The Asymptotic Behaviour of the Proportion of Hard Instances of the Halting Problem

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1 My Motivation

```
void nasty(string P){
  if(!halts(P,P)){
    while(true){}
  }
}
```

Does program P halt on input I?

The classic (*and correct!*) undecidability proof

- assume that a **halting tester** exists
- using it, build a program that predicts its own future behaviour and does precisely the opposite to the prediction

$\Rightarrow$ the prediction is incorrect by construction

$\Rightarrow$ *halting tester does not exist*

```
bool halts(string P, string I){...}
```

Many people feel this proof is cheating, "a rabbit out of the magician's hat"

```
nasty(nasty)
```

- `comp.theory` noisemakers — ignore them
- Eric C.R. Hehner — serious scientist
- some of my not worst students — *I can't ignore them!*

I wanted to write another proof that would not create such feelings

- (Halting tester proof for software engineers . . . , `comp.theory` 2012-06-15)
- got interested in this problem area

$\Rightarrow$ found some new small results in this very classic field

2 Incomplete Testers

```
bool halts2(string P, string I){
   if(P==nasty && I==nasty)
        return false;
   else return halts(P,I);
}
```

Fail on some instances (P, I)

- *3-way tester* — replies “I don’t know”
- *generic-case tester* — the tester fails to halt
- *approximating tester* — gives a wrong “yes” or “no” answer

Hard instance = tester fails on it

Examples

- always reply “I don’t know” — absolutely useless but meets the definition
- simulate 9^{9^n} steps, reply “I don’t know” if did not halt or ... by then

Any 3-way tester can be trivially converted to a generic or approximating tester

For each incomplete tester, the classic proof constructs a hard instance of it

- the tester can be **modified** to handle the instance ...
- ... but an accordingly modified `nasty` is hard for the modified tester

Every tester has ∞ many hard instances

No instance is hard for every tester

3 Proportions of Easy and Hard Instances

Notation for the number of instances of size n (of tester T)

	easy	hard	altogether
halting	$\underline{h}_T(n)$	$\overline{h}_T(n)$	$h(n)$
non-halting	$\underline{d}_T(n)$	$\overline{d}_T(n)$	$d(n)$
altogether	$\underline{p}_T(n)$	$\overline{p}_T(n)$	$p(n)$

Failure rate $= \dfrac{\overline{p}_T(n)}{p(n)} = \dfrac{\overline{h}_T(n) + \overline{d}_T(n)}{p(n)} =$ the proportion of hard instances

The hope

- the failure rate cannot be made 0, but ...
- ... perhaps it can be made small?

It proved interesting to investigate separately $\dfrac{\overline{h}_T(n)}{p(n)}$ and $\dfrac{\overline{d}_T(n)}{p(n)}$ as $n \to \infty$

Why asymptotic?

- failure rate can be made 0 for any finite set of instances with a look-up table
  - absolutely impractical and uninformative, but rules out interesting results

4 Varied Asymptotics

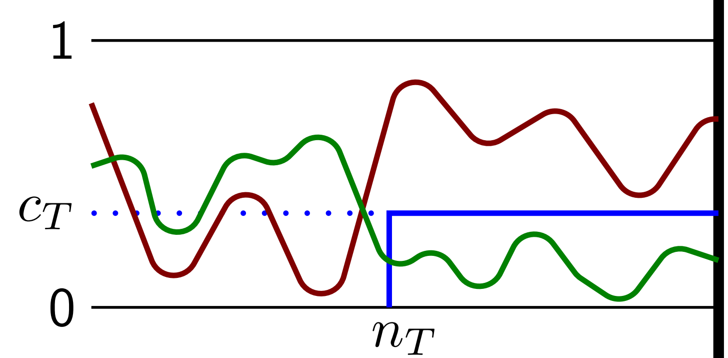
Most results in the paper are of the following kinds, with varying assumptions

