A Completeness Theorem for Ternary First-Order Logic

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1 Background Concepts

Signature

- countable sets of constant, function and relation symbols other than $=$
- arities for the latter two
- examples (arities shown as subscripts)
  - Presburger arithmetic: $\{0, 1\}$ $\{+_2\}$ $\emptyset$
  - Robinson arithmetic: $\{0, 1\}$ $\{+_2, \cdot_2\}$ $\emptyset$
  - Peano arithmetic: the same as Robinson arithmetic
  - Peano arithmetic with order: the same as Real closed field
  - Real closed field: $\{0, 1\}$ $\{+_2, \cdot_2\}$ $\{\leq_2\}$
  - finite bit strings: $\{\varepsilon\}$ $\{0_1, 1_1\}$ $\emptyset$

Countably infinite set of variable symbols x, y, x_1, ...

Term

- variable symbol
- constant symbol
- $f(t_1, \ldots, t_n)$, where f is a function symbol of arity n and the t_i are terms

Formula

- F, T — we will need these for technical reasons
- $t_1 = t_2$, where t_1 and t_2 are terms
- $R(t_1, \ldots, t_n)$, where R is a relation symbol of arity n and the t_i are terms
- $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall x : \varphi$, $\exists x : \varphi$, where φ and ψ are formulas
- $\neg$ has highest precedence, then $\wedge$, $\vee$, and quantifiers

We will often use familiar human-friendly shorthands

- e.g., $2x + 5$ for $+(\cdot(2, x), 5)$
- e.g., $x \leq 5$ for $\leq(x, 5)$
- e.g., $\varphi \rightarrow \psi$ for $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ for $(\psi \wedge \varphi) \vee \neg(\psi \vee \varphi)$
- e.g., additional $(\ )$ for clarity

An occurrence of x is bound iff within a sub-formula $\forall x : \ldots$ or $\exists x : \ldots$, otherwise free

- we denote the set of variables that occur free in formula φ with $\mathsf{fv}(\varphi)$
- φ is closed iff $\mathsf{fv}(\varphi) = \emptyset$

For the purpose of substituting terms for variables, we may write $\varphi(x_1, \ldots, x_n)$

- $\varphi(t_1, \ldots, t_n)$ denotes that every free occurrence of each x_i is replaced by t_i
- t_i is free for x_i in $\varphi(t_1, \ldots, t_n)$ iff no variable in t_i becomes bound by the substitution

Each theory has a set of (non-logical) axioms

- Countable set of closed formulas
- Presburger arithmetic
  - $\forall x : \neg(0 = x + 1)$
  - $\forall x : \forall y : (x + 1 = y + 1 \to x = y)$
  - $\forall x : x + 0 = x$
  - $\forall x : \forall y : x + (y + 1) = (x + y) + 1$
  - for every formula φ such that $x \notin \mathsf{fv}(\varphi) = \{y_1, \ldots, y_k\}$
    $\forall y_1 : \cdots \forall y_k : \big(\big(\varphi(0) \wedge \forall x : (\varphi(x) \to \varphi(x + 1)) \big) \to \forall x : \varphi(x) \big)$
- Robinson arithmetic: same as Presburger, except
  - the induction axioms are replaced by $\forall y : (y = 0 \vee \exists x : y = x + 1)$
  - $\forall x : x \cdot 0 = 0$
  - $\forall x : \forall y : x(y + 1) = xy + x$
- Peano arithmetic: same as Presburger, plus the Robinson axioms on $\cdot$
- Peano arithmetic with order: add $\forall x : \forall y : (x \le y \leftrightarrow \exists z : y = x + z)$
- Real closed field
  - familiar axioms for the real numbers, excluding the completeness axiom(s)
  - $\forall x : (0 \le x \to \exists y : y \cdot y = x)$
  - for every single-variable polynomial $P(x)$ of odd degree $\exists x : P(x) = 0$
- finite bit strings: every true closed formula on them
  - it is a recursive set   and Gödel's completeness theorem does not need this fact

