2)e^{ix\cdot\xi}(r_1+r_2+r_1r_2)\,dx\Big| \\ &\leq \|\Lambda_{q_1}-\Lambda_{q_2}\|_*\|u_1\|_{H^1}\|u_2\|_{H^1} + C(\|r_1\|_{L^2}+\|r_2\|_{L^2}+\|r_1\|_{L^2}\|r_2\|_{L^2})\end{aligned}$$

with $C = C(\Omega, n, M)$. If $\Omega \subseteq B(0,R)$ then

$$\begin{aligned}\|u_j\|_{H^1} &\leq \|e^{i\zeta_j\cdot x}(1+r_j)\|_{L^2} + \sum_{k=1}^n \|\partial_k(e^{i\zeta_j\cdot x})(1+r_j) + e^{i\zeta_j\cdot x}\partial_k r_j\|_{L^2} \\ &\leq Cse^{Rs}.\end{aligned}$$

We assume that s is so large that $s \leq e^{Rs}$, and have

$$|(\tilde q_1-\tilde q_2)\hat{}\,(\xi)| \leq C\Big(e^{4Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_* + \frac{1}{s}\Big), \tag{4.5}$$

where $\tilde q_j$ is the extension of q_j to $\mathbf{R}^n$ by zero (thus $\tilde q_j \in L^1(\mathbf{R}^n)$).

So far, we have proved that there are constants C and C', depending on Ω, n, and M, such that (4.5) holds whenever $s \geq C'$. It is possible to obtain a bound for $q_1 - q_2$ in H^{-1} by using (4.5) and the L^∞ bounds for q_j. If $\rho > 0$ is a constant which will be determined later, we have

$$\begin{aligned}\|q_1-q_2\|^2_{H^{-1}(\Omega)} \leq \|q_1-q_2\|^2_{H^{-1}(\mathbf{R}^n)} &= \Big(\int_{|\xi|\leq\rho} + \int_{|\xi|>\rho}\Big)\frac{|(q_1-q_2)\hat{}\,(\xi)|^2}{1+|\xi|^2}\,d\xi \\ &\leq C\rho^n\Big(e^{8Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_*^2 + \frac{1}{s^2}\Big) + (1+\rho^2)^{-1}\|q_1-q_2\|^2_{L^2(\mathbf{R}^n)} \\ &\leq C\rho^n e^{8Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_*^2 + \frac{C\rho^n}{s^2} + \frac{C}{\rho^2}.\end{aligned}$$

To make the last two terms of equal size, we choose

$$\rho = s^{\frac{2}{n+2}}.$$

Then

$$\|q_1-q_2\|^2_{H^{-1}(\Omega)} \leq Ce^{16Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_*^2 + Cs^{-\frac{4}{n+2}}$$

for $s \geq C'(\Omega, n, M)$. We make the final choice

$$s = \frac{1}{16R}|\log\|\Lambda_{q_1}-\Lambda_{q_2}\|_*|$$

where we assume that

$$0 < \|\Lambda_{q_1} - \Lambda_{q_2}\|_* < c'(\Omega, n, M)$$

with c' chosen so that $s \geq C'$. With this assumption, it follows that

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq C(\|\Lambda_{q_1} - \Lambda_{q_2}\|_* + |\log\|\Lambda_{q_1} - \Lambda_{q_2}\|_*|^{-\frac{4}{n+2}}).$$

The claim is an easy consequence. □

4.2. More facts on Sobolev spaces

To reduce the stability result for the conductivity equation to Theorem 4.2, we will need more properties of Sobolev spaces. There are three settings to consider: Sobolev spaces in $\mathbf{R}^n$, Sobolev spaces in bounded C^∞ domains $\Omega \subseteq \mathbf{R}^n$, and Sobolev spaces on $(n-1)$-dimensional boundaries $\partial\Omega$. Further, we want to consider the case where s may not be an integer.

The philosophy is that $H^s(\mathbf{R}^n)$ may be defined via the Fourier transform, $H^s(\Omega)$ is the restriction of $H^s(\mathbf{R}^n)$ to Ω, and $H^s(\partial\Omega)$ can be defined by locally flattening the boundary and reducing matters to $H^s(\mathbf{R}^{n-1})$. We now give some specifics, see [**7**] for more details.

Definition. If $s \geq 0$, let

$$H^s(\mathbf{R}^n) = \{u \in L^2(\mathbf{R}^n)\,;\, \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbf{R}^n)\},$$

with norm

$$\|u\|_{H^s(\mathbf{R}^n)} = \|\langle\xi\rangle^s \hat{u}\|_{L^2(\mathbf{R}^n)}.$$

Here $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbf{R}^n$.

The space $H^s(\mathbf{R}^n)$ is in fact a Hilbert space, with inner product $(u, v)_{H^s} = \int \langle\xi\rangle^{2s} \hat{u}(\xi)\overline{\hat{v}(\xi)}$. Recall from [**5**] that if $k \geq 0$ is an integer, there is the equivalent norm

$$\|u\|_{W^{k,2}(\mathbf{R}^n)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^2(\mathbf{R}^n)} \sim \|u\|_{H^k(\mathbf{R}^n)}, \tag{4.6}$$

where $A \sim B$ means that $c^{-1}B \leq A \leq cB$ for some constant $c > 0$ (independent of u).

The following properties of Sobolev spaces are the main point in this section. Recall that $C^k(\mathbf{R}^n)$ is the space of k times continuously differentiable functions on $\mathbf{R}^n$, such that all partial derivatives up to order k are bounded. The norm is $\|u\|_{C^k(\mathbf{R}^n)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\mathbf{R}^n)}$.

THEOREM 4.3.

- *(Sobolev embedding theorem) If $u \in H^{s+k}(\mathbf{R}^n)$ where $s > n/2$ and k is a nonnegative integer, then $u \in C^k(\mathbf{R}^n)$ and*
$$\|u\|_{C^k(\mathbf{R}^n)} \leq C\|u\|_{H^{s+k}(\mathbf{R}^n)}.$$
- *(Multiplication by functions) If $u \in H^s(\mathbf{R}^n)$ and $s \geq 0$, and if $f \in C^k(\mathbf{R}^n)$ where k is an integer $\geq s$, then $fu \in H^s(\mathbf{R}^n)$ and*
$$\|fu\|_{H^s(\mathbf{R}^n)} \leq \|f\|_{C^k(\mathbf{R}^n)}\|u\|_{H^s(\mathbf{R}^n)}.$$
- *(Logarithmic convexity of Sobolev norms) If $0 \leq \alpha \leq \beta$ and $0 \leq t \leq 1$, then*
$$\|u\|_{H^\gamma(\mathbf{R}^n)} \leq \|u\|_{H^\alpha(\mathbf{R}^n)}^{1-t}\|u\|_{H^\beta(\mathbf{R}^n)}^{t}, \quad u \in H^\beta(\mathbf{R}^n),$$
where $\gamma = (1-t)\alpha + t\beta$.