Easiness formulae

- a single ever-improving tester $\exists T : \forall c > 0 : \exists n_c \in \mathbb{N} : \forall n \geq n_c : \dfrac{\overline{p}_T(n)}{p(n)} \leq c$
  - that is, $\overline{p}_T(n)/p(n) \to 0$ as $n \to \infty$
- a family of better and better testers $\forall c > 0 : \exists T_c : \forall n \in \mathbb{N} : \dfrac{\overline{p}_{T_c}(n)}{p(n)} \leq c$
  - no n_c, because small inputs solved with a look-up table

Hardness formulae

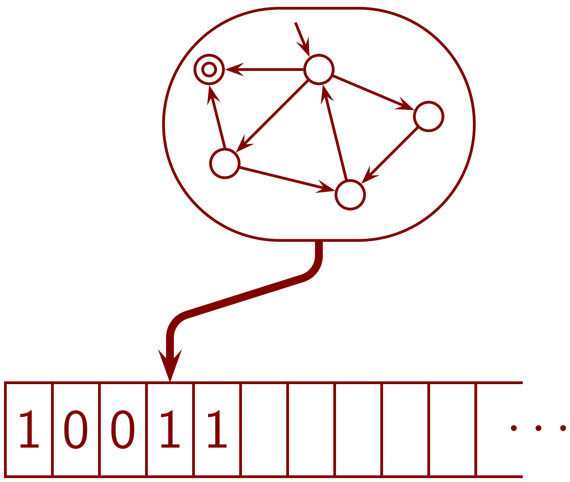
- every tester suffers a lower bound
  $\forall T : \exists c_T > 0 : \exists n_T \in \mathbb{N} : \forall n \geq n_T : \dfrac{\overline{p}_T(n)}{p(n)} \geq c_T$
- there is a common lower bound for all testers
  $\exists c > 0 : \forall T : \exists n_T \in \mathbb{N} : \forall n \geq n_T : \dfrac{\overline{p}_T(n)}{p(n)} \geq c$



Infinitely often

- the dark blue part is replaced by $\forall n_0 \in \mathbb{N} : \exists n \geq n_0$
- important, because $\neg$ "from some n on" φ $\Leftrightarrow$ "infinitely often" $\neg\varphi$

5 Literature Survey



Surprisingly few papers were found!

- and many are surprisingly recent
- many are unaware of most others

Diversity of the problem

- model of computation
  - Turing machine — which variant?
  - programming language — frequency/density assumption (next slide)
  - Gödel numbers of recursive functions — ≈ indices of programs
- type of halting problem

(A) $T(P)$ tells if P halts on the empty input

(B) $T(P)$ tells if P halts on the input P, i.e., given itself as its input

(C) $T(P, I)$ tells if P halts on the input I

  - until now we have discussed (C)
  - with (A) and (B), $p(n)$ = number of programs of size n
- failure mode: 3-way, generic-case, approximating

6 Domain-frequency

It is trivial to make many longer identically behaving copies of a program

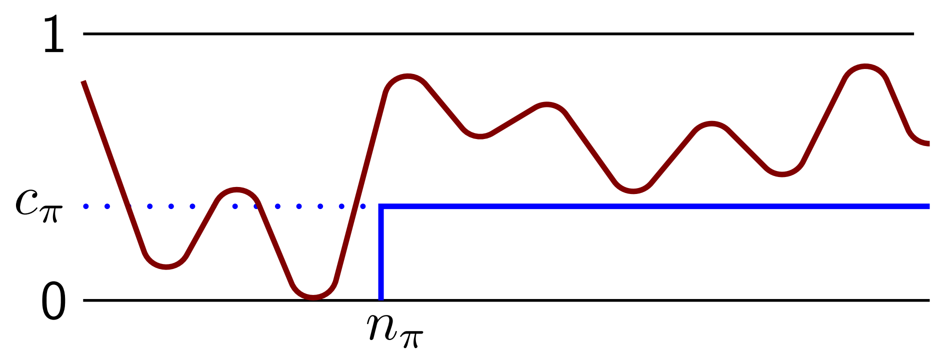
- `if( 000000000 == 123456789 ){ /* put any code here */ }`
- `bool b =` ... something very complicated yielding `false` ... `; if(b){...}`