Structure and interpretation

- assume a signature Σ is fixed
- a structure for Σ consists of a set $\mathcal{U}$ and the following
  - for each constant symbol c, an element $\underline{c}$ of $\mathcal{U}$
  - for each function symbol f, a function $\underline{f} : \mathcal{U} \times \ldots \times \mathcal{U} \to \mathcal{U}$ of the same arity
  - for each relation symbol R, a relation $\underline{R} \subseteq \mathcal{U} \times \ldots \times \mathcal{U}$ of the same arity
- a variable assignment assigns, to each variable x, a value $\underline{x} \in \mathcal{U}$
- each term evaluates to an element of $\mathcal{U}$ as follows
  - x evaluates to $\underline{x}$ and c evaluates to $\underline{c}$ and $f(t_1, \ldots, t_n)$ evaluates to $\underline{f}(\underline{t_1}, \ldots, \underline{t_n})$
  - we also say that each $\underline{u} \in \mathcal{U}$ evaluates to itself
- each formula evaluates to F or T as follows
  - F and T evaluate to themselves
  - $t_1 = t_2$ evaluates to T iff $\underline{t_1} = \underline{t_2}$
  - $R(t_1, \ldots, t_n)$ evaluates to T iff $(\underline{t_1}, \ldots, \underline{t_n}) \in \underline{R}$
  - $\neg\varphi$ evaluates to T iff φ evaluates to F
  - $\varphi \wedge \psi$ evaluates to T iff both φ and ψ do, analogously $\varphi \vee \psi$
  - $\forall x : \varphi(x)$ evaluates to T iff $\varphi(\underline{u})$ evaluates to T for every $\underline{u} \in \mathcal{U}$
  - $\exists x : \varphi(x)$ evaluates to T iff $\varphi(\underline{u})$ evaluates to T for at least one $\underline{u} \in \mathcal{U}$
- a formula holds iff it evaluates to T

Logical consequence

- assume a signature Σ is fixed
- let Γ be a set of closed formulas on Σ
- a structure is a model of Γ iff every $\varphi \in \Gamma$ holds on the structure
  - variable assignment does not matter, since Γ consists of closed formulas
- φ is a logical consequence of Γ iff for every model of Γ, and every variable assignment, φ holds
- this is denoted with $\Gamma \models \varphi$

Proof system

- a system that given a signature Σ and a set Γ of closed formulas on Σ, produces (not necessarily closed) formulas on Σ
- we say that the system proves the formula
- a proof system is sound iff every proven formula holds on every model of Γ
- a proof system is consistent iff it does not prove F
- a proof system is complete iff for every φ, it proves φ or $\neg\varphi$

2 Modern Version of Gödel's Completeness Theorem

Some sound proof system is fixed

- the theorem says that for every signature Σ and for every set Γ of closed formulas on Σ, the proof system proves all logical consequences of Γ
  - often Σ and Γ are the signature and set of axioms of some theory
- not every sound proof system is okay for the theorem, but many are
- example
  - P1: $\{\varphi\} \vdash \varphi$
  - ...
  - $\vee$-E: If $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\psi\} \vdash \chi$, then $\Gamma \cup \{\varphi \vee \psi\} \vdash \chi$
  - ...
  - $\forall$-I: If $\Gamma \vdash \varphi(x)$ and x does not occur free in Γ, then $\Gamma \vdash \forall x : \varphi(x)$

Model existence theorem

- Henkin 1949
- if Γ does not prove F, then Γ has a model
- the completeness theorem follows easily from this

Henkin's proof, step 1

- "Hilbert's hotel" the variables so that infinitely many become unused

Henkin's proof, step 2

- go through every formula φ in some order
- add φ or $\neg\varphi$ to Γ so that consistency remains
  - for simplicity, we say "add" also for the original elements of Γ
- when adding $\exists x : \psi(x)$ to Γ, choose an unused variable y and add also $\psi(y)$ to Γ
  - y is called a Henkin witness
  - it eventually becomes a constant value for which $\psi(y)$ holds
- when adding $\neg\forall x : \psi(x)$ to Γ, choose an unused variable y and add also $\neg\psi(y)$ to Γ
- let us denote the result with Γ_ω

Henkin's proof, step 3

- build a model whose $\mathcal{U}$ is the set of equivalence classes of terms
  - $[\![t]\!] = \{t' \mid \boxed{t = t'} \in \Gamma_\omega\}$
  - $\mathcal{U} = \{[\![t]\!] \mid t \text{ is a term}\}$
  - choose the $\underline{c}$, $\underline{f}$ and $\underline{R}$ as dictated by Γ_ω
- appeal to the proof system to show that it indeed is a model

3 Our Logic

How to add square root to the theory of real closed fields?