PROOF. The first statement was proved in [**5**]. The second fact follows, if s is an integer, by using the equivalent norm (4.6) and the Leibniz rule. If s is not an integer, the most convenient way to prove the result is by interpolation: if $s = l - \varepsilon$ where l is an integer and $0 < \varepsilon < 1$, the claim is true if s is replaced by $l-1$ or l, and the claim follows for s by interpolation between these two cases.

We now prove the third statement. The claim is trivial if $t = 0$ or $t = 1$, so assume that $0 < t < 1$. Then

$$\begin{aligned}
\|u\|_{H^\gamma(\mathbf{R}^n)}^2 &= \int \langle\xi\rangle^{2((1-t)\alpha+t\beta)}|\hat{u}(\xi)|^2\,d\xi \\
&= \int (\langle\xi\rangle^{2\alpha}|\hat{u}(\xi)|^2)^{1-t}(\langle\xi\rangle^{2\beta}|\hat{u}(\xi)|^2)^t\,d\xi \\
\leq \Big(\int \langle\xi\rangle^{2\alpha}|\hat{u}(\xi)|^2\,d\xi\Big)^{1-t}&\Big(\int \langle\xi\rangle^{2\beta}|\hat{u}(\xi)|^2\,d\xi\Big)^t = \|u\|_{H^\alpha(\mathbf{R}^n)}^{2(1-t)}\|u\|_{H^\beta(\mathbf{R}^n)}^{2t}
\end{aligned}$$

by using Hölder's inequality with $p = \frac{1}{1-t}$ and $p' = \frac{1}{t}$. $\square$

We would like to use similar results for Sobolev spaces in bounded domains. In the following, let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with C^∞ boundary. In [**5**] one has the Sobolev spaces in Ω, denoted here by $W^{k,2}(\Omega)$, consisting of the functions in $L^2(\Omega)$ whose all weak partial derivatives up to order k are in $L^2(\Omega)$. The norm is

$$\|u\|_{W^{k,2}(\Omega)} := \sum_{|\alpha|\leq k}\|\partial^\alpha u\|_{L^2(\Omega)}.$$

We wish to relate these to Sobolev spaces in $\mathbf{R}^n$. The main tool for doing this is the extension operator.

THEOREM 4.4. *(Extension operator) If k is a nonnegative integer, there is a bounded linear operator $E : W^{k,2}(\Omega) \to H^k(\mathbf{R}^n)$ satisfying $Eu|_\Omega = u$.*

PROOF. See the exercises. $\square$

DEFINITION. If $s \geq 0$, let $H^s(\Omega)$ be the set of those $u \in L^2(\Omega)$ such that $u = v|_\Omega$ for some $v \in H^s(\mathbf{R}^n)$. The norm is the quotient norm

$$\|u\|_{H^s(\Omega)} = \inf_{v \in H^s(\mathbf{R}^n), v|_\Omega = u} \|v\|_{H^s(\mathbf{R}^n)}.$$

The space $H^s(\Omega)$ is a Hilbert space. The definition is justified by the fact that

$$H^k(\Omega) = W^{k,2}(\Omega),$$

with equivalent norms, if $k \geq 0$ is an integer. To see this, note that

$$\|u\|_{H^k(\Omega)} \leq \|Eu\|_{H^k(\mathbf{R}^n)} \leq C\|u\|_{W^{k,2}(\Omega)}.$$

Further, if $u \in W^{k,2}(\Omega)$ and if $v \in H^k(\mathbf{R}^n)$ with $v|_\Omega = u$, then $\|u\|_{W^{k,2}(\Omega)} \leq \|v\|_{W^{k,2}(\mathbf{R}^n)}$. Therefore

$$\|u\|_{W^{k,2}(\Omega)} \leq C\|u\|_{H^k(\Omega)}.$$

On domains, the following results correspond to the ones above.

THEOREM 4.5.

- *(Sobolev embedding theorem) If $u \in H^{s+k}(\Omega)$ where $s > n/2$ and k is a nonnegative integer, then $u \in C^k(\overline{\Omega})$ and*

$$\|u\|_{C^k(\overline{\Omega})} \leq C\|u\|_{H^{s+k}(\Omega)}.$$

- *(Multiplication by functions) If $u \in H^s(\Omega)$ and $s \geq 0$, and if $f \in C^k(\overline{\Omega})$ where k is an integer $\geq s$, then $fu \in H^s(\Omega)$ and*

$$\|fu\|_{H^s(\Omega)} \leq \|f\|_{C^k(\overline{\Omega})}\|u\|_{H^s(\Omega)}.$$

- *(Logarithmic convexity of Sobolev norms) If $0 \leq \alpha \leq \beta$ and $0 \leq t \leq 1$, then*

$$\|u\|_{H^{(1-t)\alpha+t\beta}(\Omega)} \leq \|u\|_{H^\alpha(\Omega)}^{1-t}\|u\|_{H^\beta(\Omega)}^t, \quad u \in H^\beta(\Omega).$$

PROOF. The first item follows from the corresponding result on $\mathbf{R}^n$ by using the extension operator: if $u \in H^{s+k}(\Omega)$, there is $v \in H^{s+k}(\mathbf{R}^n)$ with $v|_\Omega = u$ and $\|v\|_{H^{s+k}(\mathbf{R}^n)} \leq C\|u\|_{H^{s+k}(\Omega)}$. Sobolev embedding in $\mathbf{R}^n$ implies $v \in C^k(\mathbf{R}^n)$ with $\|v\|_{C^k(\mathbf{R}^n)} \leq C\|v\|_{H^{s+k}(\mathbf{R}^n)}$. Then

$$\|u\|_{C^k(\overline{\Omega})} \leq \|v\|_{C^k(\mathbf{R}^n)} \leq C\|v\|_{H^{s+k}(\mathbf{R}^n)} \leq C\|u\|_{H^{s+k}(\Omega)}.$$

The second item is again a direct computation for integer s and in general can be proved by interpolation, and this is also true for the third item. □

Next consider the spaces $H^s(\partial\Omega)$, where $\partial\Omega$ is the boundary of a bounded C^∞ domain Ω. As described in [**5**], one can define $H^s(\partial\Omega)$ via the spaces $H^s(\mathbf{R}^{n-1})$, by using a partition of unity and diffeomorphisms which locally flatten the boundary. The above properties carry over to the spaces $H^s(\partial\Omega)$. Note that in the Sobolev embedding theorem, the condition is $s > \frac{n-1}{2}$ since $\partial\Omega$ is an $(n-1)$-dimensional manifold.

THEOREM 4.6.