$\Rightarrow$ we cannot even recognize all the copies nor design a language avoiding them

Many results assume that for any bigger size, many enough copies
can be made, but they need not necessarily be fully identically behaving

Example: *domain-frequency*

$$\forall \pi \in \text{programs} : \exists n_\pi \in \mathbb{N} : \exists c_\pi > 0 :$$
$$\forall n \geq n_\pi : \pi(n)/p(n) \geq c_\pi$$



- here $\pi(n) = \#$ programs of size n that halt on precisely the same inputs as π
- the esoteric minimalistic programming language *BF* is domain-frequent
- end-of-program maximum density raw data block implies domain-frequency
  - even if inaccessible to the actual code
- whether C++ is domain-frequent has been too difficult to find out!

$\{$`char*s="`σ`"`$\}$ $\rightsquigarrow$ $\{$`char*s="`σ`",*t="`ρ`"`$\}$

7 A Typical Hardness Result

If the programming language is domain-frequent, then

$$\forall T \in \text{three-way(B)} : \exists c_T > 0 : \exists n_T \in \mathbb{N} : \forall n \geq n_T : \frac{\overline{h}_T(n)}{p(n)} \geq c_T \wedge \frac{\overline{d}_T(n)}{p(n)} \geq c_T$$

$$\forall T \in \text{generic(B)} : \exists c_T > 0 : \exists n_T \in \mathbb{N} : \forall n \geq n_T : \frac{\overline{d}_T(n)}{p(n)} \geq c_T$$

$$\forall T \in \text{approx(B)} : \exists c_T > 0 : \exists n_T \in \mathbb{N} : \forall n \geq n_T : \frac{\overline{h}_T(n) + \overline{d}_T(n)}{p(n)} \geq c_T$$

That is, the proportion of hard instances does not vanish as $n \to \infty$

The proof is a modification of the classical one

- given T, all copies of nasty$_T$ are hard instances

A generic-case tester with $\overline{h}_T(n) = 0$ exists

- simulate the instance until it halts

$\Rightarrow$ cannot generalize $\overline{h}_T(n)/p(n) \geq c_T$ to the generic case

A (useless) approximat. tester with $\overline{h}_T(n) = 0$ exists, and another with $\overline{d}_T(n) = 0$

- always reply "yes", always reply "no"

8 A Model-Independent Easiness Result

For each programming model and variant X $\in$ {A, B, C} of the halting problem,

$$\forall c > 0 : \quad \exists T_c \in \text{approx(X)} : \forall n_0 \in \mathbb{N} : \exists n \geq n_0 : \frac{\overline{h}_{T_c}(n)}{p(n)} \leq c \wedge \overline{d}_{T_c}(n) = 0$$

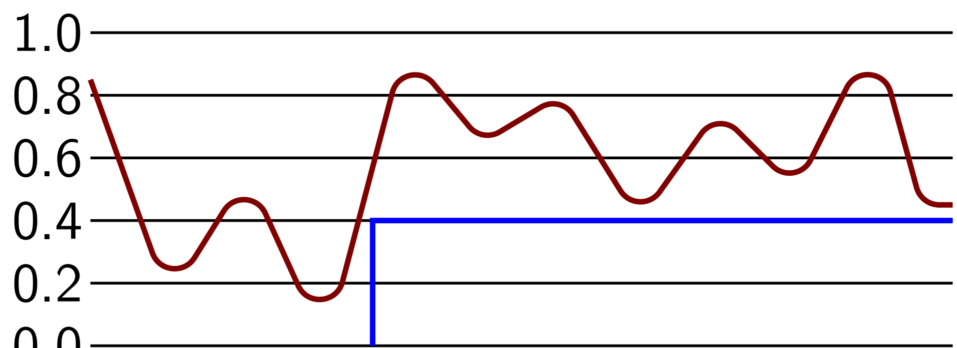
$$\forall c > 0 : \exists T_c \in \text{three-way(X)} : \forall n_0 \in \mathbb{N} : \exists n \geq n_0 : \frac{\overline{h}_{T_c}(n)}{p(n)} \leq c$$

$$\exists T \in \text{generic(X)} : \quad \forall n \in \mathbb{N} : \quad \overline{h}_T(n) = 0$$