- a symbol for it is needed, so add unary $\sqrt{\ }$ to the signature
- to specify its behaviour when defined, add the axiom
  $\forall x : (0 \leq x \rightarrow \sqrt{x} \cdot \sqrt{x} = x \wedge \sqrt{x} \geq 0)$
- need to specify when it is defined   otherwise truth of, e.g., $\sqrt{-1} = 0$ is left open

In addition to the signature Σ and set Γ of closed formulas on Σ, there is a function $\lfloor \ldots \rceil$ that assigns a formula on Σ to each function symbol in Σ

- its purpose is to specify when $f(x_1, \ldots, x_n)$ is defined
- some natural choices on some intended iterpretations
  - real numbers: $\lfloor \frac{x}{y} \rceil$ is $\neg(y = 0)$
  - real numbers: $\lfloor \sqrt{x} \rceil$ is $0 \leq x$
  - natural numbers: $\lfloor \sqrt{x} \rceil$ is $\exists y : y \cdot y = x$
  - bit strings: $\lfloor \mathsf{first}(x) \rceil$ is $\neg(x = \varepsilon)$
- if f represents a total function, then it is natural to choose T as $\lfloor f \rceil$
- $\lfloor f \rceil$ may not use function symbols whose $\lfloor f \rceil$ is not T
  - in the examples above, the only used function symbol was $\cdot$

$\Rightarrow$ we may declare that $\lfloor f \rceil$ is evaluated like above (i.e., in two-valued logic)

Signature, term, formula, structure and variable assignment are defined like above

To define how formulas are evaluated, we define an extension of $\lfloor f \rceil$ to $\lfloor t \rceil$

- we say that t is defined iff $\lfloor t \rceil$ yields T
- $\lfloor c \rceil$ and $\lfloor x \rceil$ are T
  - constant and variable symbols are always defined
  - e.g., $\lfloor 3 \rceil$ is T
- $\lfloor f(t_1, \ldots, t_n) \rceil$ is $\lfloor t_1 \rceil \wedge \cdots \wedge \lfloor t_n \rceil \wedge \lfloor f \rceil (t_1, \ldots, t_n)$
  - a function invocation is defined iff every argument is and the function itself is
  - e.g., $\left\lfloor \frac{x}{\sqrt{y}} \right\rceil$ is $\mathsf{T} \wedge (\mathsf{T} \wedge 0 \leq y) \wedge \neg(\sqrt{y} = 0)$
- that is, a term is undefined iff at least one of its sub-terms is
  - e.g., $0 \cdot (1 + \frac{x}{0})$ is undefined for every x
  - e.g., $\left\lfloor \frac{x}{\sqrt{y}} \right\rceil$ is undefined iff $y \leq 0$

Each formula evaluates to F, U or T as follows

- $t_1 = t_2$ evaluates to U iff $\lfloor t_1 \rceil$ or $\lfloor t_2 \rceil$ or both yield F
  - otherwise it evaluates like in two-valued logic
- $R(t_1, \ldots, t_n)$ evaluates to U iff at least one $\lfloor t_i \rceil$ yields F
  - otherwise it evaluates like in two-valued logic
  - so $R(t_1, \ldots, t_n)$ is undefined iff at least one t_i is undefined
- $\neg\varphi$, $\varphi \land \psi$, and $\varphi \lor \psi$ evaluate as Kleene and Łukasiewicz defined

$\neg$	
F	T
U	U
T	F

$\land$	F	U	T
F	F	F	F
U	F	U	U
T	F	U	T

$\lor$	F	U	T
F	F	U	T
U	U	U	T
T	T	T	T

- $\forall x : \varphi(x)$ yields
  - T, iff $\varphi(x)$ yields T for every x
  - F, iff $\varphi(x)$ yields F for at least one x
  - U, otherwise
- $\exists x : \varphi(x)$ yields ...
- where the definition overlaps two-valued logic, it yields the same result

We define an extension of $\lfloor f \rceil$ and $\lfloor t \rceil$ to $\lfloor \varphi \rceil$ so that $\lfloor \varphi \rceil$ yields F iff φ yields U

- a relation invocation is defined iff every argument is

$$\lfloor t = t' \rceil \text{ is } \lfloor t \rceil \wedge \lfloor t' \rceil$$
$$\lfloor R(t_1, \ldots, t_n) \rceil \text{ is } \lfloor t_1 \rceil \wedge \cdots \wedge \lfloor t_n \rceil$$