- *(Sobolev embedding theorem) If $u \in H^{s+k}(\partial\Omega)$ where $s > \frac{n-1}{2}$ and k is a nonnegative integer, then $u \in C^k(\partial\Omega)$ and*
$$\|u\|_{C^k(\partial\Omega)} \leq C\|u\|_{H^{s+k}(\partial\Omega)}.$$
- *(Multiplication by functions) If $u \in H^s(\partial\Omega)$ and $s \geq 0$, and if $f \in C^k(\partial\Omega)$ where k is an integer $\geq s$, then $fu \in H^s(\partial\Omega)$ and*
$$\|fu\|_{H^s(\partial\Omega)} \leq \|f\|_{C^k(\partial\Omega)}\|u\|_{H^s(\partial\Omega)}.$$
- *(Logarithmic convexity of Sobolev norms) If $0 \leq \alpha \leq \beta$ and $0 \leq t \leq 1$, then*
$$\|u\|_{H^{(1-t)\alpha+t\beta}(\partial\Omega)} \leq \|u\|_{H^\alpha(\partial\Omega)}^{1-t}\|u\|_{H^\beta(\partial\Omega)}^{t}, \quad u \in H^\beta(\partial\Omega).$$

Finally, a remark about negative index Sobolev spaces $H^s(\mathbf{R}^n)$, $H^s(\Omega)$, $H^s(\partial\Omega)$, where $s < 0$. These spaces contain elements which are no longer L^2 functions, and in fact are not functions at all. The most convenient definition uses the space of tempered distributions $\mathscr{S}'(\mathbf{R}^n)$, which contains for instance all L^p functions and polynomially bounded measures, and which is closed under the Fourier transform. Then, if $s \in \mathbf{R}$, one defines

$$H^s(\mathbf{R}^n) = \{u \in \mathscr{S}'(\mathbf{R}^n)\,;\, \langle\xi\rangle^s\hat{u}(\xi) \in L^2(\mathbf{R}^n)\},$$
$$\|u\|_{H^s(\mathbf{R}^n)} = \|\langle\xi\rangle^s\hat{u}(\xi)\|_{L^2(\mathbf{R}^n)}.$$

Further, $H^s(\Omega)$ can be defined via restriction as above, and $H^s(\partial\Omega)$ by reducing to $H^s(\mathbf{R}^{n-1})$. We will use the fact that multiplication by C^k functions is bounded on H^s for $|s| \leq k$, which can be proved by duality arguments.

4.3. Conductivity equation

We proceed to prove the main result on stability for the conductivity equation, which we recall here.

THEOREM 4.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_j, $j = 1, 2$, be two positive functions in $H^{s+2}(\Omega)$ with $s > n/2$, satisfying*

$$\tfrac{1}{M} \leq \gamma_j \leq M, \tag{4.7}$$

$$\|\gamma_j\|_{H^{s+2}(\Omega)} \leq M. \tag{4.8}$$

There are constants $C = C(\Omega, n, M, s) > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\log t|^{-\sigma}, \quad 0 < t < 1/e.$$

The proof will involve a stability result at the boundary. This is an easier problem, and in fact one has stability with a Lipschitz modulus of continuity.

THEOREM 4.7. *(Boundary stability) Under the assumptions in Theorem 4.1, one has*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$$

where $C = C(\Omega, n, M, s)$.

PROOF. Follows by using the same method as in the proof of the boundary determination result in [**5**], see the exercises. $\square$

We need to relate the difference of DN maps for the conductivity equation to a corresponding quantity in the Schrödinger case.

LEMMA 4.8. *Under the assumptions in Theorem 4.1, if*

$$q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}},$$

we have

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \le C(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{2}{2s+3}}),$$

with $C = C(\Omega, n, M, s)$.

Proof. We use the identity in Lemma 3.5,

$$\Lambda_{q_j} f = \gamma_j^{-1/2} \Lambda_{\gamma_j}(\gamma_j^{-1/2} f) + \frac{1}{2}\gamma_j^{-1}\frac{\partial \gamma_j}{\partial \nu} f\Big|_{\partial\Omega}.$$

If $f \in H^{1/2}(\partial\Omega)$, we obtain

$$\begin{aligned}(\Lambda_{q_1} - \Lambda_{q_2}) f &= (\gamma_1^{-1/2} - \gamma_2^{-1/2})\Lambda_{\gamma_1}(\gamma_1^{-1/2} f) \\ &\quad + \gamma_2^{-1/2}(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\gamma_1^{-1/2} f) + \gamma_2^{-1/2}\Lambda_{\gamma_2}((\gamma_1^{-1/2} - \gamma_2^{-1/2}) f) \\ &\quad + \frac{1}{2}(\gamma_1^{-1} - \gamma_2^{-1})\frac{\partial \gamma_1}{\partial \nu} f + \frac{1}{2}\gamma_2^{-1}\Big(\frac{\partial \gamma_1}{\partial \nu} - \frac{\partial \gamma_2}{\partial \nu}\Big) f.\end{aligned}$$

We estimate the $H^{-1/2}$ norm of this expression by the triangle inequality. For the first three terms we use the estimate

$$\|au\|_{H^{-1/2}(\partial\Omega)} \le \|a\|_{C^1(\partial\Omega)}\|u\|_{H^{-1/2}(\partial\Omega)},$$

and for the last two terms we use that

$$\|au\|_{H^{-1/2}(\partial\Omega)} \le \|a\|_{L^\infty(\partial\Omega)}\|u\|_{H^{1/2}(\partial\Omega)}.$$

The a priori estimates for γ_j imply

$$\begin{gathered}\|\gamma_1^p - \gamma_2^p\|_{C^1(\partial\Omega)} \le C\|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)}, \\ \|\gamma_j^p\|_{C^1(\partial\Omega)} \le C, \quad \|\Lambda_{\gamma_j}\|_* \le C,\end{gathered}$$

where $C = C(\Omega, n, M)$. Consequently

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \le C(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)}).$$

We would like to use Lemma 4.7 to estimate the last term. By Sobolev embedding, logarithmic convexity of Sobolev norms, the trace theorem, and the a priori estimates for γ_j, we have

$$\begin{aligned}\|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)} &\le C\|\gamma_1 - \gamma_2\|_{H^{s+\frac{1}{2}}(\partial\Omega)} \\ &\le C\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}}\|\gamma_1 - \gamma_2\|_{H^{s+\frac{3}{2}}(\partial\Omega)}^{\frac{2s+1}{2s+3}} \\ &\le C\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}}\|\gamma_1 - \gamma_2\|_{H^{s+2}(\Omega)}^{\frac{2s+1}{2s+3}} \\ &\le C\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}}.\end{aligned}$$

Now $\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}$, and the claim follows by Lemma 4.7. $\square$

We can now give the proof of the main stability result. It will be convenient to use the approach in Remark 3.6 to reduce matters to the Schrödinger equation.