In the approximating case, that means it is infinitely often as easy as you want

Proof for approximating testers [Köhler & al. 05]

- divide $0 \leq y \leq 1$ to strips
- there is the lowest strip i that $\frac{h(n)}{p(n)}$ visits infinitely many times
- for small n, reply "no"
- for big n, simulate instances until so many have halted that strip i is met, reply "yes" iff given instance halted



We already saw the (trivial) proof for generic-case testers

9 Anomalies Stealing the Results

For Turing machines with one-way infinite tape, it is very easy [Hamkins & al. 06]

- the probability of falling off the left end of the tape $\to 1$, as $|Q| \to \infty$

$\Rightarrow$ simulate the machine until it falls off (reply "yes")
or repeats a local state (reply "I don't know")

$$\Rightarrow \quad \exists T \in \text{three-way(X)} : \forall c > 0 : \exists n_c \in \mathbb{N} : \forall n \geq n_c : \frac{\overline{h}_T(n) + \overline{d}_T(n)}{p(n)} \leq c$$

If compile-time errors are counted, it is very easy [Köhler & al. 05], [this paper]

- the probability of syntax error $\to 1$, as $n \to \infty$

$\Rightarrow$ reply "I don't know" if compilation succeeds, otherwise "no"

By tampering the progr. lang., it can be made very easy and very hard [Lynch 74]

Each one is an *anomaly stealing the result*

- formally true, but does not tell anything about the interesting programs!
- they seem common in this research field
- make it difficult to formulate interesting results
- make it necessary to be very careful with the details of the language, etc.

10 A Difference Between A- and B-types

A program may have lots of information that it cannot access

- comments, junk after the end of a self-delimiting program, ...

If the language allows dense junk, an arbitrarily good empty-input tester exists

$$\forall c > 0 : \exists T_c \in \text{three-way(A)} : \forall n \in \mathbb{N} : \frac{\overline{h}_{T_c}(n) + \overline{d}_{T_c}(n)}{p(n)} \leq c$$

- reason: as n grows, a growing proportion of big programs are copies of programs of size $\leq n$ (yet another anomaly)
- (the claim for B in the paper is wrong, sorry ...)

A modified proof (not in the paper) of Theorem 7 yields

$$\exists c > 0 : \forall T \in \text{three-way(B)} : \forall n_0 \in \mathbb{N} : \exists n \geq n_0 : \frac{\overline{h}_T(n)}{p(n)} \geq c \wedge \frac{\overline{d}_T(n)}{p(n)} \geq c$$

- T is not in the program, but is obtained from the size of the input
  - if $|I| \in \{0, 1, 3, 6, 10, \ldots\}$, then T is P_1
  - if $|I| \in \{2, 4, 7, 11, \ldots\}$, then T is P_2, and so on

($\overline{h}_T(n)$ of any good B-tester oscillates)

So with dense junk, A is strictly easier than B

- intuitive reason: with B, the program gets the junk as part of its input

11 Discussion

There are more theorems in the paper

- even so, the results leave many questions open

⇒ lots of room for future work

Hardness proofs rely on the ability to pack raw data densely

- string constants do not seem dense enough!

⇒ theorems assumed, e.g., any byte string as the input or at the end of program

Many known easiness results arise as anomalies

- uninteresting in themselves, but make it hard to find interesting results

Ideas for future work

- perhaps it would be better to study $\overline{h}_T(n)/h(n)$ and $\overline{d}_T(n)/d(n)$?
- [Lynch 74] gives a very strong result, how do its assumptions relate to ours?

$$\exists c > 0 : \forall T \in \text{three-way(B)} : \exists n_T \in \mathbb{N} : \forall n \geq n_T : \frac{\overline{H}_T(n) + \overline{D}_T(n)}{P(n)} \geq c$$

Thank you for attention!