- propositional rules
  - $\lfloor \mathsf{F} \rceil$ is $\lfloor \mathsf{T} \rceil$ is T — and $\lfloor \mathsf{U} \rceil$ would be F
  - $\lfloor \neg\varphi \rceil$ is $\lfloor \varphi \rceil$ — ¬U is U, but ¬T and ¬F are not
  - $\lfloor \varphi \wedge \psi \rceil$ is $(\lfloor \varphi \rceil \wedge \lfloor \psi \rceil) \vee (\lfloor \varphi \rceil \wedge \neg\varphi) \vee (\lfloor \psi \rceil \wedge \neg\psi)$
  - $\lfloor \varphi \vee \psi \rceil$ is $(\lfloor \varphi \rceil \wedge \lfloor \psi \rceil) \vee (\lfloor \varphi \rceil \wedge \varphi) \vee (\lfloor \psi \rceil \wedge \psi)$
- quantifier rules
  - $\lfloor \forall x : \varphi(x) \rceil$ is $(\forall x : \lfloor \varphi(x) \rceil) \vee \exists x : \lfloor \varphi(x) \rceil \wedge \neg\varphi(x)$
  - $\lfloor \exists x : \varphi(x) \rceil$ is $(\forall x : \lfloor \varphi(x) \rceil) \vee \exists x : \lfloor \varphi(x) \rceil \wedge \varphi(x)$

Theorem On any interpretation and assignment to variables, φ yields U if and only if $\lfloor \varphi \rceil$ yields F.

What are we doing?

- the $\lfloor f \rceil$ come from the user, similarly to Σ and Γ
- the $\lfloor t \rceil$ and $\lfloor \varphi \rceil$ are obtained algorithmically
  $\Rightarrow$ they can be used in a proof system without sacrificing effectiveness
- the $\lfloor t \rceil$ and $\lfloor \varphi \rceil$ reduce dealing with undefined terms and U to two-valued logic
- $\lfloor \cdots \rceil$ is not an operator in the logic, but in meta-language
  - e.g., $\lfloor \frac{x}{y} \rceil$ in an axiom: $\forall x : \forall y : \neg(\neg(y = 0)) \vee \dfrac{x}{y} \cdot y = x$
  - e.g., $\lfloor \frac{x}{y} \rceil$ in a proof rule: $\{\neg(x = 0)\} \vdash \dfrac{x+1}{x} = \dfrac{x+1}{x}$

Logical consequence may now be defined like above

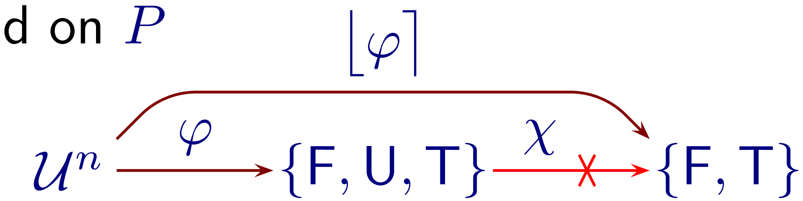
Our theorem says that for every signature Σ, for every $\lfloor f \rceil$ on Σ, and for every set Γ of closed formulas on Σ, the proof system proves all logical consequences of Γ

- we will soon present our proof system

Kleene defined that a propositional formula is regular iff either $\varphi(\ldots, \mathsf{U}, \ldots)$ yields U or $\varphi(\ldots, P, \ldots)$ does not depend on P

$\Rightarrow$ cannot express "P yields U"

- the notion extends naturally to predicate logic
- we exploit later the fact that our predicate logic is regular (not important)



4 Sound and Complete Proof System

Notation

- φ, ψ, χ are formulas
- Γ, Δ are sets of formulas
- x, x_i, y are variable symbols
- t, t_i, t'_i are terms

Rules about reasoning in general:

P1: $\{\varphi\} \vdash \varphi$

P2: If $\Gamma \vdash \varphi$ then $\Gamma \cup \Delta \vdash \varphi$

P3: If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \psi$

The Law of the Excluded Fourth and the concept of contradiction:

C1: $\emptyset \vdash \varphi \lor \neg\varphi \lor \neg\lfloor\varphi\rceil$ (this replaces the Law of Excluded Middle)

C2: $\{\mathsf{F}\} \vdash \varphi$

C3: $\{\varphi,\ \neg\varphi\} \vdash \mathsf{F}$

(C2 and C3) or $\{\varphi,\ \neg\varphi\} \vdash \psi$

If a formula is true, then it is also defined:

D1: $\{\varphi\} \vdash \lfloor\varphi\rceil$ (this does not exist in classical logic)