Proof of Theorem 4.1. As in Remark 3.6, we introduce the function

$$v = \log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}} = \frac{1}{2}(\log \gamma_1 - \log \gamma_2).$$

This is a C^2 function in $\overline{\Omega}$, and satisfies

$$\left\{ \begin{array}{rll} \nabla \cdot (\gamma_1\gamma_2)^{1/2} \nabla v &= (\gamma_1\gamma_2)^{1/2}(q_1 - q_2) & \text{in } \Omega, \\ v &= \frac{1}{2}(\log \gamma_1 - \log \gamma_2) & \text{on } \partial\Omega, \end{array} \right.$$

where $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. Therefore

$$\begin{aligned} &\frac{1}{2}\|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)} = \|v\|_{H^1(\Omega)} \\ &\leq \|(\gamma_1\gamma_2)^{1/2}(q_1 - q_2)\|_{H^{-1}(\Omega)} + \frac{1}{2}\|\log \gamma_1 - \log \gamma_2\|_{H^{1/2}(\partial\Omega)} \\ &\leq C\|q_1 - q_2\|_{H^{-1}(\Omega)} + C\|\log \gamma_1 - \log \gamma_2\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

By Theorem 4.2 and Lemma 4.8, if $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$ is small we have

$$\begin{aligned} \|q_1 - q_2\|_{H^{-1}(\Omega)} &\leq C|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_*|^{-\frac{2}{n+2}} \\ &\leq C|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{2}{2s+3}}|^{-\frac{2}{n+2}} \\ &\leq C|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}}. \end{aligned}$$

We obtain

$$\begin{aligned} \|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)} \leq C|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}} \\ + C\|\log \gamma_1 - \log \gamma_2\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

We would like to change the norms of $\log \gamma_1 - \log \gamma_2$ on both sides to L^∞ norms. As before, this can be done by Sobolev embedding, logarithmic convexity of Sobolev norms, and the a priori bounds on γ_j:

$$\begin{aligned} \|\log \gamma_1 - \log \gamma_2\|_{L^\infty(\Omega)} &\leq C\|\log \gamma_1 - \log \gamma_2\|_{H^s(\Omega)} \\ &\leq C\|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)}^{\frac{2}{s+1}} \|\log \gamma_1 - \log \gamma_2\|_{H^{s+2}(\Omega)}^{\frac{s-1}{s+1}} \\ &\leq C\|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)}^{\frac{2}{s+1}}, \end{aligned}$$

and

$$\begin{aligned}\|\log\gamma_1-\log\gamma_2\|_{H^{1/2}(\partial\Omega)} &\leq C\|\log\gamma_1-\log\gamma_2\|_{L^2(\partial\Omega)}^{\frac{2s}{2s+3}}\|\log\gamma_1-\log\gamma_2\|_{H^{s+\frac{3}{2}}(\Omega)}^{\frac{1}{2s+3}}\\ &\leq C\|\log\gamma_1-\log\gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s}{2s+3}}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\log\gamma_1-\log\gamma_2\|_{L^\infty(\Omega)}^{\frac{s+1}{2}} &\leq C|\log\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}}\\ &\qquad + C\|\log\gamma_1-\log\gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s}{2s+3}}.\end{aligned}$$

Finally, we obtain bounds in terms of $\gamma_1-\gamma_2$ by using that

$$\begin{aligned}\log\gamma_1-\log\gamma_2 &= \int_0^1 \frac{d}{dt}\log\left((1-t)\gamma_2+t\gamma_1\right)dt\\ &= \Big(\int_0^1 \frac{1}{(1-t)\gamma_2+t\gamma_1}\Big)(\gamma_1-\gamma_2).\end{aligned}$$

The a priori bounds on γ_j imply that

$$\begin{aligned}\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} &\leq C\|\log\gamma_1-\log\gamma_2\|_{L^\infty(\Omega)},\\ \|\log\gamma_1-\log\gamma_2\|_{L^\infty(\partial\Omega)} &\leq C\|\gamma_1-\gamma_2\|_{L^\infty(\partial\Omega)}.\end{aligned}$$

This shows that

$$\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)}^{\frac{s+1}{2}} \leq C|\log\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}} + C\|\gamma_1-\gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s}{2s+3}}.$$

The result follows by Theorem 4.7. □

CHAPTER 5

Partial data

In Chapter 3, we showed that if the boundary measurements for two C^2 conductivities coincide on the whole boundary, then the conductivities are equal. Here we consider the case where measurements are made only on part of the boundary.

The first result in this direction was proved by Bukhgeim and Uhlmann. It involves a unit vector α in $\mathbf{R}^n$ and the subset of the boundary

$$\partial\Omega_{-,\varepsilon} = \{x \in \partial\Omega\,;\, \alpha \cdot \nu(x) < \varepsilon\}.$$

The theorem is as follows.

THEOREM 5.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. If $\alpha \in \mathbf{R}^n$ is a unit vector, if $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$, and if for some $\varepsilon > 0$ one has*

$$\Lambda_{\gamma_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{\gamma_2} f|_{\partial\Omega_{-,\varepsilon}} \quad \textit{for all } f \in H^{1/2}(\partial\Omega),$$

then $\gamma_1 = \gamma_2$ in Ω.

The proof is based on complex geometrical optics solutions, but requires new elements since we need some control of the solutions on parts of the boundary. The main tool is a weighted norm estimate known as a *Carleman estimate*. This estimate also gives rise to a new construction of complex geometrical optics solutions, which does not involve Fourier analysis.

5.1. Carleman estimates

Again, we first consider the Schrödinger equation, $(-\Delta+q)u = 0$ in Ω, where $q \in L^\infty(\Omega)$ and $\Omega \subseteq \mathbf{R}^n$ is a bounded open set with smooth boundary.

MOTIVATION. Recall from Theorem 3.8 that in the construction of complex geometrical optics solutions, which depend on a large vector

$\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$, we needed to solve equations of the form

$$(D^2 + 2\zeta \cdot D + q)r = f \qquad \text{in } \Omega,$$

or written in another way,

$$e^{-i\zeta \cdot x}(-\Delta + q)e^{i\zeta \cdot x} r = f \qquad \text{in } \Omega.$$

In particular, Theorem 3.8 shows the existence of a solution and implies the estimate

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|f\|_{L^2(\Omega)}.$$

We write

$$\zeta = \frac{1}{h}(\beta + i\alpha),$$

where α and β are orthogonal unit vectors in $\mathbf{R}^n$, and $h > 0$ is a *small parameter*. The estimate for r may be written as

$$\|r\|_{L^2(\Omega)} \leq C_0 h \|e^{\frac{1}{h}\alpha \cdot x}(-\Delta + q)e^{-\frac{1}{h}\alpha \cdot x} r\|_{L^2(\Omega)}.$$

It is possible to view this as a uniqueness result: if the right hand side is zero, then the solution r also vanishes. It turns out that such a uniqueness result can be proved directly without Fourier analysis, and this is sufficient to imply also existence of a solution.

Remark 5.2. We will systematically use a small parameter h instead of a large parameter $|\zeta|$ (these are related by $h = \frac{\sqrt{2}}{|\zeta|}$). This is of course just a matter of convention, but has the benefit of being consistent with *semiclassical calculus* which is a well-developed theory for the analysis of certain asymptotic limits. We will also arrange so that our basic partial derivatives will be hD_j instead of $\frac{\partial}{\partial x_j}$. The usefulness of these choices will hopefully be evident below.

5.1.1. Carleman estimates for test functions. We begin with the simplest Carleman estimate, which is valid for test functions and does not involve boundary terms.

Theorem 5.3. *(Carleman estimate) Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbf{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, we have*

$$\|u\|_{L^2(\Omega)} \leq Ch\|e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}u\|_{L^2(\Omega)}, \quad u \in C_c^\infty(\Omega).$$

We introduce some notation which will be used in the proof and also later. If $u, v \in L^2(\Omega)$ we write

$$(u|v) = \int_\Omega u\bar{v}\,dx,$$
$$\|u\| = (u|u)^{1/2} = \|u\|_{L^2(\Omega)}.$$

Consider the semiclassical Laplacian

$$P_0 = -h^2\Delta = (hD)^2,$$

and the corresponding Schrödinger operator

$$P = h^2(-\Delta + q) = P_0 + h^2 q.$$

The operators conjugated with exponential weights will be denoted by

$$P_{0,\varphi} = e^{\varphi/h} P_0 e^{-\varphi/h},$$
$$P_\varphi = e^{\varphi/h} P e^{-\varphi/h} = P_{0,\varphi} + h^2 q.$$

We will also need the concept of adjoints of differential operators. If

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is a differential operator in Ω, with $a_\alpha \in W^{|\alpha|,\infty}(\Omega)$ (that is, all partial derivatives up to order $|\alpha|$ are in $L^\infty(\Omega)$), then L^* is the differential operator which satisfies

$$(Lu|v) = (u|L^* v), \qquad u, v \in C_c^\infty(\Omega).$$

For L of the above form, an integration by parts shows that

$$L^* v = \sum_{|\alpha| \leq m} D^\alpha(\overline{a_\alpha(x)} v).$$

Proof of Theorem 5.3. Using the notation above, the desired estimate can be written as

$$h\|u\| \leq C\|P_\varphi u\|, \qquad u \in C_c^\infty(\Omega).$$

First consider the case $q = 0$, that is, the estimate

$$h\|u\| \leq C\|P_{0,\varphi} u\|, \qquad u \in C_c^\infty(\Omega).$$

We need an explicit expression for $P_{0,\varphi}$. On the level of operators, one has

$$e^{\varphi/h} hD_j e^{-\varphi/h} = hD_j + i\partial_j \varphi.$$

Since $\varphi(x) = \alpha \cdot x$ where α is a unit vector, we obtain

$$\begin{aligned} P_{0,\varphi} &= \sum_{j=1}^{n} (e^{\varphi/h} hD_j e^{-\varphi/h})(e^{\varphi/h} hD_j e^{-\varphi/h}) = \sum_{j=1}^{n} (hD_j + i\alpha_j)^2 \\ &= (hD)^2 - 1 + 2i\alpha \cdot hD. \end{aligned}$$

The objective is to prove a positive lower bound for

$$\|P_{0,\varphi} u\|^2 = (P_{0,\varphi} u | P_{0,\varphi} u).$$

To this end, we decompose $P_{0,\varphi}$ in a way which is useful for determining which parts in the inner product are positive and which may be negative. Write

$$P_{0,\varphi} = A + iB$$

where $A^* = A$ and $B^* = B$. Here, A and iB are the self-adjoint and skew-adjoint parts of $P_{0,\varphi}$. Since

$$\begin{aligned} P_{0,\varphi}^* &= (e^{\varphi/h} P_0 e^{-\varphi/h})^* = e^{-\varphi/h} P_0 e^{\varphi/h} = P_{0,-\varphi} \\ &= (hD)^2 - 1 - 2i\alpha \cdot hD, \end{aligned}$$

we obtain A and B from the formulas (cf. the real and imaginary parts of a complex number)

$$\begin{aligned} A &= \frac{P_{0,\varphi} + P_{0,\varphi}^*}{2} = (hD)^2 - 1, \\ B &= \frac{P_{0,\varphi} - P_{0,\varphi}^*}{2i} = 2\alpha \cdot hD. \end{aligned}$$

Now we have

$$\begin{aligned} \|P_{0,\varphi} u\|^2 &= (P_{0,\varphi} u | P_{0,\varphi} u) = ((A + iB)u | (A + iB)u) \\ &= (Au|Au) + (Bu|Bu) + i(Bu|Au) - i(Au|Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u|u), \end{aligned}$$

where $[A, B] = AB - BA$ is the commutator of A and B. This argument used integration by parts and the fact that $A^* = A$ and $B^* = B$. There are no boundary terms since $u \in C_c^\infty(\Omega)$.

The terms $\|Au\|^2$ and $\|Bu\|^2$ are nonnegative, so the only negative contributions could come from the commutator term. But in our case A and B are constant coefficient differential operators, and these operators always satisfy

$$[A, B] \equiv 0.$$

Therefore

$$\|P_{0,\varphi}u\|^2 = \|Au\|^2 + \|Bu\|^2.$$

By the Poincaré inequality (see [**5**]) [1],

$$\|Bu\| = 2h\|\alpha \cdot Du\| \geq ch\|u\|,$$

where c depends on Ω. This shows that for any $h > 0$, one has

$$h\|u\| \leq C\|P_{0,\varphi}u\|, \qquad u \in C_c^\infty(\Omega).$$

Finally, consider the case where q may be nonzero. The last estimate implies that for $u \in C_c^\infty(\Omega)$, one has

$$\begin{aligned} h\|u\| &\leq C\|P_{0,\varphi}u\| \leq C\|(P_{0,\varphi} + h^2 q)u\| + C\|h^2 qu\| \\ &\leq C\|P_\varphi u\| + Ch^2\|q\|_{L^\infty(\Omega)}\|u\|. \end{aligned}$$

Choose h_0 so that $C\|q\|_{L^\infty(\Omega)}h_0 = \frac{1}{2}$, that is,

$$h_0 = \frac{1}{2C\|q\|_{L^\infty(\Omega)}}.$$

Then, if $0 < h \leq h_0$,

$$h\|u\| \leq C\|P_\varphi u\| + \frac{1}{2}h\|u\|.$$

The last term may be absorbed in the left hand side, which completes the proof. □

5.1.2. Complex geometrical optics solutions. Here, we show how the Carleman estimate gives a new method for constructing complex geometrical optics solutions. We first establish an existence result for an inhomogeneous equation, analogous to Theorem 3.8.

THEOREM 5.4. *Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbf{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, the equation*

$$e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}r = f \quad \textit{in } \Omega$$

has a solution $r \in L^2(\Omega)$ for any $f \in L^2(\Omega)$, satisfying

$$\|r\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.$$

[1] In fact, if $\alpha \in \mathbf{R}^n$ is a unit vector, then the proof given in [**5**] implies the following Poincaré inequality in the unbounded strip $S = \{x \in \mathbf{R}^n \,;\, a < x \cdot \alpha < b\}$:

$$\|u\|_{L^2(S)} \leq \frac{b-a}{\sqrt{2}}\|\alpha \cdot Du\|_{L^2(S)}, \quad u \in C_c^\infty(S).$$

REMARK 5.5. With some knowledge of unbounded operators on Hilbert space, the proof is immediate. Consider $P_\varphi^* : L^2(\Omega) \to L^2(\Omega)$ with domain $C_c^\infty(\Omega)$. It is a general fact that

$$\left\{\begin{array}{l} T \text{ injective} \\ \text{range of } T \text{ closed} \end{array}\right. \implies T^* \text{ surjective.}$$

Since the Carleman estimate is valid for P_φ^* one obtains injectivity and closed range for P_φ^*, and thus solvability for P_φ. Below we give a direct proof based on duality and the Hahn-Banach theorem, and also obtain the norm bound.