For instance

- D1: $\left\{\frac{\sqrt{x}}{x-1} > 0\right\} \vdash x \geq 0 \land \neg(x-1=0)$
- C1: $\emptyset \vdash \frac{\sqrt{x}}{x-1} > 0 \lor \neg\left(\frac{\sqrt{x}}{x-1} > 0\right) \lor \neg(x \geq 0 \land \neg(x-1=0))$

If the system is not contradictory

- that is, if $\Gamma \not\vdash \mathsf{F}$
- please recall that $\lfloor\neg\varphi\rceil$ is $\lfloor\varphi\rceil$

$\Rightarrow$ C1 and D1 make precisely one of φ, $\neg\varphi$, and $\neg\lfloor\varphi\rceil$ hold

$\Rightarrow$ each claim yields precisely one of T, F, and U for each assignment to variables

Rules for conjunction and disjunction:

$\wedge$-**I:** $\{\varphi, \psi\} \vdash \varphi \wedge \psi$

$\wedge$-**E1:** $\{\varphi \wedge \psi\} \vdash \varphi$

$\wedge$-**E2:** $\{\varphi \wedge \psi\} \vdash \psi$

$\vee$-**I1:** $\{\varphi\} \vdash \varphi \vee \psi$

$\vee$-**I2:** $\{\psi\} \vdash \varphi \vee \psi$

$\vee$-**E:** If $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\psi\} \vdash \chi$, then $\Gamma \cup \{\varphi \vee \psi\} \vdash \chi$

Rules of equality:

=-1: $\{\lfloor t \rceil\} \vdash t = t$

=-2: If f is an n-ary function symbol and $1 \le i \le n$, then

$$\{t_i = t'_i,\ \lfloor f(t_1, \ldots, t_n) \rceil\} \vdash f(t_1, \ldots, t_n) = f(t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n)$$

=-3: If $\varphi(x_1, \ldots, x_n)$ is a formula, $1 \le i \le n$ and t_i and t'_i are free for x_i in φ, then

$$\{t_i = t'_i,\ \varphi(t_1, \ldots, t_n)\} \vdash \varphi(t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n)$$

For instance

- =-1: $\{x \ge 0 \wedge \neg(x - 1 = 0)\} \vdash \frac{\sqrt{x}}{x-1} = \frac{\sqrt{x}}{x-1}$
- =-2: $\{x = z + 1,\ x \ge 0 \wedge \neg(x - 1 = 0)\} \vdash \frac{\sqrt{x}}{x-1} = \frac{\sqrt{x}}{z+1-1}$

Comments

- =-1 and =-2 were tailored to not prove an undefined term equivalent to something
- by definition, $\lfloor f(t_1, \ldots, t_n) \rceil$ yields $\lfloor t_1 \rceil, \ldots, \lfloor t_n \rceil$
- by D1, $t_i = t'_i$ implies $\lfloor t_i = t'_i \rceil$, which is $\lfloor t_i \rceil \wedge \lfloor t'_i \rceil$, so $\lfloor t'_i \rceil$
- =-3 need only be assumed for relations, but proving that is too long and dull

Rules for quantifiers:

$\forall$-E: If t is free for x in φ, then $\{\lfloor t \rceil, \forall x : \varphi(x)\} \vdash \varphi(t)$

$\forall$-I: If $\Gamma \vdash \varphi(x)$ and x does not occur free in Γ, then $\Gamma \vdash \forall x : \varphi(x)$

$\exists$-I: If t is free for x in φ, then $\{\varphi(t)\} \vdash \exists x : \varphi(x)$

$\exists$-E: If $\Gamma \cup \{\varphi(y)\} \vdash \psi$ and y does not occur in Γ, $\exists x : \varphi(x)$, nor in ψ,
then $\Gamma \cup \{\exists x : \varphi(x)\} \vdash \psi$

Comments

- $\forall$-E was tailored to not prove anything about undefined terms
- variable symbols are never undefined, so $\forall$-I and $\exists$-E need not be tailored
- $\exists$-I need not be $\{\varphi(t), \lfloor t \rceil\} \vdash \exists x : \varphi(x)$
  - if t is undefined but $\varphi(t)$ is not, then by regularity $\forall x : \varphi(x)$ holds

Only 5 differences from two-valued logic!