PROOF OF THEOREM 5.4. Note that $P_\varphi^* = P_{0,-\varphi} + h^2\bar{q}$. If h_0 is as in Theorem 5.3, for $h \leq h_0$ we have

$$\|u\| \leq \frac{C}{h}\|P_\varphi^* u\|, \quad u \in C_c^\infty(\Omega).$$

Let $D = P_\varphi^* C_c^\infty(\Omega)$ be a subspace of $L^2(\Omega)$, and consider the linear functional

$$L : D \to \mathbf{C}, \quad L(P_\varphi^* v) = (v|f), \quad \text{for } v \in C_c^\infty(\Omega).$$

This is well defined since any element of D has a unique representation as $P_\varphi^* v$ with $v \in C_c^\infty(\Omega)$, by the Carleman estimate. Also, the Carleman estimate implies

$$|L(P_\varphi^* v)| \leq \|v\|\|f\| \leq \frac{C}{h}\|f\|\|P_\varphi^* v\|.$$

Thus L is a bounded linear functional on D.

The Hahn-Banach theorem ensures that there is a bounded linear functional $\hat{L} : L^2(\Omega) \to \mathbf{C}$ satisfying $\hat{L}|_D = L$ and $\|\hat{L}\| \leq Ch^{-1}\|f\|$. By the Riesz representation theorem, there is $\tilde{r} \in L^2(\Omega)$ such that

$$\hat{L}(w) = (w|\tilde{r}), \quad w \in L^2(\Omega),$$

and $\|\tilde{r}\| \leq Ch^{-1}\|f\|$. Then, for $v \in C_c^\infty(\Omega)$, by the definition of weak derivatives we have

$$(v|P_\varphi \tilde{r}) = (P_\varphi^* v|\tilde{r}) = \hat{L}(P_\varphi^* v) = L(P_\varphi^* v) = (v|f),$$

which shows that $P_\varphi \tilde{r} = f$ in the weak sense.

Finally, set $r = h^2\tilde{r}$. This satisfies $e^{\varphi/h}(-\Delta + q)e^{-\varphi/h} r = f$ in Ω, and $\|r\| \leq Ch\|f\|$. $\square$

We now give a construction of complex geometrical optics solutions to the equation $(-\Delta + q)u = 0$ in Ω, based on Theorem 5.4. This is slightly more general than the discussion in Chapter 3, and is analogous to the *WKB construction* used in finding geometrical optics solutions for the wave equation.

Our solutions are of the form

$$u = e^{-\frac{1}{h}(\varphi + i\psi)}(a + r). \tag{5.1}$$

Here $h > 0$ is small and $\varphi(x) = \alpha \cdot x$ as before, ψ is a real valued phase function, a is a complex amplitude, and r is a correction term which is small when h is small.

Writing $\rho = \varphi + i\psi$ for the complex phase, using the formula

$$e^{\rho/h} hD_j e^{-\rho/h} = hD_j + i\partial_j \rho$$

which is valid for operators, and inserting (5.1) in the equation, we have

$$\begin{aligned} & (-\Delta + q)u = 0 \\ \Leftrightarrow \quad & e^{\rho/h}((hD)^2 + h^2 q)e^{-\rho/h}(a + r) = 0 \\ \Leftrightarrow \quad & e^{\rho/h}((hD)^2 + h^2 q)e^{-\rho/h} r = -((hD + i\nabla\rho)^2 + h^2 q)a \end{aligned}$$

The last equation may be written as

$$e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}(e^{-i\psi/h} r) = f$$

where

$$f = -e^{-i\psi/h}(-h^{-2}(\nabla\rho)^2 + h^{-1}[2\nabla\rho \cdot \nabla + \Delta\rho] + (-\Delta + q))a.$$

Now, Theorem 5.4 ensures that one can find a correction term r satisfying $\|r\| \leq Ch$, thus showing the existence of complex geometrical optics solutions, provided that

$$\|f\| \leq C$$

with C independent of h. Looking at the expression for f, we see that it is enough to choose ψ and a in such a way that

$$(\nabla\rho)^2 = 0,$$
$$2\nabla\rho \cdot \nabla a + (\Delta\rho)a = 0.$$

Since $\varphi(x) = \alpha \cdot x$ with α a unit vector, expanding the square in $(\nabla\rho)^2 = 0$ gives the following equations for ψ:

$$|\nabla\psi|^2 = 1, \qquad \alpha \cdot \nabla\psi = 0.$$

This is an *eikonal equation* (a certain nonlinear first order PDE) for ψ. We obtain one solution by choosing $\psi(x) = \beta \cdot x$ where $\beta \in \mathbf{R}^n$ is a unit vector satisfying $\alpha \cdot \beta = 0$. It would be possible to use other solutions ψ, but this choice is close to the discussion in Chapter 3.

If $\psi(x) = \beta \cdot x$, then the second equation becomes

$$(\alpha + i\beta) \cdot \nabla a = 0.$$

This is a complex *transport equation* (a first order linear equation) for a, analogous to the equation for a in Theorem 3.9. One solution is given by $a \equiv 1$. Again, other choices would be possible.

This ends the construction of complex geometrical optics solutions based on Carleman estimates. There is one additional difference with the analogous result in Theorem 3.9: the correction term r given by this argument is only in $L^2(\Omega)$, not in $H^1(\Omega)$. The same is true for the solution u. One can in fact obtain r and u in $H^1(\Omega)$ (and even in $H^2(\Omega)$), but this requires a slightly stronger Carleman estimate and some additional work. Some details for this were given in the exercises and lectures.

5.1.3. Carleman estimates with boundary terms. We will continue by deriving a Carleman estimate for functions which vanish at the boundary but are not compactly supported. The estimate will include terms involving the normal derivative. We will use the notation

$$(u|v)_{\partial\Omega} = \int_{\partial\Omega} u\bar{v}\,dS,$$
$$\partial_\nu u = \nabla u \cdot \nu|_{\partial\Omega}$$

and

$$\partial\Omega_\pm = \partial\Omega_\pm(\alpha) = \{x \in \partial\Omega\,;\, \pm\alpha \cdot \nu(x) \geq 0\}.$$

THEOREM 5.6. *(Carleman estimate with boundary terms) Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbf{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, we have*

$$\begin{aligned} &- h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + \|u\|^2_{L^2(\Omega)} \\ &\qquad \leq Ch^2\|e^{\varphi/h}(-\Delta+q)e^{-\varphi/h}u\|^2_{L^2(\Omega)} + Ch((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+} \end{aligned}$$

for any $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$.

Note that the sign of $\alpha \cdot \nu$ on $\partial\Omega_\pm$ ensures that all terms in the Carleman estimate are nonnegative.

Proof. We first claim that

$$ch^2\|u\|^2 - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega} \leq \|P_{0,\varphi}u\|^2 \tag{5.2}$$

for $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$. It is easy to see that this implies the desired estimate in the case $q = 0$.

As in the proof of Theorem 5.3, we decompose

$$P_{0,\varphi} = A + iB$$

where $A = (hD)^2 - 1$ and $B = 2\alpha\cdot hD$, and $A^* = A$, $B^* = B$. Then

$$\begin{aligned}\|P_{0,\varphi}u\|^2 &= (P_{0,\varphi}u|P_{0,\varphi}u) = ((A+iB)u|(A+iB)u) \\ &= \|Au\|^2 + \|Bu\|^2 + i(Bu|Au) - i(Au|Bu).\end{aligned}$$

We wish to integrate by parts to obtain the commutator term involving $i[A,B]$, but this time boundary terms will arise. We have

$$\begin{aligned}i(Bu|(hD)^2u) &= \sum_{j=1}^n i(Bu|(hD_j)^2u) \\ &= \sum_{j=1}^n \Big[i(Bu|\frac{h}{i}\nu_j hD_j u)_{\partial\Omega} + i(hD_jBu|hD_ju)\Big] \\ = -2h^3(\alpha\cdot\nabla u|\partial_\nu u)_{\partial\Omega} &+ \sum_{j=1}^n \Big[i(hD_jBu|\frac{h}{i}\nu_j u)_{\partial\Omega} + i((hD_j)^2Bu|u)\Big].\end{aligned}$$

But $u|_{\partial\Omega} = 0$, so the boundary term involving $\frac{h}{i}\nu_j u$ is zero. For the first boundary term we use the decomposition

$$\nabla u|_{\partial\Omega} = (\partial_\nu u)\nu + (\nabla u)_{\tan}$$

where $(\nabla u)_{\tan} := \nabla u - (\nabla u\cdot\nu)\nu|_{\partial\Omega}$ is the tangential part of ∇u, which vanishes since $u|_{\partial\Omega} = 0$. By these facts, we obtain

$$i(Bu|Au) = i(ABu|u) - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega}.$$

Similarly, using that $u|_{\partial\Omega} = 0$,

$$\begin{aligned}i(Au|Bu) &= i(Au|2\alpha\cdot\frac{h}{i}\nu u)_{\partial\Omega} + i(BAu|u) \\ &= i(BAu|u).\end{aligned}$$

We have proved that

$$\|P_{0,\varphi}u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u) - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega}.$$

Again, since A and B are constant coefficient operators, we have $[A, B] = AB - BA \equiv 0$. The Poincaré inequality gives $\|Bu\| \geq ch\|u\|$, which proves (5.2).

Writing (5.2) in a different form, we have

$$\begin{aligned}
&-2h((\alpha \cdot \nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + c\|u\|^2 \\
&\qquad \leq h^2\|e^{\varphi/h}(-\Delta)e^{-\varphi/h}u\|^2 + 2h((\alpha \cdot \nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}.
\end{aligned}$$

Adding a potential, it follows that

$$\begin{aligned}
&-2h((\alpha \cdot \nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + c\|u\|^2 \\
&\qquad \leq h^2\|e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}u\|^2 + h^2\|q\|^2_{L^\infty(\Omega)}\|u\|^2 \\
&\qquad\qquad + 2h((\alpha \cdot \nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}.
\end{aligned}$$

Choosing h small enough (depending on $\|q\|_{L^\infty(\Omega)}$), the term involving $\|u\|^2$ on the right can be absorbed to the left hand side. This concludes the proof. $\square$

5.2. Uniqueness with partial data

Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary, where $n \geq 3$. If $\alpha \in \mathbf{R}^n$, recall the subsets of the boundary

$$\begin{aligned}
\partial\Omega_\pm &= \{x \in \partial\Omega\,;\, \pm\alpha \cdot \nu(x) > 0\}, \\
\partial\Omega_{-,\varepsilon} &= \{x \in \partial\Omega\,;\, \alpha \cdot \nu(x) < \varepsilon\}.
\end{aligned}$$

Also, let $\partial\Omega_{+,\varepsilon} = \{x \in \partial\Omega\,;\, \alpha \cdot \nu(x) > \varepsilon\}$. We first consider a partial data uniqueness result for the Schrödinger equation.

THEOREM 5.7. *Let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ are well-posed. If α is a unit vector in $\mathbf{R}^n$, and if*

$$\Lambda_{q_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{q_2} f|_{\partial\Omega_{-,\varepsilon}} \quad \text{for all } f \in H^{1/2}(\partial\Omega),$$

then $q_1 = q_2$ in Ω.

Given this result, it is easy to prove the corresponding theorem for the conductivity equation.

PROOF THAT THEOREM 5.7 IMPLIES THEOREM 5.1. Define $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. By Lemma 3.5 we have the relation

$$\Lambda_{q_j} f = \gamma_j^{-1/2}\Lambda_{\gamma_j}(\gamma_j^{-1/2} f) + \frac{1}{2}\gamma_j^{-1}\frac{\partial\gamma_j}{\partial\nu} f\Big|_{\partial\Omega}.$$

Since $\Lambda_{\gamma_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{\gamma_2} f|_{\partial\Omega_{-,\varepsilon}}$ for all f, boundary determination results (see [**5**]) imply that

$$\gamma_1|_{\partial\Omega_{-,\varepsilon}} = \gamma_2|_{\partial\Omega_{-,\varepsilon}}, \quad \frac{\partial\gamma_1}{\partial\nu}|_{\partial\Omega_{-,\varepsilon}} = \frac{\partial\gamma_2}{\partial\nu}|_{\partial\Omega_{-,\varepsilon}}.$$

Thus $\Lambda_{q_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{q_2} f|_{\partial\Omega_{-,\varepsilon}}$ for all $f \in H^{1/2}(\partial\Omega)$. Theorem 5.7 then implies $q_1 = q_2$, or

$$\frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \quad \text{in } \Omega.$$

Now also $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$, so the arguments in Section 3.1 imply that $\gamma_1 = \gamma_2$ in Ω. $\square$

We proceed to the proof of Theorem 5.7. The main tool is the Carleman estimate in Theorem 5.6, which will be applied with the weight $-\varphi$ instead of φ. The estimate then has the form

$$\begin{aligned} &h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+} + \|u\|^2_{L^2(\Omega)} \\ &\qquad \leq Ch^2\|e^{-\varphi/h}(-\Delta+q)e^{\varphi/h}u\|^2_{L^2(\Omega)} - Ch((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} \end{aligned}$$

with $u \in C^\infty(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$. Choosing $v = e^{\varphi/h}u$ and noting that $v|_{\partial\Omega} = 0$, this may be written as

$$\begin{aligned} (5.3)\quad &h((\alpha\cdot\nu)e^{-\varphi/h}\partial_\nu v|e^{-\varphi/h}\partial_\nu v)_{\partial\Omega_+} + \|e^{-\varphi/h}v\|^2_{L^2(\Omega)} \\ &\leq Ch^2\|e^{-\varphi/h}(-\Delta+q)v\|^2_{L^2(\Omega)} - Ch((\alpha\cdot\nu)e^{-\varphi/h}\partial_\nu v|e^{-\varphi/h}\partial_\nu v)_{\partial\Omega_-}. \end{aligned}$$

This last estimate is valid for all $v \in H^2 \cap H^1_0(\Omega)$, which follows by an approximation argument (or can be proved directly).