5 Completeness Proof

We use Henkin's strategy: prove that every consistent theory has a model

$\Rightarrow$ if $\Gamma \not\vdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ has a model, so φ is not a semantic consequence of Γ

- therefore, we assume from now on $\Gamma \not\vdash \mathsf{F}$

Lemma There is Γ' such that

- $\Gamma' \not\vdash \mathsf{F}$
- both or neither of Γ and Γ' have a model
- infinitely many variable symbols are unused in Γ'
- for every bound x in Γ' there is an x' such that its only occurrence in Γ' is $x = x'$

Proof Replace each v_i in Γ by v_{3i} and add the $v_{3i} = v_{3i-1}$. $\square$

Choose true formulas, introduce witnesses

- let $\Gamma_0 := \Gamma'$
- for every formula φ_i, construct Γ_i by applying the first that matches

if	φ_i form	$\Gamma_i := \Gamma_{i-1} \cup$
$\Gamma_{i-1} \cup \{\lfloor \varphi_i \rceil\} \vdash$ F		$\{\neg \lfloor \varphi_i \rceil\}$
$\Gamma_{i-1} \cup \{\varphi_i\} \quad \nvdash$ F	is $\exists x : \psi(x)$	$\{\varphi_i,\ \psi(y)\}$
$\Gamma_{i-1} \cup \{\neg\varphi_i\} \nvdash$ F	is $\forall x : \psi(x)$	$\{\neg\varphi_i, \neg\psi(y)\}$
$\Gamma_{i-1} \cup \{\varphi_i\} \quad \nvdash$ F	not $\exists x : \psi(x)$	$\{\varphi_i\}$
$\Gamma_{i-1} \cup \{\neg\varphi_i\} \nvdash$ F	not $\forall x : \psi(x)$	$\{\neg\varphi_i\}$

- let $\Gamma_\omega := \Gamma_0 \cup \Gamma_1 \cup \cdots$

Lemma

- $\Gamma' = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_\omega$
- $\Gamma_\omega \nvdash$ F
- for each φ, precisely one of φ, $\neg\varphi$ and $\neg\lfloor \varphi \rceil$ is in Γ_ω
- for each φ, precisely one of $\lfloor \varphi \rceil$ and $\neg\lfloor \varphi \rceil$ is in Γ_ω
- for each t, precisely one of $\lfloor t \rceil$ and $\neg\lfloor t \rceil$ is in Γ_ω
- $\Gamma_\omega \vdash \varphi$ if and only if $\varphi \in \Gamma_\omega$

Theorem Γ_ω has a model

Proof

- elements of the universe are
  - equivalence classes of terms for which $\lfloor t \rceil \in \Gamma_\omega$, induced by the $t = t'$ in Γ_ω
  - a single element for the remaining terms
- nothing depends on the choice of the representative of each equivalence class
  - where necessary, use v_{3i-1} to make terms free for x

$\in \Gamma_\omega$	φ	$\neg\varphi$	$\neg\lfloor\varphi\rceil$
truth value of φ	T	F	U

- some routine arguments
- lots of dull reasoning using the proof system □

Corollary Both Γ' and Γ have a model

6 Extension to Łukasiewicz Logic

Łukasiewicz: $\mathsf{U} \xrightarrow{\text{Ł}} \mathsf{U}$ yields T and $\mathsf{U} \xleftrightarrow{\text{Ł}} \mathsf{U}$ yields T

- that P yields U can be expressed as $(P \xrightarrow{\text{Ł}} \neg P) \wedge (\neg P \xrightarrow{\text{Ł}} P)$
- all truth functions $\{\mathsf{F}, \mathsf{U}, \mathsf{T}\}^n \to \{\mathsf{F}, \mathsf{U}, \mathsf{T}\}$ can be expressed

This reduces to the earlier case by replacing each

$$\varphi \xrightarrow{\text{Ł}} \psi$$

by

$$\neg\varphi \vee \psi \vee \neg(\lfloor \varphi \rceil \vee \lfloor \psi \rceil)$$

7 Conclusions

Key ideas

- $\frac{1}{0}$, etc., are not treated as values
  - variables are never undefined, terms may be
  - $\frac{1}{0} = \frac{1}{0}$, $\frac{1}{0} \neq \frac{1}{0}$, and $\frac{1}{0} > \frac{1}{0}$ yield U
- the intuitive notion "is defined" is encoded as mechanical rules
  - "is defined" is itself always defined
- for each φ, the model contains precisely one of φ, $\neg\varphi$ and $\neg\lfloor\varphi\rceil$
  - correspondingly φ yields T, F or U

Many practical reasoning laws have been developed

- were a topic of another talk
- regularity simplifies things

Thank you for attention! Questions?