Proof of Theorem 5.7. Recall from Lemma 3.8 that

$$\int_\Omega (q_1 - q_2)u_1 u_2\,dx = \langle(\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega}), u_2|_{\partial\Omega}\rangle_{\partial\Omega}, \tag{5.4}$$

whenever $u_j \in H^1(\Omega)$ are solutions of $(-\Delta + q_j)u_j = 0$ in Ω. By the assumption on the DN maps, the boundary integral is really over $\partial\Omega_{+,\varepsilon}$. If further $u_1 \in H^2(\Omega)$, then

$$\Lambda_{q_1}(u_1|_{\partial\Omega}) = \partial_\nu u_1|_{\partial\Omega}$$

since $\nabla u_1 \in H^1(\Omega)$ and $\partial_\nu u_1|_{\partial\Omega} = (\operatorname{tr}\nabla u_1)\cdot\nu|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Also,

$$\Lambda_{q_2}(u_1|_{\partial\Omega}) = \partial_\nu \tilde{u}_2|_{\partial\Omega},$$

where $\tilde{u}_2$ solves

$$\left\{ \begin{array}{ll} (-\Delta + q_2)\tilde{u}_2 = 0 & \text{in } \Omega, \\ \tilde{u}_2 = u_1 & \text{on } \partial\Omega. \end{array} \right.$$

We have $\tilde{u}_2 \in H^2(\Omega)$ since $u_1|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$. Therefore, (5.4) implies

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = \int_{\partial\Omega_{+,\varepsilon}} \partial_\nu (u_1 - \tilde{u}_2) u_2 \, dS$$

for any $u_j \in H^2(\Omega)$ which solve $(-\Delta + q_j)u_j = 0$ in Ω.

Given the unit vector $\alpha \in \mathbf{R}^n$, let $\xi \in \mathbf{R}^n$ be a vector orthogonal to α, and let $\beta \in \mathbf{R}^n$ be a unit vector such that $\{\alpha, \beta, \xi\}$ is an orthogonal triplet. Write $\varphi(x) = \alpha \cdot x$ and $\psi(x) = \beta \cdot x$. Theorem 3.9 ensures that there exist CGO solutions to $(-\Delta + q_j)u_j = 0$ of the form

$$\begin{aligned} u_1 &= e^{\frac{1}{h}(\varphi + i\psi)} e^{ix\cdot\xi}(1 + r_1), \\ u_2 &= e^{-\frac{1}{h}(\varphi + i\psi)}(1 + r_2), \end{aligned}$$

where $\|r_j\| \leq Ch$, $\|\nabla r_j\| \leq C$, and $u_j \in H^2(\Omega)$ (the part that $r_j \in H^2(\Omega)$ was in the exercises). Then, writing $u := u_1 - \tilde{u}_2 \in H^2 \cap H_0^1(\Omega)$, we have

$$\int_\Omega e^{ix\dot{\xi}}(q_1 - q_2)(1 + r_1 + r_2 + r_1 r_2) \, dx = \int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u) u_2 \, dS. \tag{5.5}$$

By the estimates for r_j, the limit as $h \to 0$ of the left hand side is $\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)\, dx$. We wish to show that the right hand side converges to zero as $h \to 0$.

By Cauchy-Schwarz, one has

$$\begin{aligned} \Big| \int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u) u_2 \, dS \Big|^2 &= \Big| \int_{\partial\Omega_{+,\varepsilon}} e^{-\varphi/h}(\partial_\nu u) e^{\varphi/h} u_2 \, dS \Big|^2 \\ &\leq \Big(\int_{\partial\Omega_{+,\varepsilon}} |e^{-\varphi/h}\partial_\nu u|^2 \, dS \Big) \Big(\int_{\partial\Omega_{+,\varepsilon}} |e^{\varphi/h} u_2|^2 \, dS \Big). \end{aligned} \tag{5.6}$$

To use the Carleman estimate, we note that $\varepsilon \leq \alpha \cdot \nu$ on $\partial\Omega_{+,\varepsilon}$, By (5.3) applied to u and with potential q_2, and using that $\partial_\nu u|_{\partial\Omega_{-,\varepsilon}} = 0$ by the assumption on DN maps, we obtain for small h that

$$\begin{aligned} \int_{\partial\Omega_{+,\varepsilon}} |e^{-\varphi/h}\partial_\nu u|^2 &\leq \frac{1}{\varepsilon} \int_{\partial\Omega_{+,\varepsilon}} (\alpha \cdot \nu) |e^{-\varphi/h}\partial_\nu u|^2 \, dS \\ &\leq \frac{1}{\varepsilon} Ch \|e^{-\varphi/h}(-\Delta + q_2)u\|_{L^2(\Omega)}^2. \end{aligned}$$

The reason for choosing the potential q_2 is that

$$(-\Delta + q_2)u = (-\Delta + q_2)u_1 = (q_2 - q_1)u_1.$$

Thus, the solution $\tilde{u}_2$ goes away, and we are left with an expression involving only u_1 for which we know exact asymptotics. We have

$$\int_{\partial\Omega_{+,\varepsilon}} |e^{-\varphi/h}\partial_\nu u|^2 \leq \frac{1}{\varepsilon} Ch \|(q_2 - q_1)e^{i\psi/h}e^{ix\cdot\xi}(1 + r_1)\|^2_{L^2(\Omega)} \leq Ch.$$

This takes care of the first term on the right hand side of (5.6). For the other term we compute

$$\begin{aligned}\int_{\partial\Omega_{+,\varepsilon}} |e^{\varphi/h}u_2|^2 \, dS &= \int_{\partial\Omega_{+,\varepsilon}} |1 + r_2|^2 \, dS \\ &\leq \frac{1}{2}\int_{\partial\Omega_{+,\varepsilon}} (1 + r_2^2) \, dS \leq C(1 + \|r_2\|^2_{L^2(\partial\Omega)}).\end{aligned}$$

By the trace theorem, $\|r_2\|_{L^2(\partial\Omega)} \leq C\|r_2\|_{H^1(\Omega)} \leq C$. Combining these estimates, we have for small h that

$$\Big| \int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u)u_2 \, dS \Big| \leq C\sqrt{h}.$$

Taking the limit as $h \to 0$ in (5.5), we are left with

$$\int_\Omega e^{ix\cdot\xi}(q_1 - q_2) \, dx = 0. \tag{5.7}$$

This is true for all $\xi \in \mathbf{R}^n$ orthogonal to α. However, since the DN maps agree on $\partial\Omega_{-,\varepsilon}(\alpha)$ for a fixed constant $\varepsilon > 0$, they also agree on $\partial\Omega_{-,\varepsilon'}(\alpha')$ for α' sufficiently close to α on the unit sphere and for some smaller constant ε'. Thus, in particular, (5.7) holds for ξ in an open cone in $\mathbf{R}^n$. Writing q for the function which is equal to $q_1 - q_2$ in Ω and which is zero outside of Ω, this implies that the Fourier transform of q vanishes in an open set. But since q is compactly supported, the Fourier transform is analytic by the Paley-Wiener theorem, and this implies that $q \equiv 0$. We have proved that $q_1 \equiv q_2$. $\square$

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